ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 4, 2005

A-COMPACTNESS AND MINIMAL SUBALGEBRAS OF $C(X)$

S.K. ACHARYYA AND D. DE

ABSTRACT. Let $\sum(X)$ be the set of all subalgebras of $C(X)$ containing $C^*(X)$, where X is a Tychonoff space. Given $A(X) \equiv A \in \sum(X)$ there is associated a subset $\upsilon_A X$ of βX which is an A-analogue of the Hewitt real compactification υX of X. X is called A-compact if and only if $\upsilon_A X = X$. Redlin and Watson asked whether, for any real compact space X, there exists in some sense a minimal $A_m \in \sum(X)$ for which X becomes A_m-compact. Acharyya, Chattopadhyay and Ghosh answered this question in affirmative by defining a suitable preorder on $\sum(X)$, and they made the following conjecture that there does not exist any minimal subalgebra $A(\mathbf{N})$ of $C(\mathbf{N})$ containing $C^*(\mathbf{N})$, in the usual set inclusion sense for which $\mathbf{N}$ is A-compact. In this paper we have shown that, given any real compact space X there does not exist any minimal member $A_m \in \sum(X)$, in the usual set inclusion sense for which X becomes A_m-compact and thereby proving the conjecture as a special case of it. From this result it has been further shown that for any $A(X) \neq C^*(X)$ in $\sum(X)$ there does not exist any minimal member $B(X) \in \sum(X)$ in the usual set inclusion sense for which $\upsilon_A X = \upsilon_B X$.

1. Introduction. It is a remarkable fact in the theory of rings of continuous functions that the Stone-Čech compactification βX of a Tychonoff space X could be realized as the set of all maximal ideals of an arbitrary subalgebra $A(X)$ of $C(X)$ containing $C^*(X)$, equipped with hull kernel topology. Such subalgebras of $C(X)$ were initiated and investigated in detail by Plank [**7**]. Let $\sum(X)$ be the family of all such subalgebras of $C(X)$. Following Redlin and Watson [**8**], for any $A(X) \equiv A \in \sum(X)$, a maximal ideal M of A is called *real* provided that the quotient field A/M is isomorphic to the real field $\mathbf{R}$, otherwise

2000 AMS *Mathematics Subject Classification.* Primary 54X10, 58Y30, Secondary 55Z10.
Key words and phrases. Algebra of continuous functions, real compactification, Stone-Čech compactification, convex ring.
This paper constitutes a portion of D. De's doctoral dissertation under the supervision of Prof. S.K. Acharyya. This research was supported by a CSIR fellowship.
Received by the editors on July 20, 2002.

M is called *hyper-real*. X is called A-compact if every real maximal ideal M of A is fixed in the sense that there is a point x in X at which every function in M vanishes. With every $A(X) \in \sum(X)$ we have associated a subset $\upsilon_A X$ of βX, which is an A-analogue of υX, the Hewitt real compactification of X. X is called A-compact if and only if $\upsilon_A X = X$. In this terminology every compact space is C^*-compact and every real compact space is C-compact. The last two authors proved that if $A, B \in \sum(X)$ and $A \subset B$ then the assumption X is A-compact implies that X is B-compact, in particular every A-compact space is real compact; quite naturally they asked the following question. Given any real compact space X, does there exist any minimal member A of $\sum(X)$ in some sense for which X becomes A-compact? Acharyya, Chattopadhyay and Ghosh [**1**] offered an affirmative answer to this question after defining a suitable preorder on the class $\sum(X)$. But the minimal members of $\sum(X)$ thus discovered are no longer minimal in the usual set inclusion sense, and these three authors conjectured in the same paper that there does not exist any minimal subalgebra $A(\mathbf{N})$ of $C(\mathbf{N})$ containing $C^*(\mathbf{N})$ in the usual set inclusion sense for which $\mathbf{N}$ is A-compact. In the present paper we have shown that if X is a noncompact real compact space then there does not exist any minimal member $A_m(X)$ of $\sum(X)$ in the usual set inclusion sense for which X becomes A_m-compact. The conjecture mentioned in the last sentence is therefore an immediate corollary to this fact. Furthermore it has been deduced in this paper that for any $A(X) \neq C(X)$ in $\sum(X)$, there does not exist in the usual set inclusion sense any minimal member $B(X)$ of $\sum(X)$ with the property $\upsilon_A X = \upsilon_B X$. Nevertheless the problem posed by Redlin and Watson [**8**] does not end with these results. It is still unknown whether there exists a minimal member A of $\sum(X)$ with respect to a suitable partial ordering on it, necessarily different from the usual set inclusion, for which X becomes A-compact. To conclude the authors would like to refer to [**2**] for the benefit of a reader who is new to the subject.

2. A-compactifications. In what follows X will stand for a Tychonoff space and for any f in $C(X)$, $Z(f) \equiv \{x \in X : f(x)\} = 0$ will be called the *zero set* of f. For any ideal I in $C(X)$, $Z[I]$ consists of the sets $Z[f]$ for f in I. Each member f of $C(X)$ has a unique continuous extension $f^* : \beta X \to \mathbf{R}^* = \mathbf{R} \bigcup \{\infty\}$, where $\mathbf{R}^*$ is the onepoint compactification of $\mathbf{R}$. If $f \in C^*(X)$, f^* is the same as f^β, the unique continuous extension of f over βX. For our convenience we record the following known result.

Theorem 2.1. *If $p \in \beta X - X$ and f, g are in $C(X)$ such that $f^*(p)$ and $g^*(p)$ are both real, then $(f+g)^*(p) = f^*(p) + g^*(p)$ and $(f.g)^*(p) = f^*(p).g^*(p)$.*

Plank has shown in [**7**] that, given $A(X) \in \sum(X)$, the family of all the maximal ideals in A is exactly the set $\{M_A^p : p \in \beta X\}$ where $M_A^p = \{(f.g)^*(p) = 0$ for all $g \in A\}$; and M_A^p is a free ideal if and only if $p \in \beta X - X$. Following Redlin and Watson [**8**], Acharyya, Chattopadhyay and Ghosh [**1**] defined a maximal ideal M in A to be real provided that the quotient field A/M is isomorphic to $\mathbf{R}$, otherwise M is called hyper-real. X is called A-compact if every real maximal ideal of A is fixed. Set $\upsilon_A X = \{p \in \beta X : M_A^p$ is real$\}$. We call it A-compactification of X. Then $\upsilon_C X$ and $\upsilon_{C^*} X$ are clearly the Hewitt real compactification υX and the Stone-Čech compactification βX of X, respectively. Also X is A-compact if and only if $\upsilon_A X = X$. Now as in [**6**] we have established the following results in [**2**].

Theorem 2.2. *For any $p \in \beta X$, M_A^p is a real maximal ideal in A, if and only if $f^*(p) \neq \infty$ for all f in $A(X)$.*

Theorem 2.3. *For any $A(X) \in \sum(X)$, $\upsilon_A X$ is the largest subspace of βX to which each member of $A(X)$ can be extended continuously.*

Let f^{υ_A} be the unique continuous extension over $\upsilon_A X$ of f in $A(X)$. Then $A^\upsilon = \{f^{\upsilon_A} : f \in A(X)\}$ is an algebra over $\mathbf{R}$ lying between $C^*(\upsilon_A X)$ and $C(\upsilon_A X)$, furthermore $\upsilon_A X$ is an A^υ-compact space, see [**2**]. Also from Corollary 3.7 of [**3**] we get the following result.

Theorem 2.4. *Let $A_1(X)$, $A_2(X)$ in $\sum(X)$ be such that $A_2(X) \subset A_1(X)$ and $p \in \upsilon_{A_2} X$. Then $M_{A_1}^p$ is hyperreal if and only if $M_{A_2}^p$ contains a unit of $A_1(X)$.*

The result of this theorem can now be stated in a straightforward manner to establish the following:

Theorem 2.5. *Let $A_1, A_2 \in \sum(X)$ with $A_2 \subset A_1$. Then $\upsilon_{A_1}X \subset \upsilon_{A_2}X$. In particular therefore if X is A_2-compact, then it is A_1-compact.*

Remark 2.6. From the above theorem it clearly follows that every A-compact space X is real compact and therefore $\upsilon_A X$ is always real compact.

We define a relation '$\sim$' among the members of $\sum(X)$ as follows: $A \sim B$ if and only if $\upsilon_A X = \upsilon_B X$. Then '$\sim$' is an equivalence relation on $\sum(X)$; let us denote the equivalence class of $A(X)$ by $[A(X)]$. Set $B(X) = \{h|_X : h \in C(\upsilon_A X)\}$. Then in view of Theorems 2.3 and 2.5, $B(X)$ is a member of $\sum(X)$ containing $A(X)$ with $\upsilon_A X = \upsilon_B X$. Also, if $B_1(X) \in \sum(X)$ is such that $\upsilon_{B_1}X = \upsilon_A X$ it is plain that $B_1(X) \subset B(X)$. Thus we have the following:

Theorem 2.7. *Each equivalence class $[A(X)]$ has a largest member, which consists precisely of those functions in $C(X)$ which have continuous extensions over $\upsilon_A X$.*

Remark 2.8. Since each function in $C^*(X)$ has a continuous extension over βX and no function in $C(X) \setminus C^*(X)$ has any such extension over βX it is clear that $[C^*(X)] = \{C^*(X)\}$.

We write down the following known fact from [**1**].

Theorem 2.9. *A space X is A-compact if and only if for every $p \in \beta X \setminus X$, there exists an f in $C^*(X)$ such that f is a unit of $A(X)$ and $f^\beta(p) = 0$ or, equivalently, X is A-compact if and only if for every $p \in \beta X \setminus X$, there exists a unit g of $A(X)$ such that $g^{-1} \in C^*(X)$ and $g^*(p) = \infty$.*

We also take down the following known fact as given in Proposition 3.3 of [**4**].

Theorem 2.10. *Each $A(X)$ in $\sum(X)$ is absolutely convex, in particular, a lattice ordered ring.*

3. The question of existence of minimal member of $\sum(X)$.

Theorem 3.1. *Let X be a noncompact real compact space. Then there does not exist any minimal member $A(X) \in \sum(X)$ in the usual set inclusion sense for which X is A-compact.*

Proof. Let $A(X) \in \sum(X)$ be such that X is A-compact. Since X is a noncompact real compact space it cannot be pseudocompact. Hence by result 1.21 of [**6**], X contains a closed discrete set $\{x_n : n \in \omega\}$ C-*embedded* in it. We fix any point $p \in \beta X$ which is a limit point of $\{x_n : n \in \omega\}$. Since X is A-compact, by Theorem 2.9 we can select an $f \in A$ such that $f \geq 1$ and $f^*(p) = \infty$. By using the continuity of f^* at p and the fact that p is in the closure of $\{x_n : n \in \omega\}$ in βX, we can construct an increasing sequence $\{k_n : n \in \omega\}$ of natural numbers so that $f(x_{k_n}) > n$ for each $n \in \omega$. Let N denote the subspace $\{x_{k_n} : n \in \omega\}$ of X, which is obviously a copy of $\mathbf{N}$ in X. Set $B = \mathcal{A}(C^*(N) \cup \{f|_N\})$, the smallest subalgebra of $C(N)$ containing $C^*(N)$ and $f|_N$, and $D = \mathcal{A}(C^*(N) \cup \{\log_e(1+f)|_N\})$. Then, as in 4.1 of [**1**], $D \subseteq B$.

Let $D' = \{h \in A : h|_N \in D\}$. Clearly $C^*(X) \subset D'$ and D' is a proper subset of A as $f \in A \setminus D'$, also D' is an algebra over $\mathbf{R}$ so that $D' \in \sum(X)$. For each $q \in \mathrm{cl}_{\beta X} N \setminus X$, $f^*(q) = \infty$, hence $h^*(q) = \infty$ where $h = \log_e(1+f)$. We note that $h \in D'$. On the other hand, for each $q \in \beta X \setminus X \setminus cl_{\beta X} N$, by Theorem 2.9, we can choose an $a \in A$ with $a \geq 1$ and $a^*(q) = \infty$; also by using just the complete regularity of βX we can have a $g \in C^*(X)$ with $g^*(q) = 1$ and $g(N) = 0$. It is clear that $a.g \in D'$ and $(a.g)^*(q) = \infty$. Altogether for any point $q \in \beta X \setminus X$ we can select an $h' \in D'$ with $h'^*(q) = \infty$. Hence by Theorem 2.9 X becomes D'-compact. The theorem is completely proved. □

We conclude this article after observing that each equivalence class $[A(X)]$ with $A(X) \neq C^*(X)$, in Theorem 2.7 is devoid of any minimal member with respect to the usual set inclusion relation.

Theorem 3.2. *Let X be a noncompact Tychonoff space not necessarily real compact. Suppose $A(X) \neq C^*(X)$ is any member of $\sum(X)$. Then there exists an $A_0(X) \in \sum(X)$ such that $A_0(X) \subsetneqq A(X)$ and $\upsilon_{A_0}X = \upsilon_A X$.*

Proof. The case with X A-compact has already been proved in Theorem 3.1. Assume therefore that X is not A-compact. We recall the fact as mentioned after Theorem 2.3 that $\upsilon_A X$ is an A^υ-compact space where $\{f^{\upsilon_A} : f \in A\} = A^\upsilon$ with f^{υ_A} being the unique continuous extension of f to $\upsilon_A X$. Since X is not A-compact, $X \subsetneqq \upsilon_A X$ and as $A(X) \neq C^*(X)$ in view of Remark 2.8 it is clear that $\upsilon_A X \neq \beta X$. Therefore $\upsilon_A X$ is a noncompact real compact space. Hence, by Theorem 3.1 there exists a subalgebra $B(\upsilon_A X)$ of $C(\upsilon_A X)$ with $C^*(\upsilon_A X) \subsetneqq B(\upsilon_A X) \subsetneqq A^\upsilon$ such that $\upsilon_A X$ is $B(\upsilon_A X)$-compact. Now set $A_0(X) = \{f|_X : f \in B(\upsilon_A X)\}$. Then $A_0(X) \in \sum(X)$ and $A_0(X) \subsetneqq A(X)$. To complete the theorem we shall show that $\upsilon_{A_0}X = \upsilon_A X$. As $A_0(X) \subset A(X)$ by Theorem 2.5 we have $\upsilon_A X \subset \upsilon_{A_0} X$. Now from Theorem 6.7 of [**6**] we can write $\beta(\upsilon_A X) = \beta X$. Then for any $p \in \beta X \backslash \upsilon_A X$, as $\upsilon_A X$ is $B(\upsilon_A X)$-compact, by Theorem 2.9 there exists $f_p \in B(\upsilon_A X)$ such that $f_p^*(p) = \infty$. Let $f = f_p|_X$, so that $f \in A_0(X)$. It is clear that $f^*(p) = \infty$, i.e., $p \notin \upsilon_{A_0} X$ by Theorem 2.2. Since this is true for any $p \in \beta(\upsilon_A X) \setminus \upsilon_A X$ it follows that $\upsilon_{A_0} X \subset \upsilon_A X$. Hence $\upsilon_{A_0} X = \upsilon_A X$. □

REFERENCES

1. S.K. Acharyya, K.C. Chattopadhyay and D.P. Ghosh, *On a class of subalgebras of $C(X)$ and the intersection of their free maximal ideals*, Proc. Amer. Math. Soc. **125** (1997), 611–615.

2. ———, *A class of subalgebras of $C(X)$ and the associated compactness*, Kyungpook Math. J. **41** (2001), 323–324.

3. H.L. Byun and S. Watson, *Prime and maximal ideals of $C(X)$*, Topology Appl. **40** (1991), 45–62.

4. J.M. Dominguege, J. Gomez and M.A. Mulero, *Intermediate algebras between $C^*(X)$ and $C(X)$ as rings of fractions of $C^*(X)$*, Topology Appl. **77** (1997), 115–130.

5. D.P. Ghosh, *Compactification, real compactification, and subsets of $C(X)$*, Ph.D. Thesis, Burdwan University, 1991.

6. L. Gillman and M. Jerison, *Rings of continuous functions*, Springer-Verlag, New York, 1976.

7. D. Plank, *On a class of subalgebras of $C(X)$ with application to $\beta X - X$*, Fund. Math. **64** (1969), 41–54.

8. L. Redlin and S. Watson, *Maximal ideals in subalgebras of $C(X)$*, Proc. Amer. Math. Soc. **100** (1987), 763–766.

Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata-700019, India
E-mail address: sudipkumaracharyya@yahoo.co.in

Department of Mathematics, Krishnagar Women's College, Krishnagar, Nadia-741101, India
E-mail address: d1dey_72@rediffmail.com