ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 2, 2005

CONJUGACY SEPARABILITY OF CERTAIN HNN EXTENSIONS OF GROUPS

GOANSU KIM AND C.Y. TANG

ABSTRACT. In this paper we improve a criterion for the conjugacy separability of HNN extensions of conjugacy separable groups with cyclic associated subgroups. Using this result, we characterize the conjugacy separability of such HNN extensions of finitely generated nilpotent groups.

1. Introduction. A group G is said to be *conjugacy separable* if, for each pair $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\overline{G}$ of G such that the images of x and y in $\overline{G}$ are not conjugate in $\overline{G}$. Conjugacy separability is related to the conjugacy problem for groups as observed by Mal'cev [**11**] and Mostowski [**13**]. In this paper we consider the conjugacy separability of HNN extensions of a conjugacy separable group A with cyclic associated subgroups $\langle h \rangle$ and $\langle k \rangle$:

$$G = \langle A, t : t^{-1}ht = k \rangle.$$

In general, residual and separability properties of HNN extensions depend very much on the choice of the associated subgroups. Meskin [**12**] showed that the Baumslag-Solitar group, $\langle a, t : t^{-1}a^{\alpha}t = a^{\beta} \rangle$, is residually finite if and only if $|\alpha| = 1$ or $|\beta| = 1$ or $\alpha \pm \beta = 0$. In this paper we show that this is also true for conjugacy separability. Baumslag and Tretkoff [**3**] showed that HNN extensions of residually finite groups with finite associated subgroups are residually finite. Collins [**4**] proved a similar result for HNN extensions of conjugacy separable groups. We will make use of these results in our paper. Shirvani [**18**], Andreadakis, Raptis and Varsos [**1, 17**], considered residual finiteness

Received by the editors on April 17, 2001, and in revised form on July 17, 2003.
1991 AMS *Mathematics Subject Classification.* Primary 20E26, 20E06, 20F10.
Key words and phrases. HNN-extension, conjugacy separable, residually finite.
The first author was partly supported by KOSEF through the GARC at S.N.U. and KOSEF 985-0100-002-2.
The second author gratefully acknowledges the partial support by the Natural Science and Engineering Research Council of Canada, Grant No. A-4064.

of HNN extensions of residually finite groups. Raptis, Talelli and Varsos [**16**] proved an interesting result that if G is an HNN extension of a finitely generated abelian group then G is conjugacy separable if and only if G is residually finite. In this paper we obtain a similar result for HNN extensions of finitely generated nilpotent groups with cyclic associated subgroups. In [**7**], Kim and Tang characterized the conjugacy separability of HNN extensions of finitely generated abelian groups with cyclic associated subgroups. In this paper we improve the criterion given by [**8**, Theorem 4.5] for the conjugacy separability of HNN extensions with cyclic associated subgroups, Theorem 3.9. Using this we characterize the conjugacy separability of HNN extensions of finitely generated nilpotent groups with nontrivially intersecting cyclic associated subgroups, Theorem 3.15. It turns out that the result is similar to the case of finitely generated abelian groups. If the associated subgroups intersect nontrivially, a similar result cannot be obtained because Example 1 [**9**] shows that such HNN extensions need not be residually finite. However Corollary 3.12 gives a similar characterization if the associated subgroups are in the center of the base group.

2. Preliminaries. Throughout this paper we use standard notations and terminology for this topic. The letter G always denotes a group. In addition:

$N \lhd_f G$ means that N is a normal subgroup of finite index in G;

Let S be a subset of G. Then $x \sim_S y$ means x is conjugate to y by an element of S;

We use $\|x\|$ to denote the length of x in HNN-extensions.

We shall make extensive use of the following two results by Collins.

Theorem 2.1 [**5**, Theorem 3]. *Let x and y be cyclically reduced elements of the* HNN-*extension $G = \langle B, t : t^{-1}Ht = K\rangle$. Suppose that $x \sim_G y$. Then $\|x\| = \|y\|$, and one of the following holds.*

(1) $\|x\| = \|y\| = 0$ *and there is a finite sequence $z_1, z_2, \dots, z_m$ of elements in $H \cup K$ such that $x \sim_B z_1 \sim_{B,t^*} z_2 \sim_{B,t^*} \cdots \sim_{B,t^*} z_m \sim_B y$, where $u \sim_{B,t^*} v$ means one of*:

(i) $u \sim_B v$, *or*

(ii) $u \in H$ *and* $v = t^{-1}ut (\in K)$, *or*

(iii) $u \in K$ *and* $v = tut^{-1} (\in H)$.

(2) $\|x\| = \|y\| \geq 1$ *and* $y \sim_{H \cup K} x^*$ *where* x^* *is a cyclic permutation of* x.

Let S be a subset of a group G. Then G is said to be S-*separable* if for each $x \in G \setminus S$ there exists $N \lhd_f G$ such that $x \notin NS$. In particular, G is *residually finite* if G is $\{1\}$-separable. A group G is said to be *cyclic subgroup separable*, if G is $\langle x \rangle$-separable for each $x \in G$. Such a group is also called π_c by Stebe [**19**].

A group G is *conjugacy separable* if, for each $x \in G$, G is $\{x\}^G$-separable, where $\{x\}^G$ is the set of all conjugates of x in G.

Theorem 2.2 [**5**, Theorem 13]. *If A is conjugacy separable and H, K are finite, then the* HNN-*extension* $\langle A, t : t^{-1}Ht = K \rangle$ *is conjugacy separable.*

In [**14**], Niblo introduced the concept of regular quotient at $\{h, k\}$. For our purpose, we define the following which is the same as "quasi-regular" in [**8**] and [**9**].

Definition 2.3. Let A be a group, and let $h, k \in A$ be of infinite order. Then A is said to be *regular at* $\{h, k\}$ if, for each given integer $\varepsilon > 0$, there exist an integer $\lambda_\varepsilon > 0$ and $N_\varepsilon \lhd_f A$, depending on ε, such that $N_\varepsilon \cap \langle h \rangle = \langle h^{\varepsilon \lambda_\varepsilon} \rangle$ and $N_\varepsilon \cap \langle k \rangle = \langle k^{\varepsilon \lambda_\varepsilon} \rangle$.

Remark 2.4. Let $G = \langle A, t : t^{-1}ht = k \rangle$. For $N \lhd_f A$ such that $N \cap \langle h \rangle = \langle h^s \rangle$ and $N \cap \langle k \rangle = \langle k^s \rangle$, there is a natural homomorphism

$$\pi_N : \langle A, t : t^{-1}ht = k \rangle \longrightarrow \langle \overline{A}, \tau : \tau^{-1}\bar{h}\tau = \bar{k} \rangle,$$

where $\overline{A} = A/N$, $a\pi_N = \bar{a}$ for $a \in A$ and $t\pi_N = \tau$. Then, by Theorem 2.2, the homomorphic image $G\pi_N$ of G is conjugacy separable.

Regularity plays an important role in the study of HNN extensions as seen in the next result.

Theorem 2.5 [**9**]. *Let A be cyclic subgroup separable, and let $h, k \in A$ be of infinite order. Then $G = \langle A, t : t^{-1}ht = k\rangle$ is cyclic subgroup separable if and only if A is regular at $\{h, k\}$.*

Definition 2.6 [**8**]. A group A is said to be *double coset separable* at $\{h, k\}$ if, for each $u \in A$ and for each integer $\varepsilon > 0$, A is $\langle h^\varepsilon\rangle u\langle h^\varepsilon\rangle$-separable, $\langle h^\varepsilon\rangle u\langle k^\varepsilon\rangle$-separable and $\langle k^\varepsilon\rangle u\langle k^\varepsilon\rangle$-separable.

Lemma 2.7 [**8**]. *Let G be $\langle a^\varepsilon\rangle x\langle b^\varepsilon\rangle$-separable, where $x, a, b \in G$ and a, b are of infinite order. If $\langle x^{-1}ax\rangle \cap \langle b\rangle = 1$, then there exists $N \triangleleft_f G$ such that $\bar{x}^{-1}\bar{a}^i\bar{x} = \bar{b}^j$ only if $\varepsilon \mid i, j$, where $\overline{G} = G/N$.*

Lemma 2.8 [**8**]. *Let A be regular at $\{h, k\}$ and double coset separable at $\{h, k\}$. Then $G = \langle A, t : t^{-1}ht = k\rangle$ is double coset separable at $\{h, k\}$.*

3. A criterion. In this section we consider the conjugacy separability of HNN extensions of the type

$$G = \langle A, t : t^{-1}ht = k\rangle,$$

where A is conjugacy separable. In [**8**, Theorem 4.5] we proved a criterion for the conjugacy separability of G when $\langle h\rangle \cap \langle k\rangle = 1$. In this section we prove our main result Theorem 3.9 which generalizes the criterion to include the case $\langle h\rangle \cap \langle k\rangle \neq 1$. Applying this result we characterize the conjugacy separability of G of finitely generated nilpotent groups A when $\langle h\rangle \cap \langle k\rangle \neq 1$. If $|h| = |k|$ is finite then, by Theorem 2.2, the above G is conjugacy separable. Thus, throughout this section, we assume that h, k are of infinite order and if $h^n = k^{\pm n}$ for some $n > 0$ then we assume that n is the smallest such integer. In this case, we have $\langle h\rangle \cap \langle k\rangle = \langle h^n\rangle = \langle k^n\rangle$.

Lemma 3.1. *Let A be double coset separable at $\langle h, k\rangle$ and $h^n = k^{\pm n}$ for some $n > 0$. Let $G = \langle A, t : t^{-1}ht = k\rangle$ and let $x \in G$ be reduced. Then, for each $M \triangleleft_f A$ and for each $s > 0$, there exist $N_s \triangleleft_f A$ and $\lambda_s > 0$ such that $N_s \subset M$, $N_s \cap \langle h\rangle = \langle h^{s\lambda_s}\rangle$, $N_s \cap \langle k\rangle = \langle k^{s\lambda_s}\rangle$, $\langle \bar{h}\rangle \cap \langle \bar{k}\rangle = \langle \bar{h}^n\rangle$, and $\|\bar{x}\| = \|x\|$ in $\overline{G} = G\pi_{N_s}$.*

Proof. Let $x = u_1 t^{\varepsilon_1} u_2 t^{\varepsilon_2} \cdots t^{\varepsilon_m} u_{m+1} \in G$ be reduced where $u_i \in A$ and $\varepsilon_i = \pm 1$. Since x is reduced, we have $u_{i+1} \notin \langle h \rangle$ if $\varepsilon_i = -\varepsilon_{i+1} = -1$ and $u_{i+1} \notin \langle k \rangle$ if $\varepsilon_i = -\varepsilon_{i+1} = 1$. By double coset separability, A is $\langle h^\varepsilon \rangle$-separable and $\langle k^\varepsilon \rangle$-separable for any $\varepsilon > 0$. Thus we can find $N_1 \lhd_f A$ such that, for each i, if $\varepsilon_i = -\varepsilon_{i+1} = -1$ then $u_{i+1} \notin N_1 \langle h \rangle$, or if $\varepsilon_i = -\varepsilon_{i+1} = 1$ then $u_{i+1} \notin N_1 \langle k \rangle$. Since n is the smallest integer such that $h^n = k^{\pm n}$, we have $h^i k^{-j} \notin \langle h^n \rangle = \langle k^n \rangle$ for all $0 \leq i, j < n$ except $i = j = 0$. Thus there exists $N_2 \lhd_f A$ such that $h^i k^{-j} \notin N_2 \langle h^n \rangle$ for all $0 \leq i, j < n$ except $i = j = 0$. Let $N_1 \cap N_2 \cap M \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $N_1 \cap N_2 \cap M \cap \langle k \rangle = \langle k^{s_2} \rangle$. Let $\varepsilon = n s_1 s_2 s$, where $s > 0$ is a given integer. Since A is $\langle h^\varepsilon \rangle$-separable and $\langle k^\varepsilon \rangle$-separable, there exists $N_3 \lhd_f A$ such that $h^i \notin N_3 \langle h^\varepsilon \rangle$ and $k^i \notin N_3 \langle k^\varepsilon \rangle$ for all $1 \leq i < \varepsilon$. This implies $N_3 \cap \langle h \rangle = \langle h^{\lambda_1 \varepsilon} \rangle$ and $N_3 \cap \langle k \rangle = \langle k^{\lambda_2 \varepsilon} \rangle$ for some $\lambda_1, \lambda_2 > 0$. Since $h^n = k^{\pm n}$ and $n \mid \varepsilon$, we have $N_3 \cap \langle h \rangle = N_3 \cap \langle h \rangle \cap \langle h^n \rangle = N_3 \cap \langle h^n \rangle = N_3 \cap \langle k^n \rangle = N_3 \cap \langle k \rangle$, hence $\lambda_1 = \lambda_2$. Let $N_s = N_1 \cap N_2 \cap N_3 \cap M$ and $\lambda_s = n s_1 s_2 \lambda_1$. Clearly $N_s \lhd_f A$ and $N_s \subset M$. Also, we have $N_s \cap \langle h \rangle = \langle h^{s \lambda_s} \rangle$, $N_s \cap \langle k \rangle = \langle k^{s \lambda_s} \rangle$, $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = \langle \bar{h}^n \rangle$, and $\|\bar{x}\| = \|x\|$ in $\overline{G} = G \pi_{N_s}$. ◻

Lemmas 3.1 and 2.8 imply the following:

Corollary 3.2. *Let A be double coset separable at $\{h, k\}$ and $h^n = k^{\pm n}$ for some $n > 0$. Then A is regular at $\{h, k\}$, hence $G = \langle A, t : t^{-1} h t = k \rangle$ is double coset separable at $\{h, k\}$.*

For convenience, we say $N \in \mathcal{N}$ if $N \lhd_f A$ such that $N \cap \langle h \rangle = \langle h^{\varepsilon \lambda} \rangle$ and $N \cap \langle k \rangle = \langle k^{\varepsilon \lambda} \rangle$. In this case, for each $N \in \mathcal{N}$ we can construct $\overline{G} = G \pi_N$ as in Remark 2.4.

Theorem 3.3. *Let A be double coset separable at $\{h, k\}$ and $h^n = k^{\pm n}$ for some $n > 0$. Let $x \in G = \langle A, t : t^{-1} h t = k \rangle$ be cyclically reduced. If $\|x\| \geq 1$ then, for each $y \in G$ such that $x \not\sim_G y$, there exists $N \lhd_f A$ such that $\bar{x} \not\sim_{\overline{G}} \bar{y}$, where $\overline{G} = G \pi_N$.*

Proof. The proof is similar to [**8**, Theorem 3.7]. Instead of using [**8**, Lemma 4.3], we use Lemma 3.1. ◻

Niblo [**14**] introduced the concept of regular quotient at $X = \{a_i; i \in I\}$ to prove the subgroup separability of HNN extensions of free groups. However to prove the conjugacy separability of HNN extensions we need certain conjugacy conditions to be satisfied by the images of h and k. We introduce the concept of conjugacy-regular at $\{h, k\}$.

Definition 3.4. Let A be a group, and let $h, k \in A$ be of infinite order. Then A is said to be *conjugacy-regular* at $\{h, k\}$ if, for each integer $\varepsilon > 0$, there exist an integer $\lambda_\varepsilon > 0$ and $N_\varepsilon \lhd_f A$ such that (1) $N_\varepsilon \cap \langle h \rangle = \langle h^{\varepsilon\lambda_\varepsilon} \rangle$ and $N_\varepsilon \cap \langle k \rangle = \langle k^{\varepsilon\lambda_\varepsilon} \rangle$ and, in $\overline{A} = A/N_\varepsilon$, (2) $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = \overline{\langle h \rangle \cap \langle k \rangle}$, (3) if $\bar{h}^i \sim_{\overline{A}} \bar{h}^j$, then $\bar{h}^j = \bar{h}^{\pm i}$, (4) if $\bar{k}^i \sim_{\overline{A}} \bar{k}^j$, then $\bar{k}^j = \bar{k}^{\pm i}$, and (5) if $\langle h \rangle \cap \langle k \rangle = 1$ and $\bar{h}^i \sim_{\overline{A}} \bar{k}^j$ then $\bar{h}^i = \bar{k}^j = 1$ and if $\langle h \rangle \cap \langle k \rangle \neq 1$ and $\bar{h}^i \sim_{\overline{A}} \bar{k}^j$ then $\bar{h}^i = \bar{h}^{\pm j}$.

Remark 3.5. (1) Since $|\bar{h}| = |\bar{k}| = \varepsilon\lambda_\varepsilon$, in (5), $\bar{h}^i = \bar{h}^{\pm j}$ implies $\bar{k}^i = \bar{k}^{\pm j}$.

(2) If A is conjugacy-regular at $\{h, k\}$ and $\langle h \rangle \cap \langle k \rangle = 1$ then A is c-quasi regular at $\{h, k\}$ in [**8**, Definition 4.1].

(3) Suppose A is conjugacy-regular at $\{h, k\}$. Then clearly A is regular at $\{h, k\}$. Hence, as observed in [**9**, Remark 4], if $\langle h \rangle \cap \langle k \rangle \neq 1$, then $h^\alpha = k^{\pm\alpha}$ for some $\alpha > 0$.

Theorem 3.6. *Let A be residually finite and regular at $\{h, k\}$. If A is $\langle h \rangle$-separable and $\langle k \rangle$-separable, then $G = \langle A, t : t^{-1}ht = k \rangle$ is residually finite.*

Proof. Let $1 \neq g \in G$.

Case 1. $g \in A$. Since A is residually finite, there exists $N_1 \lhd_f A$ such that $g \notin N_1$. Let $N_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $N_1 \cap \langle k \rangle = \langle k^{s_2} \rangle$. Since A is regular at $\{h, k\}$, there exist an integer λ and $N_2 \lhd_f A$ such that $N_2 \cap \langle h \rangle = \langle h^{\lambda s_1 s_2} \rangle$ and $N_2 \cap \langle k \rangle = \langle k^{\lambda s_1 s_2} \rangle$. Let $N = N_1 \cap N_2$. Then $N \lhd_f A$ and $g \notin N$. Moreover $N \cap \langle h \rangle = \langle h^{\lambda s_1 s_2} \rangle$ and $N \cap \langle k \rangle = \langle k^{\lambda s_1 s_2} \rangle$. Hence $\bar{g} \neq 1$ in $\overline{G} = G\pi_N$. Since $\overline{G}$ is residually finite, there exists $\overline{L} \lhd_f \overline{G}$ such that $\bar{g} \notin \overline{L}$. Let L be the preimage of $\overline{L}$ in G. Then $L \lhd_f G$ and $g \notin L$.

Case 2. $\|g\| \geq 1$. Since A is $\langle h\rangle$-separable, $\langle k\rangle$-separable and regular at $\{h,k\}$, as in the proof of Lemma 3.1, we can find $N \lhd_f A$ such that $\|\overline{g}\| = \|g\|$ in $\overline{G} = G\pi_N$. Hence $\bar{g} \neq 1$. Since $\overline{G}$ is residually finite, as before we can find $L \lhd_f G$ and $g \notin L$.

This completes the proof. $\square$

Definition 3.7. A group G is said to be cyclic conjugacy separable for $\langle h\rangle$ if, whenever $x \in G$ and $\{x\}^G \cap \langle h\rangle = \varnothing$, there exists $N \lhd_f G$ such that, in $\overline{G} = G/N$, $\{\bar{x}\}^{\overline{G}} \cap \langle \bar{h}\rangle = \varnothing$.

The following is a slight modification of [**8**, Theorem 4.5].

Theorem 3.8. *Let A be conjugacy separable, conjugacy-regular and double coset separable at $\{h,k\}$, where $\langle h\rangle \cap \langle k\rangle = 1$. If A is cyclic conjugacy separable for $\langle h\rangle$ and $\langle k\rangle$, then $G = \langle A, t : t^{-1}ht = k\rangle$ is conjugacy separable.*

Proof. By Remark 3.5 (2), A is c-quasi regular at $\{h,k\}$. By Theorem 3.6, G is residually finite. Thus the proof is the same as the proof of [**8**, Theorem 4.5]. $\square$

We can now improve the criterion given by [**8**, Theorem 4.5]. We note that it is difficult to relax the conditions in Theorem 3.9 because of the Baumslag-Solitar group [**2**] $\langle a, t : t^{-1}a^2t = a^3\rangle$ satisfies all the conditions of the theorem except conjugacy regularity and it is not even residually finite.

Theorem 3.9. *Let A be conjugacy separable, conjugacy-regular and double coset separable at $\{h,k\}$. If A is cyclic conjugacy separable for $\langle h\rangle$ and $\langle k\rangle$, then $G = \langle A, t : t^{-1}ht = k\rangle$ is conjugacy separable.*

Proof. As we mentioned in Remark 3.5 (3), since A is conjugacy-regular at $\{h,k\}$, we have either $\langle h\rangle \cap \langle k\rangle = 1$ or $h^\alpha = k^{\pm\alpha}$ for some $\alpha > 0$. If $\langle h\rangle \cap \langle k\rangle = 1$ then, by Theorem 3.8, G is conjugacy separable. So we assume $h^\alpha = k^{\pm\alpha}$, where $\alpha > 0$ is the smallest such integer. We shall show that G is conjugacy separable. Let $x, y \in G$ be such that $x \not\sim_G y$ and that x, y are of minimal lengths in their respective conjugacy classes. Since A is conjugacy-regular at $\{h, k\}$, A is regular at $\{h, k\}$. Thus, by Theorem 3.6, G is residually finite. Therefore we may assume $x \neq 1 \neq y$. Because of Theorem 3.3, we need only consider the case $\|x\| = \|y\| = 0$, that is, $x, y \in A$. By Theorem 2.2, $\overline{G} = G\pi_N$ is conjugacy separable for each $N \in \mathcal{N}$. Hence, throughout the proof, we shall find a suitable $N \in \mathcal{N}$ such that, in $\overline{G} = G\pi_N$, $\bar{x} \not\sim_{\overline{G}} \bar{y}$ for the following cases:

Case 1. $x, y \in \langle h \rangle$ (similarly $x, y \in \langle k \rangle$).

(a) $x = h^m$ and $y = h^n$, where $m \neq \pm n$.

(b) $x = h^m$ and $y = h^{-m}$.

Case 2. $x \in \langle h \rangle$ and $y \in \langle k \rangle$ (similarly $x \in \langle k \rangle$ and $y \in \langle h \rangle$).

Case 3. $x \in A$ and $\{x\}^A \cap \langle h \rangle = \varnothing$ (or $\{x\}^A \cap \langle k \rangle = \varnothing$).

We shall only prove Case 1 (b) since the proofs for the other cases are similar to the proofs in [**8**, Theorem 4.5]. To prove Case 1 (b), we note that $x \not\sim_G y$ implies $h^m \not\sim_A h^{-m}$, $k^m \not\sim_A k^{-m}$ and $h^m \not\sim_A k^{-m}$. By double coset separability and conjugacy separability of A, there exists $N_1 \lhd_f A$ such that $h^i k^{-j} \notin N_1 \langle h^\alpha \rangle \langle k^\alpha \rangle = N_1 \langle h^\alpha \rangle$ for all $0 \leq i$, $j < \alpha$, except $i = 0 = j$, $N_1 h^m \not\sim_{A/N_1} N_1 h^{-m}$, $N_1 k^m \not\sim_{A/N_1} N_1 k^{-m}$ and $N_1 h^m \not\sim_{A/N_1} N_1 k^{-m}$. Since $h^i \notin N_1 \langle h^\alpha \rangle$ for all $1 \leq i < \alpha$, $N_1 \cap \langle h \rangle = \langle h^{\alpha s} \rangle$ for some $s > 0$. Similarly, $N_1 \cap \langle k \rangle = \langle k^{\alpha s_1} \rangle$ for some $s_1 > 0$. Since $h^\alpha = k^{\pm \alpha}$, $k^{\alpha s_1} = h^{\pm \alpha s_1} \in N_1 \cap \langle h \rangle = \langle h^{\alpha s} \rangle$. Hence αs divides αs_1. Similarly αs_1 divides αs. Therefore $s = \pm s_1$ and $N_1 \cap \langle h \rangle = \langle h^{\alpha s} \rangle = N_1 \cap \langle k \rangle$. Since A is conjugacy-regular at $\{h, k\}$, there exist $N_2 \lhd_f A$ and λ such that (1) $N_2 \cap \langle h \rangle = \langle h^{\lambda \alpha s} \rangle$ and $N_2 \cap \langle k \rangle = \langle k^{\lambda \alpha s} \rangle$ and, in $\tilde{A} = A/N_2$, (2) $\langle \tilde{h} \rangle \cap \langle \tilde{k} \rangle = \langle \tilde{h}^\alpha \rangle = \langle \tilde{k}^\alpha \rangle$, (3) if $\tilde{h}^i \sim_{\tilde{A}} \tilde{h}^j$, then $\tilde{h}^j = \tilde{h}^{\pm i}$, (4) if $\tilde{k}^i \sim_{\tilde{A}} \tilde{k}^j$, then $\tilde{k}^j = \tilde{k}^{\pm i}$, and (5) if $\tilde{h}^i \sim_{\tilde{A}} \tilde{k}^j$, then $\tilde{h}^j = \tilde{h}^{\pm i}$. Let $N = N_1 \cap N_2$. Then $N \cap \langle h \rangle = \langle h^{\lambda \alpha s} \rangle$ and $N \cap \langle k \rangle = \langle k^{\lambda \alpha s} \rangle$. Let $\overline{G} = G\pi_N$ as in Remark 2.4. Then we shall show that $\bar{x} \not\sim_{\overline{G}} \bar{y}$. For, if $\bar{x} \sim_{\overline{G}} \bar{y}$ then, by Theorem 2.1, there exist $z_1, z_2, \ldots, z_r$ of elements in $\langle \bar{h} \rangle \cup \langle \bar{k} \rangle$ such that

$$\bar{h}^m \sim_{\overline{A}} z_1 \sim_{\overline{A}, \tau^*} z_2 \sim_{\overline{A}, \tau^*} \cdots \sim_{\overline{A}, \tau^*} z_r \sim_{\overline{A}} \bar{h}^{-m}.$$

If $z_1 = \bar{h}^i \in \langle \bar{h} \rangle$, then $\bar{h}^m \sim_{\overline{A}} z_1$ implies $\tilde{h}^m \sim_{\tilde{A}} \tilde{h}^i$. Thus, by (3) above, $\tilde{h}^i = \tilde{h}^{\pm m}$, whence $h^{i \mp m} \in N_2 \cap \langle h \rangle \subset N_1 \cap \langle h \rangle$. Thus $h^{i \mp m} \in N$, that is, $z_1 = \bar{h}^i = \bar{h}^{\pm m}$. Similarly, if $z_1 = \bar{k}^i \in \langle \bar{k} \rangle$, then $\bar{h}^m \sim_{\overline{A}} z_1$ implies

$\tilde{h}^m \sim_{\tilde{A}} \tilde{k}^i$. Hence, by (5) above, $\tilde{h}^i = \tilde{h}^{\pm m}$ and $\tilde{k}^i = \tilde{k}^{\pm m}$ by (1) in Remark 3.5. Hence $z_1 = \bar{k}^i = \bar{k}^{\pm m}$ as before. Now, by the choice of N_1, since $\bar{h}^m \not\sim_{\overline{A}} \bar{h}^{-m}$ and $\bar{h}^m \not\sim_{\overline{A}} \bar{k}^{-m}$, we have either $z_1 = \bar{h}^m$ or $z_1 = \bar{k}^m$. Similarly, since $z_1 = \bar{h}^m$ or $z_1 = \bar{k}^m$, $z_1 \sim_{\overline{A},\tau^*} z_2$ implies either $z_2 = \bar{h}^m$ or $z_2 = \bar{k}^m$. Repeating this process, we have either $z_r = \bar{h}^m$ or $z_r = \bar{k}^m$. Since $z_r \sim_{\overline{A}} \bar{h}^{-m}$, both cases $z_r = \bar{h}^m$ and $z_r = \bar{k}^m$ contradict to the choice of N_1. Hence $\bar{x} \not\sim_{\overline{G}} \bar{y}$. $\square$

Lemma 3.10. (1) *Let $h, k \in Z(A)$ be of infinite order such that A is $\langle h^\varepsilon \rangle \langle k^\varepsilon \rangle$-separable for any $\varepsilon > 0$. Then A is conjugacy regular at $\{h, k\}$ if and only if $\langle h \rangle \cap \langle k \rangle = 1$ or $h^n = k^{\pm n}$ for some $n > 0$.*

(2) *Let A be a finitely generated nilpotent or free group. Let $h, k \in A$ be of infinite order such that $h^n = k^{\pm n}$ for some $n > 0$. Then A is conjugacy regular at $\{h, k\}$.*

Remark. In (2) one might suspect whether A is conjugacy-regular at $\{h, k\}$ if A is finitely generated nilpotent and $\langle h \rangle \cap \langle k \rangle = 1$. But this is not true in general, see [**9**, Example 1].

Proof. (1) As we mentioned in Remark 3.5, if A is conjugacy regular at $\{h, k\}$, then $\langle h \rangle \cap \langle k \rangle = 1$ or $h^n = k^{\pm n}$ for some $n > 0$.

Conversely, suppose $\langle h \rangle \cap \langle k \rangle = 1$. Let $\varepsilon > 0$ be a given integer. Since $h, k \in Z(A)$, we consider $\tilde{A} = A/\langle h^\varepsilon \rangle \langle k^\varepsilon \rangle$. Then $|\tilde{h}| = |\tilde{k}| = \varepsilon$ and $\langle \tilde{h} \rangle \cap \langle \tilde{k} \rangle = 1$. Since A is $\langle h^\varepsilon \rangle \langle k^\varepsilon \rangle$-separable, $\tilde{A}$ is residually finite. Hence there exists $\tilde{N} \triangleleft_f \tilde{A}$ such that $\tilde{h}^i \tilde{k}^{-j} \notin \tilde{N}$ for all $\tilde{h}^i \tilde{k}^{-j} \neq 1$. Let N be the preimage of $\tilde{N}$ in A. Then $N \triangleleft_f A$ such that $N \cap \langle h \rangle = \langle h^\varepsilon \rangle$ and $N \cap \langle k \rangle = \langle k^\varepsilon \rangle$. In $\overline{A} = A/N$, we have $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1 = \overline{\langle h \rangle \cap \langle k \rangle}$. Moreover, since $h, k \in Z(A)$, if $\bar{h}^i \sim_{\overline{A}} \bar{h}^j$, then $\bar{h}^i = \bar{h}^j$ and if $\bar{k}^i \sim_{\bar{A}} \bar{k}^j$ then $\bar{k}^i = \bar{k}^j$. Also, if $\bar{h}^i \sim_{\overline{A}} \bar{k}^j$, then $\bar{h}^i = \bar{k}^j \in \langle \bar{h} \rangle \cap \langle \bar{k} \rangle = 1$. This shows that A is conjugacy-regular at $\{h, k\}$.

We now show that A is conjugacy-regular at $\{h, k\}$ if $h^n = k^{\pm n}$ for $n > 0$. Without loss of generality, we assume that n is the smallest positive integer such that $h^n = k^{\pm n}$. Thus $\langle h \rangle \cap \langle k \rangle = \langle h^n \rangle$. Let $\varepsilon > 0$ be a given integer. Consider $\tilde{A} = A/\langle h^{n\varepsilon} \rangle$. Then $|\tilde{h}| = |\tilde{k}| = n\varepsilon$ and $\langle \tilde{h} \rangle \cap \langle \tilde{k} \rangle = \langle \tilde{h}^n \rangle = \langle \tilde{k}^n \rangle$. Since $\tilde{A}$ is residually finite, there exists $\tilde{N} \triangleleft_f \tilde{A}$ such that $\tilde{h}^i \tilde{k}^{-j} \notin \tilde{N}$ for all $\tilde{h}^i \tilde{k}^{-j} \neq 1$. Let N be the preimage of $\tilde{N}$ in

A. Then $N \triangleleft_f A$ with $N \cap \langle h \rangle = \langle h^{n\varepsilon} \rangle$ and $N \cap \langle k \rangle = \langle k^{n\varepsilon} \rangle$. Moreover, in $\bar{A} = A/N$, we have $\langle \bar{h} \rangle \cap \langle \bar{k} \rangle = \langle \bar{h}^n \rangle = \overline{\langle h \rangle \cap \langle k \rangle}$. Since $h, k \in Z(A)$, (3) and (4) in Definition 3.4 trivially hold. For (5), if $\bar{h}^i \sim_{\overline{A}} \bar{k}^j$, then $\bar{h}^i = \bar{k}^j \in \langle \bar{h} \rangle \cap \langle \bar{k} \rangle = \langle \bar{h}^n \rangle = \langle \bar{k}^n \rangle$. Hence $n \mid j$ and $k^j = h^{\pm j}$. Thus $\bar{h}^i = \bar{h}^{\pm j}$. Therefore A is conjugacy-regular at $\{h, k\}$.

(2) Suppose A is finitely generated nilpotent and $h^n = k^{\pm n}$. Let $\varepsilon > 0$ be a given integer. Since A is $\langle h^n \rangle$-separable [**10**], there exists $N_1 \triangleleft_f A$ such that $h^i k^{-j} \notin N_1 \langle h^n \rangle$ for all $0 \leq i, j < n$ except $i = 0 = j$. Then, as in the proof of Theorem 3.9, $N_1 \cap \langle h \rangle = \langle h^{ns_1} \rangle$ and $N_1 \cap \langle k \rangle = \langle k^{ns_1} \rangle$ for some $s_1 > 0$. Since A is nilpotent, let c be the largest integer such that $\langle h \rangle \cap Z_c(A) = 1$ and $\langle h \rangle \cap Z_{c+1}(A) = \langle h^\lambda \rangle \neq 1$. Let $\tilde{A} = A/Z_c(A)$. Then $|\tilde{h}| = |\tilde{k}| = \infty$ and $\tilde{h}^{n\lambda} = \tilde{k}^{\pm n\lambda} \in Z(\tilde{A})$. Let $\overline{A} = \tilde{A}/\langle \tilde{h}^d \rangle$, where $d = n\lambda s_1 \varepsilon$. Then $|\bar{h}| = d = |\bar{k}|$. Now, if $\bar{h}^i \sim_{\overline{A}} \bar{h}^j$, then $\tilde{h}^{i+ds} = \tilde{a}^{-1} \tilde{h}^j \tilde{a}$ for some $a \in A$ and some integer s. Since $\tilde{A}$ is finitely generated nilpotent and $|\tilde{h}| = \infty$, $i + ds = j$. Hence $\bar{h}^j = \bar{h}^{i+ds} = \bar{h}^i$. Similarly, if $\bar{k}^i \sim_{\overline{A}} \bar{k}^j$, then $\bar{k}^i = \bar{k}^j$. If $\bar{h}^i \sim_{\overline{A}} \bar{k}^j$, then $\tilde{h}^{i+ds} = \tilde{a}^{-1} \tilde{k}^j \tilde{a}$ for some $a \in A$ and some integer s. Then $\tilde{h}^{n(i+ds)} = \tilde{a}^{-1} \tilde{k}^{nj} \tilde{a} = \tilde{a}^{-1} \tilde{h}^{\pm nj} \tilde{a}$, thus $n(i + ds) = \pm nj$ which implies $i + ds = \pm j$. Hence $\bar{h}^{\pm j} = \bar{h}^{i+ds} = \bar{h}^i$. Since $\overline{A}$ is conjugacy separable and $|\bar{h}| = |\bar{k}| < \infty$, there exists $\overline{N}_2 \triangleleft_f \overline{A}$ such that $\bar{k}^{-j} \bar{h}^i \notin \overline{N}_2$ for all $\bar{k}^{-j} \bar{h}^i \neq 1$, $\overline{N}_2 \bar{h}^i \not\sim_{\overline{A}/\overline{N}_2} \overline{N}_2 \bar{h}^j$ for all $\bar{h}^i \not\sim_{\overline{A}} \bar{h}^j$, $\overline{N}_2 \bar{k}^i \not\sim_{\overline{A}/\overline{N}_2} \overline{N}_2 \bar{k}^j$ for all $\bar{k}^i \not\sim_{\overline{A}} \bar{k}^j$, and $\overline{N}_2 \bar{h}^i \not\sim_{\overline{A}/\overline{N}_2} \overline{N}_2 \bar{k}^j$ for all $\bar{h}^i \not\sim_{\overline{A}} \bar{k}^j$. Let N_2 be the preimage of $\overline{N}_2$ in A and $N = N_1 \cap N_2$. Then $N \triangleleft_f A$ and $N \cap \langle h \rangle = \langle h^d \rangle = \langle k^d \rangle = N \cap \langle k \rangle$ and, in $\hat{A} = A/N$, we have $\langle \hat{h} \rangle \cap \langle \hat{k} \rangle = \langle \hat{h}^n \rangle$ by the choice of N_1. Moreover, by the choice of N_2, if $\hat{h}^i \sim_{\hat{A}} \hat{h}^j$, then $\hat{h}^i = \hat{h}^j$, if $\hat{k}^i \sim_{\hat{A}} \hat{k}^j$, then $\hat{k}^i = \hat{k}^j$, and if $\hat{h}^i \sim_{\hat{A}} \hat{k}^j$, then $\hat{h}^i = \hat{h}^{\pm j}$. This proves that A is conjugacy-regular at $\{h, k\}$.

Suppose A is a free group and $h^n = k^{\pm n}$. Then $h = k^{\pm 1}$. Let $h \in \Gamma_{i-1}(A) \backslash \Gamma_i(A)$, where $\Gamma_i(A)$ denotes the ith term of the lower central series of A. Let $\overline{A} = A/\Gamma_i(A)$. Clearly $\overline{A}$ is a finitely generated torsion-free nilpotent group and $\bar{h}$ and $\bar{k}$ are of infinite order with $\bar{h} = \bar{k}^{\pm 1} \in Z(\overline{A})$. By (1) above $\overline{A}$ is conjugacy-regular at $\{\bar{h}, \bar{k}\}$. Hence A is conjugacy-regular at $\{h, k\}$. □

We apply Theorem 3.9 to prove the following:

Theorem 3.11. *Let A be conjugacy separable, and let $h, k \in Z(A)$ be of infinite order. Suppose A is $\langle h^\varepsilon\rangle\langle k^\varepsilon\rangle$-separable for any $\varepsilon > 0$. Let $G = \langle A, t : t^{-1}ht = k\rangle$.*

(1) If $A = \langle b\rangle$ and $h = b^\alpha$, $k = b^\beta$, then G is conjugacy separable if and only if $|\alpha| = 1$ or $|\beta| = 1$ or $\alpha \pm \beta = 0$.

(2) If A is not cyclic, then G is conjugacy separable if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n > 0$.

Proof. (1) This follows from [**7**, Theorem 3.6].

To prove (2), suppose A is not cyclic. If G is conjugacy separable then G is residually finite. Let $C = \langle h, k\rangle$. Then C is abelian. If C is cyclic, say $C = \langle w\rangle$, then $\langle C, t\rangle = \langle w, t : t^{-1}w^\alpha t = w^\beta\rangle$, where $h = w^\alpha$ and $k = w^\beta$ for some α, β. Since $\langle C, t\rangle$ is residually finite, by [**9**, Theorem 2.16] we have $|\alpha| = 1$ or $|\beta| = 1$ or $\alpha \pm \beta = 0$. Consider $G = \langle A, t : t^{-1}w^\alpha t = w^\beta\rangle$. By [**9**, Lemma 2.14], it is easy to see $|\alpha| = |\beta|$, that is, $h = k^{\pm 1}$. If C is not cyclic, then consider $\langle C, t\rangle = \langle C, t : t^{-1}ht = k\rangle$. By [**9**, Theorem 2.16], $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n > 0$.

Conversely, we shall show that if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n > 0$ then G is conjugacy separable. Since $h, k \in Z(A)$, $g \notin \langle h^\varepsilon\rangle u\langle k^\varepsilon\rangle$ if and only if $gu^{-1} \notin \langle h^\varepsilon\rangle\langle k^\varepsilon\rangle$ for all $u \in A$. By assumption, A is $\langle h^\varepsilon\rangle\langle k^\varepsilon\rangle$-separable for any $\varepsilon > 0$. Hence A is $\langle h^\varepsilon\rangle u\langle k^\varepsilon\rangle$-separable for all $u \in A$. Similarly, A is $\langle h^\varepsilon\rangle u\langle h^\varepsilon\rangle$-separable and $\langle k^\varepsilon\rangle u\langle k^\varepsilon\rangle$-separable for all $u \in A$. Thus A is double coset separable at $\{h, k\}$. Also, since A is $\langle h\rangle$-separable and $h \in Z(A)$, A is cyclic conjugacy separable for $\langle h\rangle$. Similarly A is cyclic conjugacy separable for $\langle k\rangle$. By Lemma 3.10, A is conjugacy-regular at $\{h, k\}$. Hence by Theorem 3.9, G is conjugacy separable. $\square$

By the above result we can easily prove the following generalization of [**7**, Theorem 3.6] and [**8**, Corollary 4.6].

Corollary 3.12. *Let A be polycyclic-by-finite and let $h, k \in Z(A)$ be of infinite order. Let $G = \langle A, t : t^{-1}ht = k\rangle$.*

(1) If $A = \langle b\rangle$ and $h = b^\alpha$, $k = b^\beta$, then G is conjugacy separable if and only if $|\alpha| = 1$ or $|\beta| = 1$ or $\alpha \pm \beta = 0$.

(2) *If A is not cyclic, then G is conjugacy separable if and only if* $\langle h\rangle \cap \langle k\rangle = 1$ *or* $h^n = k^{\pm n}$ *for some* $n > 0$.

Proof. It is well known that polycyclic-by-finite groups are conjugacy separable [**6**], whence residually finite. Since homomorphic images of polycyclic-by-finite groups are polycyclic-by-finite, $A/\langle h^\varepsilon\rangle\langle k^\varepsilon\rangle$ is residually finite for any $\varepsilon > 0$. Hence A is $\langle h^\varepsilon\rangle\langle k^\varepsilon\rangle$-separable for any $\varepsilon > 0$. Thus the corollary follows directly from Theorem 3.11. □

It is well known that free groups are residually finite. But even infinitely generated abelian groups need not be residually finite, for example, the Prüfer group $\mathbf{Z}(p^\infty)$ is not residually finite. Because of this situation, in the study of residual properties of groups, we usually assume that the groups involved are finitely generated.

Theorem 3.13. *Let A be a free group or a finitely generated torsion-free nilpotent group. Let* $h, k \in \Gamma_{i-1}(A)\backslash\Gamma_i(A)$ *and* $\overline{A} = A/\Gamma_i(A)$. (1) *If* $\langle \bar{h}\rangle \cap \langle \bar{k}\rangle = 1$ *or* (2) *if* $\langle h\rangle \cap \langle k\rangle \neq 1$ *and* $\bar{h}^n = \bar{k}^{\pm n}$ *for some* $n > 0$, *then* $G = \langle A, t : t^{-1}ht = k\rangle$ *is conjugacy separable.*

Proof. Free groups and finitely generated nilpotent groups are double coset separable [**10, 15**], conjugacy separable [**6, 20**] and cyclic conjugacy separable [**5**]. To apply Theorem 3.9, we shall prove that A is conjugacy-regular at $\{h, k\}$. We note that in the case of infinitely generated free groups we need only consider the finitely generated case. This is because if A is infinitely generated and h, k are reduced words on the free generators of A, then $A = B * C$, where B is the finitely generated free factor of A generated by the free generators of A involved in h and k, and C is the free factor generated by the rest of the free generators of A. Clearly B is finitely generated. Therefore we can consider the corresponding HNN extension of B.

(1) Suppose $\langle \bar{h}\rangle \cap \langle \bar{k}\rangle = 1$. Clearly $\langle h\rangle \cap \langle k\rangle = 1$. Since $\bar{h}, \bar{k} \in Z(\overline{A})$ and $\bar{h}, \bar{k}$ are of infinite order, by Lemma 3.10 (1), $\overline{A}$ is conjugacy-regular at $\{\bar{h}, \bar{k}\}$. Since $\langle h\rangle \cap \langle k\rangle = 1$, this implies that A is conjugacy-regular at $\{h, k\}$.

(2) Let $h^\lambda = k^\mu$. Then $\bar{h}^\lambda = \bar{k}^\mu$. Since $\bar{h}^n = \bar{k}^{\pm n}$, $\bar{h}^{\lambda n} = \bar{k}^{\mu n} = \bar{h}^{\mp \mu n}$. Thus $\lambda = \mp\mu$. Hence $h^\lambda = k^{\mp\lambda}$. Since A is free or finitely generated

torsion-free nilpotent, $h = k^{\pm 1}$. By Lemma 3.10 (2), A is conjugacy-regular at $\{h, k\}$.

Thus, by Theorem 3.9, G is conjugacy separable. □

As shown in the examples in [**9**], HNN extensions $\langle A, t : t^{-1}ht = k\rangle$ of a free nilpotent group A may not be residually finite, whence not conjugacy separable, if $\langle h\rangle \cap \langle k\rangle = 1$. However, we can derive some results if $\langle h\rangle \cap \langle k\rangle \neq 1$.

In a free group or a finitely generated torsion-free nilpotent group A, if $h, k \in A$ such that $h^n = k^{\pm n}$ for some $n > 0$, then $h = k^{\pm 1}$. Thus we have the following:

Corollary 3.14. *Let A be a free group or a finitely generated torsion-free nilpotent group. Then $\langle A, t : t^{-1}ht = h\rangle$ and $\langle A, t : t^{-1}ht = h^{-1}\rangle$ are conjugacy separable.*

Finally, we have the following characterization of conjugacy separability of HNN extensions of finitely generated nilpotent groups when $\langle h\rangle \cap \langle k\rangle \neq 1$:

Theorem 3.15. *Let A be a finitely generated nilpotent group. Let $h, k \in A$ be of infinite order such that $\langle h\rangle \cap \langle k\rangle \neq 1$. Let $G = \langle A, t : t^{-1}ht = k\rangle$.*

(1) *If $A = \langle b\rangle$ and $h = b^{\alpha}$, $k = b^{\beta}$, then G is conjugacy separable if and only if $|\alpha| = 1$ or $|\beta| = 1$ or $\alpha \pm \beta = 0$.*

(2) *If A is not cyclic, then G is conjugacy separable if and only if $h^n = k^{\pm n}$ for some $n > 0$.*

Proof. By [**7**, Theorem 3.6], (1) holds.

For (2), suppose A is not cyclic. If G is conjugacy separable, then G is residually finite. Hence $h^n = k^{\pm n}$ for some $n > 0$ by [**9**, Theorem 2.15].

For the converse of (2), suppose $h^n = k^{\pm n}$ for some $n > 0$, where n is the smallest such integer. Then A is conjugacy-regular at $\{h, k\}$ by Lemma 3.10 (2). Since A is a finitely generated nilpotent, as we mentioned in the proof of Theorem 3.13, A is conjugacy separable, double coset separable and cyclic conjugacy separable. Hence, by Theorem 3.9, G is conjugacy separable. $\square$

Remark. In general if $\langle h\rangle \cap \langle k\rangle = 1$ then the above HNN extension G need not be residually finite whence not conjugacy separable, [**9**, Example 1]. However, if $h, k \in Z(A)$ then, by Corollary 3.12, G is conjugacy separable.

Applying [**9**, Theorem 2.15] and Theorem 3.15 we obtain a result similar to Theorem A [**16**] for HNN extensions of finitely generated nilpotent groups with cyclic associated subgroups.

Theorem 3.16. *Let A be a finitely generated nilpotent group. Let $h, k \in A$ be of infinite order such that $\langle h\rangle \cap \langle k\rangle \neq 1$. Then $G = \langle A, t : t^{-1}ht = k\rangle$ is conjugacy separable if and only if G is residually finite.*

Acknowledgments. We would like to thank the referee for suggesting part (2) of Lemma 3.10, which simplifies some of the proofs subsequent to Lemma 3.10.

REFERENCES

1. S. Andreadakis, E. Raptis and D. Varsos, *A characterization of residually finite* HNN-*extensions of finitely generated abelian groups*, Arch. Math. **50** (1988), 495–501.

2. G. Baumslag and D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **38** (1962), 199–201.

3. B. Baumslag and M. Tretkoff, *Residually finite* HNN *extensions*, Comm. Algebra **6** (1978), 179–194.

4. D.J. Collins, *Recursively enumerable degrees and the conjugacy problem*, Acta Math. **122** (1969), 115–160.

5. J.L. Dyer, *Separating conjugates in amalgamated free products and* HNN *extensions*, J. Austral. Math. Soc. **29** (1980), 35–51.

6. E. Formanek, *Conjugate separability in polycyclic groups*, J. Algebra **42** (1976), 1–10.

7. G. Kim and C.Y. Tang, *Conjugacy separability of* HNN-*extensions of abelian groups*, Arch. Math. **67** (1996), 353–359.

8. ———, *A criterion for the conjugacy separability of certain* HNN-*extensions of groups*, J. Algebra **222** (1999), 574–594.

9. ———, *Cyclic subgroup separability of* HNN-*extensions with cyclic associated subgroups*, Canad. Math. Bull. **42** (1999), 335–343.

10. John C. Lennox and John S. Wilson, *On products of subgroups in polycyclic groups*, Arch. Math. **33** (1979), 305–309.

11. A.I. Mal'cev, *Homomorphisms of finite groups*, Ivanov Gos. Ped. Inst. Učen. Zap. Uchen. Zap. Karel. Ped. Inst. Ser. Fiz.-Mat. Nauk **18** (1958), 49–60.

12. S. Meskin, *Non residually finite one-relator groups*, Trans. Amer. Math. Soc. **164** (1972), 105–114.

13. A.W. Mostowski, *On the decidability of some problems in special classes of groups*, Fund. Math. **59** (1966), 123–135.

14. G.A. Niblo, H.N.N. *extensions of a free group by Z which are subgroup separable*, Proc. London Math. Soc. **61** (1990), 18–32.

15. ———, *Separability properties of free groups and surface groups*, J. Pure Appl. Algebra **78** (1992), 77–84.

16. E. Raptis, O. Talelli and D. Varsos, *On the conjugacy separability of certain graphs of groups*, J. Algebra **199** (1998), 327–336.

17. E. Raptis and D. Varsos, *The residual finiteness of* HNN-*extensions and generalized free products of nilpotent groups*: *A characterization*, J. Austral. Math. Soc. **53** (1992), 408–420.

18. M. Shirvani, *On residually finite* HNN-*extensions*, Arch. Math. **44** (1985), 110–115.

19. P.F. Stebe, *Residual finiteness of a class of knot groups*, Comm. Pure. Appl. Math. **21** (1968), 563–583.

20. ———, *A residual property of certain groups*, Proc. Amer. Math. Soc. **26** (1970), 37–42.

Department of Mathematics, Yeungnam University, Kyongsan, 712-749, Korea
E-mail address: gskim@yu.ac.kr

University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
E-mail address: fcytang@math.uwaterloo.ca