ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 2, 2005

A NOTE ON SOME MODULAR SUBGROUPS

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ABSTRACT. In this study, we introduce some modular subgroups and we calculate parabolic class numbers of those groups. In addition to this, we gave a nice formula for the parabolic class number of $\Gamma_0(n^2)$.

1. Introduction. Let Γ denote the modular group, which consist of those mappings of the form

$$T(z) = \frac{az+b}{cz+d}, \quad ad-bc=1, \quad a,b,c,d \in \mathbf{Z}$$

and let Γ^m be its subgroup such that

$$\Gamma^m = \{T \in \Gamma : a \equiv d \equiv 0 \pmod{m} \text{ or } c \equiv b \equiv 0 \pmod{m}\}$$

where $m \geq 2$, is a natural number. Let $\Gamma_0(n)$ be the congruence subgroup of the modular group Γ such that $c \equiv 0 \pmod{n}$ and let $\Gamma_0^m(n) = \Gamma^m \cap \Gamma_0(n)$. Our main purpose is to calculate parabolic class number of $\Gamma_0^m(n)$ for m to be a power of a prime number.

2. Some definitions and theorems. Let Λ be a discrete subgroup of $PSL(2,\mathbf{R})$. When $x \in \mathbf{R} \cup \{\infty\}$ is a fixed point of a parabolic element of Λ, we say that x is a parabolic point of Λ. We also call a parabolic point of Λ a cusp of Λ.

We now give a theorem from [**6**], which is related to the cusps of Λ.

Theorem 1. *Let x be a cusp of Λ, and let $\Lambda_x = \{T \in \Lambda : T(x) = x\}$. Then Λ_x is an infinite cyclic group. Moreover, any element of Λ_x is either identity or parabolic.*

The parabolic subgroups of Λ are defined to be those nonidentity cyclic subgroups $C \subset \Lambda$, which consist of parabolic elements, together

AMS *Mathematics Subject Classification.* Primary 20H05, 20H10.
Received by the editors on January 31, 2002, and in revised form on May 12, 2003.

with the identity, and which are maximal with respect to this property. The parabolic class number s of Λ is the number of conjugacy classes of parabolic subgroup of Λ, see [**1**].

Lemma 1. *Let Λ be a discrete subgroup of $PSL(2,\mathbf{R})$ and assume that Λ^* is a subgroup of finite index in Λ. Let P be the set of the parabolic points of Λ. Then P is the set of the parabolic points of Λ^* and the parabolic class number of Λ^* is the number of orbits of Λ^* on P.*

Proof. Let x be a point in P. Then $S(x) = x$ for some parabolic element of Λ. Since Λ^* is a subgroup of finite index in Λ, we see that $S^k \in \Lambda^*$ for some natural number k. Then S^k is a parabolic mapping and $S^k(x) = x$. Thus x is a cusp of Λ^*. By the above theorem, Λ_x^* is an infinite cyclic group and any nonidentity element of Λ_x^* is a parabolic element. It can be seen that Λ_x^* is maximal with respect to this property. That is, Λ_x^* is a parabolic subgroup. Now let C be a parabolic subgroup of Λ^*. Then every element of C has a fixed point x and therefore $C \subset \Lambda_x^*$. This shows that $C = \Lambda_x^*$. In addition, x and y lie in the same orbit if and only if Λ_x^* and Λ_y^* are conjugate. Then the proof follows. $\square$

By a Fuchsian group Λ we will mean a finitely generated discrete subgroup of $PSL(2,\mathbf{R})$ the group of conformal homeomorphisms of the upper-half plane. The most general presentation for Λ is

Generators;

$$\begin{array}{ll} a_1, b_1, \dots, a_g, b_g & \text{(Hyperbolic)} \\ x_1, x_2, \dots, x_r & \text{(Elliptic)} \\ p_1, p_2, \dots, p_s & \text{(Parabolic)} \end{array}$$

Relations;

$$x_1^{m_1} = x_2^{m_2} = \cdots x_r^{m_r} = \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{s} p_k = 1$$

where $[a_i, b_i]$ denotes the commutator of a_i and b_i.

We then say Λ has signature, see [**7**],

$$(g, m_1, m_2, \dots, m_r; s).$$

s is the number of parabolic classes, i.e., of conjugacy classes of maximal parabolic subgroup, and r is the number of elliptic classes. The m_i are the periods of Λ.

In the following we calculate the indexes of some subgroups of Γ in Γ. Before giving our lemmas, we give a lemma from [**5**].

Lemma 2. $|\Gamma : \Gamma_0(n)| = n \prod_{p\,|\,n}(1 + 1/p)$.

Lemma 3. $|\Gamma : \Gamma^m| = m\,|\Gamma : \Gamma_0(m)|\,/2$.

Proof. Let $H = \{T \in \Gamma^m :\ b \equiv c \equiv 0 \pmod m\}$, and let

$$K = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbf{Z}) : b \equiv c \equiv 0 \pmod m \right\}.$$

Now let $\lambda_m : SL(2, \mathbf{Z}) \to SL(2, \mathbf{Z}/m\mathbf{Z})$ be an epimorphism defined in [**4**, p. 104], as follows:

$$\lambda_m \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(\begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} \right),$$

where $a \equiv a^* \pmod m$, $b \equiv b^* \pmod m$, $c \equiv c^* \pmod m$, $d \equiv d^* \pmod m$. Since

$$\lambda_m(K) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in SL(2, \mathbf{Z}/m\mathbf{Z}) : a \in SL(2, \mathbf{Z}/m\mathbf{Z}) \text{ is a unit} \right\},$$

we see that $|\lambda_m(K)| = \varphi(m)$. Therefore

$$\begin{aligned} |\Gamma : H| &= |SL(2, \mathbf{Z}) : K| = |SL(2, \mathbf{Z}/m\mathbf{Z}) : \lambda_m(K)| \\ &= \frac{m^3 \prod_{p|m}(1 - 1/p^2)}{\varphi(m)} \end{aligned}$$

and thus $|\Gamma : H| = m\,|\Gamma : \Gamma_0(m)|$. Since $H \subset \Gamma^m \subset \Gamma$, we have

$$|\Gamma : \Gamma^m| = \frac{|\Gamma : H|}{|\Gamma^m : H|} = \frac{|\Gamma : H|}{2} = \frac{m}{2}\,|\Gamma : \Gamma_0(m)|\,. \qquad \square$$

Lemma 4. *Let* $(m,n)=1$. *Then* $|\Gamma^m : \Gamma_0^m(n)| = |\Gamma : \Gamma_0(n)|$ *and*

$$|\Gamma : \Gamma_0^m(n)| = |\Gamma : \Gamma^m|\,|\Gamma^m : \Gamma_0^m(n)|\,,$$

if m *is not prime to* n, *then* $|\Gamma : \Gamma_0^m(n)| = \left|\Gamma : \Gamma_0(m^2n)\right|$.

Proof. Assume that $(m,n)=1$. Let $H_0 = \{T \in H : c \equiv 0 \pmod{n}\}$. Then $H_0 \subset H$ and $H_0 \subset \Gamma_0^m(n)$. Moreover, it can be seen that H is conjugate to $\Gamma_0(m^2)$ and that H_0 is conjugate to $\Gamma_0(m^2n)$. This follows from the fact that $H = S^{-1}\Gamma_0(m^2)S$ and $H_0 = S^{-1}\Gamma_0(m^2n)S$ where $S(z) = z/m$ and the conjugation is being done in $SL(2,\mathbf{R})$ and not in $SL(2,\mathbf{Z})$. Since $|\Gamma^m : H| = |\Gamma_0^m(n) : H| = 2$, we have

$$|\Gamma^m : \Gamma_0^m(n)| = \frac{|\Gamma^m : H_0|}{|\Gamma_0^m(n) : H_0|} = \frac{|\Gamma^m : H|\,|H : H_0|}{|\Gamma_0^m(n) : H_0|} = |H : H_0|\,.$$

On the other hand, since

$$|H : H_0| = \left|S^{-1}\Gamma_0(m^2)S : S^{-1}\Gamma_0(m^2n)S\right| = \left|\Gamma_0(m^2) : \Gamma_0(m^2n)\right|,$$

and

$$\left|\Gamma_0(m^2) : \Gamma_0(m^2n)\right| = |\Gamma : \Gamma_0(n)|\,,$$

we see that $|\Gamma^m : \Gamma_0^m(n)| = |\Gamma : \Gamma_0(n)|$. Since

$$|\Gamma : \Gamma_0^m(n)| = |\Gamma : \Gamma^m|\,|\Gamma^m : \Gamma_0^m(n)|\,,$$

the proof follows. If m is not prime to n, then $\Gamma_0^m(n) = H_0$ and the proof is similar. $\square$

In view of the above lemmas, we see that Γ^m and $\Gamma_0^m(n)$ are finitely generated discrete subgroup of $PSL(2,\mathbf{R})$ and therefore they are Fuchsian groups.

3. Main theorems. Let $T \in \Gamma$ and

$$T(z) = \frac{az+b}{cz+d}, \quad ad-bc=1.$$

Since

$$d(ak+bs) - b(ck+ds) = k$$

and

$$a(ck+ds)-c(ak+bs)=s,$$

we see that if k/s is in the reduced form, then $(ak+bs)/(ck+ds)$ is in the reduced form. Now we give the following lemma without proof. The proof is easy and can be found in [**1**].

Lemma 5. *Let Γ be the modular group. Then the set of parabolic points of Γ is the set $\hat{\mathbf{Q}}=\mathbf{Q}\cup\{\infty\}=\{k/s:(k,s)=1\}\cup\{\infty\}$ and Γ acts transitively on $\hat{\mathbf{Q}}$.*

If we represent ∞ as $1/0=-1/0$, we have

$$\hat{\mathbf{Q}}=\{k/s:(k,s)=1 \text{ and } k,s \text{ are integers}\}.$$

From the above lemmas, it follows that the parabolic class number of $\Gamma_0^m(n)$ is the number of orbits of $\Gamma_0^m(n)$ on $\hat{\mathbf{Q}}$. Since Γ acts transitively on $\hat{\mathbf{Q}}$ and $\Gamma_0^m(n)$ is of finite index in Γ, it can be seen that the parabolic class number of $\Gamma_0^m(n)$ is finite, see [**1**].

In the following, we give a lemma without proof, which we use later. The lemma appears in [**8**, p. 73], as a problem.

Lemma 6. *Let m and k be positive numbers. Then the number of the positive integers $\leq mk$ that are prime to m is $k\varphi(m)$.*

From now on, we will assume that $m=p^r$ for some prime number p.

Lemma 7. *Let $(m,n)=1$ and assume that $k/s\in\hat{\mathbf{Q}}$ with $p\mid s$. Then there exist some $T\in\Gamma_0^m(n)$ such that $T(k/s)=k_1/s_1$ with $(m,s_1)=1$.*

Proof. Since $(m,n)=1$, there exist two integers a and b such that $am^2-nb=1$. Let

$$T(z)=\frac{amz+b}{nz+m}.$$

Then $T\in\Gamma_0^m(n)$ and $T(k/s)=k_1/nk+ms=k_1/s_1$ with $(m,s_1)=1$ where $s_1=nk+ms$. $\square$

Lemma 8. *Let $k/s \in \mathbf{Q}$ with $(k,s) = (s,m) = 1$. Then there exist some $T \in \Gamma_0^m(n)$ such that*

$$T(k/s) = k_1/s_1$$

with $s_1 \mid n$.

Proof. Let $s_1 = (s,n)$. Then $s_1 = (s,\, kn) = (s,\ kmn)$. Therefore there exist some integers c_1 and d_1 such that $knmc_1 + sd_1 = s_1$. Since $(d_{1,}\ knm/s_1) = 1$, there exists an integer k_0 such that $(d_1 - (knm/s_1)k_0,\, nm) = 1$. Let $d = d_1 - (knm/s_1)k_0$ and $c = c_1 + (s/s_1)k_0$. Then $knmc + sd = s_1$. Furthermore, $(d,\, cm^2n) = 1$, since $(d,\, c) = (d,\, mn) = 1$. Thus there exist two integers x and y such that $xd - cm^2ny = 1$. Let

$$T(z) = \frac{xz + my}{cnmz + d}.$$

Then $T \in \Gamma_0^m(n)$ and

$$T\left(\frac{k}{s}\right) = \frac{k_1}{cnmk + ds} = \frac{k_1}{s_1} \quad \text{with } s_1 \mid n. \qquad \square$$

Lemma 9. *Let $(a_1,\, d_1) = (a_2, d_1) = 1$ and $d_1 \,|\, n$. Then a_1/d_1 is conjugate to a_2/d_1 under $\Gamma_0^m(n)$ if and only if $a_1 \equiv a_2 \pmod{mt}$ where $t = (d_1,\, n/d_1)$.*

Proof. Suppose that a_1/d_1 is conjugate to a_2/d_1 by some $T \in \Gamma_0^m(n)$. Then $T(a_1/d_1) = a_2/d_1$ for some $T \in \Gamma_0^m(n)$. Let T be of the form

$$T(z) = \frac{az + bm}{cnmz + d}, \quad ad - bcnm^2 = 1.$$

Thus we have

$$\frac{aa_1 + bmd_1}{cnma_1 + dd_1} = \frac{a_2}{d_1}$$

and therefore there exist some $u = \pm 1$ such that

$$aa_1 + bmd_1 = ua_2$$

and

$$cnma_1 + dd_1 = ud_1.$$

It follows that $aa_1 \equiv ua_2 \pmod{mt}$, $ad \equiv 1 \pmod{mt}$, and $d \equiv u \pmod{mt}$, which implies that $a_1 \equiv a_2 \pmod{mt}$. If T is of the form

$$T(z) = \frac{amz + b}{cnz + dm}, \quad adm^2 - bcn = 1,$$

we obtain

$$ama_1 + bd_1 = ua_2$$

and

$$cna_1 + dmd_1 = ud_1$$

for some $u = \pm 1$. Therefore we have

$$bcn \equiv 1 \pmod{m}, \quad bd_1 \equiv ua_2 \pmod{m},$$

and $cna_1 \equiv ud_1 \pmod{m}$. From those equations we see that $a_1 \equiv a_2 \pmod{m}$. In the same way, it can be seen that $adm^2 \equiv 1 \pmod{t}$, $ama_1 \equiv ua_2 \pmod{t}$, and $dm \equiv u \pmod{t}$, which implies that $a_1 \equiv a_2 \pmod{t}$. Thus $a_1 \equiv a_2 \pmod{mt}$.

Conversely, assume that $a_1 \equiv a_2 \pmod{mt}$ where $t = (d_1, n/d_1)$. Let $n_1 = n/d_1$. Then $t = (d_1, n_1) = (d_1, n_1a_1a_2)$ and therefore $tm = (d_1m, n_1a_1a_2m)$. Since $mt \mid a_2 - a_1$, there exist two integers x and y such that

$$n_1a_1a_2mx + d_1my = a_2 - a_1.$$

Thus we obtain

$$a_1(n_1a_2mx + 1) + d_1my = a_2.$$

If we take $b = y$ and $a = n_1a_2mx + 1$, we have

$$aa_1 + d_1mb = a_2.$$

Let $c = n_1d_1x$ and $d = 1 - mn_1a_1x$. Then

$$cma_1 + dd_1 = d_1.$$

On the other hand, we see that

$$ad - bcm^2 = a(1 - mn_1a_1x) - bn_1d_1xm^2 = a - (aa_1 + d_1mb)mn_1x = 1.$$

Let

$$T(z) = \frac{az + bm}{cmz + d}.$$

Then $T \in \Gamma_0^m(n)$ and

$$T(a_1/d_1) = \frac{aa_1 + bmd_1}{cma_1 + dd_1} = \frac{a_2}{d_1}. \qquad \square$$

We give a lemma whose proof is easy and will be omitted.

Lemma 10. *Let $(a_1, d_1) = (a_2, d_2) = 1$, $d_1 \mid n$, and $d_2 \mid n$ where d_1 and d_2 are positive integers. If a_1/d_1 is conjugate to a_2/d_2 under $\Gamma_0^m(n)$, then $d_1 = d_2$.*

Now we can give our main theorem.

Theorem 2. *Let $m = p^r$, and let $(m, n) = 1$. Then the parabolic class number of $\Gamma_0^m(n)$ is*

$$m \sum_{d \mid n} \varphi\left(\left(d, \frac{n}{d}\right)\right).$$

Proof. It suffices to calculate the number of orbits of $\Gamma_0^m(n)$ on $\hat{\mathbf{Q}}$. By Lemma 6 and Lemma 9, there exist $m\varphi((d, n/d))$ different orbits for each $d \mid n$. Then from Lemma 6 to Lemma 10, it follows that the number of orbits of $\Gamma_0^m(n)$ on $\hat{\mathbf{Q}}$ is

$$\sum_{d \mid n} m\varphi\left(\left(d, \frac{n}{d}\right)\right) = m \sum_{d \mid n} \varphi\left(\left(d, \frac{n}{d}\right)\right).$$

We can deduce the following easily.

Theorem 3. *The number of the orbits of* $\Gamma_0(n)$ *on* $\hat{Q}$ *is*

$$\sum_{d|n} \varphi\left(\left(d,\ \frac{n}{d}\right)\right),$$

which is the parabolic class number of $\Gamma_0(n)$.

Theorem 4. *Let* m *be not prime to* n. *Then the parabolic class number of* $\Gamma_0^m(n)$ *is*

$$\sum_{d|nm^2} \varphi\left(\left(d,\ \frac{nm^2}{d}\right)\right).$$

Proof. Since m is not prime to n, $\Gamma_0^m(n) = H_0$ and therefore $\Gamma_0^m(n)$ is conjugate to $\Gamma_0(nm^2)$. Then the proof follows. □

We now give a nice formula for the parabolic class number of $\Gamma_0(n^2)$ for any natural number $n \geq 2$. Before giving our main theorem we will give some lemmas.

Let $(m, n) = 1$, and let $\Lambda = \Gamma_0(n) \cap H$. Then it can be seen that Λ consists of all those mappings of the form

$$T(z) = \frac{az + bm}{cnmz + d}, \quad ad - bcnm^2 = 1.$$

Furthermore, it is easily seen that Λ is of finite index in Γ and that Λ is conjugate to $\Gamma_0(m^2n)$. Thus the parabolic class number of Λ and the parabolic class number of $\Gamma_0(m^2n)$ are the same.

Lemma 11. *Let* $k/s \in \hat{\mathbf{Q}}$ *with* $(k, s) = 1$. *Then there exist some* $T \in \Lambda$ *such that* $T(k/s) = k_1/s_1q$ *with* $s_1 \,|\, n$ *and* $q \,|\, m$.

Proof. Let $q = (s, m)$ and $s_1 = (s, n)$. Then $s = s^*q$ and $m = m^*q$ with $(s^*, m^*) = 1$. Then $s_1 = (s, n) = (s^*q, n) = (s^*, n) = (s^*, kn) = (s^*, knm^*)$. Thus $s_1q = (s^*q, knm^*q) = (s, knm)$. Therefore, there exist two integers c_1 and d_1 such that $s_1q = kmnc_1 +$

sd_1. Since $(d_1, kmn/s_1q) = 1$, there exists an integer k_0 such that $(d_1 - (knm/s_1q)k_0,\, nm^2) = 1$. Let $d = d_1 - (knm/s_1q)k_0$ and $c = c_1 + (s/s_1q)k_0$. Then $knmc + sd = s_1q$. Furthermore, $(d,\, cm^2n) = 1$, since $(d, c) = (d, nm^2) = 1$. Thus there exist two integers x and y such that $xd - cm^2ny = 1$. Let

$$T(z) = \frac{xz + my}{cnmz + d}.$$

Then $T \in \Lambda$ and

$$T(k/s) = \frac{k_1}{cnmk + ds} = \frac{k_1}{s_1q} \quad \text{with } s_1 \,|\, n \text{ and } q \,|\, m. \qquad \square$$

Lemma 12. *Let $d_1 \,|\, n$, $q \,|\, m$ and $(a_1, d_1q) = (a_2, d_1q) = 1$. Then $a_1/d_1\, q$ is conjugate to $a_2/d_1\, q$ under Λ if and only if $a_1 \equiv a_2 \pmod{t(m/q)}$ where $t = (d_1,\, n/d_1)$.*

Proof. Suppose that $(a_1, d_1q) = (a_2, d_1q) = 1$ and $a_1 \equiv a_2 \pmod{t(m/q)}$. Let $m/q = q^*$ and $n_1 = n/d_1$. Then it can be easily seen that $tq^* = (d_1qm,\, a_1a_2n_1q^*)$. Since $tq^* \,|\, a_2 - a_1$, there exist two integers x and y such that

$$n_1a_1a_2q^*x + d_1qmy = a_2 - a_1.$$

That is,

$$a_1(n_1a_2q^*x + 1) + d_1qmy = a_2.$$

Taking $b = y$ and $a = n_1a_2q^*x + 1$, we obtain

$$aa_1 + d_1qmb = a_2.$$

Let $c = n_1d_1x$ and $d = 1 - q^*n_1a_1x$. Then $n \,|\, c$ and

$$cma_1 + dqd_1 = d_1q.$$

On the other hand, we see that

$$\begin{aligned} ad - bcm^2 &= a(1 - q^*n_1a_1x) - bn_1d_1xm^2 \\ &= a - (aa_1 + d_1qmb)q^*n_1x = 1. \end{aligned}$$

Let

$$T(z) = \frac{az + bm}{cmz + d}.$$

Then $T \in \Lambda$ and

$$T(a_1/d_1) = \frac{aa_1 + bmd_1}{cma_1 + dqd_1} = \frac{a_2}{d_1 q}.$$

Let a_1/d_1q be conjugate to a_2/d_1q by some $T \in \Lambda$. Then it follows that

$$aa_1 + bmqd_1 = ua_2$$

and

$$cnma_1 + dd_1q = ud_1q$$

for some $u = \pm 1$, where $ad - bcm^2n = 1$. This shows that $aa_1 \equiv ua_2 \pmod{t(m/q)}$, $d \equiv u \pmod{t(m/q)}$ and $ad \equiv 1 \pmod{t(m/q)}$, which implies that $a_1 \equiv a_2 \pmod{t(m/q)}$. $\square$

Lemma 13. *Let $q \mid m$, $q^* \mid m$ and $d_1 \mid n$, $d_2 \mid n$. If a_1/d_1q is conjugate to a_2/d_1q by some $T \in \Lambda$, then $d_1 = d_2$ and $q = q^*$ where $d_1, d_2, q, q^* \geq 1$.*

Lemma 14. *Let $m = p^r$ for some prime number p and let $(m, n) = 1$. Then the parabolic class number of $\Gamma_0(m^2n)$ is $(2m - \varphi(m))\nu_\infty(\Gamma_0(n))$ where $\nu_\infty(\Gamma_0(n))$ denotes the parabolic class number of $\Gamma_0(n)$.*

Proof. For the divisor 1 of m and for each divisor d of n, we have $m\varphi((d, n/d))$ different orbits. Let $1 \leq i \leq r$. Then for each divisor p^i of m and for each divisor d of n, we have $\varphi(p^{r-i}(d, n/d))$ different orbits. Therefore, we have

$$\begin{aligned} m \sum_{d|n} \varphi\left(\left(d,\, \frac{n}{d}\right)\right) &+ \left[\sum_{i=1}^{r} \left[\sum_{d|n} \varphi\left(p^{r-i}\left(d,\, \frac{n}{d}\right)\right)\right]\right] \\ &= \left(\sum_{i=1}^{r} m + \varphi(p^{r-i})\right) \sum_{d|n} \varphi\left(\left(d,\, \frac{n}{d}\right)\right) \end{aligned}$$

different orbits, which is equal to

$$(2m-\varphi(m))\sum_{d|n}\varphi\left(\left(d,\,\frac{n}{d}\right)\right).$$

The proof then follows. $\square$

Theorem 5. *Let $n \geq 2$ be a natural number. Then the parabolic class number of $\Gamma_0(n^2)$ is*

$$|\Gamma:\Gamma_0(n)| = n\prod_{p\,|\,n}\left(1+\frac{1}{p}\right).$$

Proof. Let $m_i = p_i^{\alpha_i}$, and let $n = m_1 m_2 \cdots m_r$ with $(p_i, p_j) = 1$ for $i \neq j$. Then by using induction we have

$$\begin{aligned}\nu_\infty(\Gamma_0(n^2)) &= \nu_\infty(\Gamma_0(m_1^2\cdots m_r^2)) = \nu_\infty(\Gamma_0((m_1^2)(m_2^2\cdots m_r^2)) \\ &= \nu_\infty(\Gamma_0(m_1^2))\,\nu_\infty(\Gamma_0(m_2^2\cdots m_r^2)) \\ &= \nu_\infty(\Gamma_0(m_1^2))\cdots\,\nu_\infty(\Gamma_0(m_r^2)).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\nu_\infty(\Gamma_0(n^2)) &= (2m_1-\varphi(m_1))(2m_2-\varphi(m_2))\cdots(2m_r-\varphi(m_r)) \\ &= (p_1^{\alpha_1}+p_1^{\alpha_1-1})(p_2^{\alpha_2}+p_2^{\alpha_2-1})\cdots(p_r^{\alpha_r}+p_r^{\alpha_r-1}),\end{aligned}$$

which is equal to $n(1+1/p_1)(1+1/p_2)\cdots(1+1/p_r)$. Then the proof follows. $\square$

Corollary 1. *Let $n \geq 2$ be a natural number. Then*

$$\sum_{d|n^2}\varphi\left(\left(d,\,\frac{n^2}{d}\right)\right) = n\prod_{p\,|\,n}\left(1+\frac{1}{p}\right).$$

REFERENCES

1. G.A. Jones and D. Singerman, *Complex functions: An algebraic and geometric viewpoint*, Cambridge Univ. Press, Cambridge, 1987.

2. R. Keskin, *On the parabolic class number of some subgroups of Hecke groups*, Turkish J. Math. **22** (1998), 199–205.

3. ———, *On the parabolic class numbers of some Fuchsian groups*, Note Mat. **19** (1999), 275–283.

4. T. Miyake, *Modular forms*, Springer Verlag, New York, 1989.

5. B. Schoeneberg, *Elliptic modular functions*, Springer Verlag, New York, 1974.

6. G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton Univ. Press, Princeton, 1971.

7. D. Singerman, *Subgroups of Fuchsian groups and finite permutation groups*, Bulletin London Math. Soc. **2** (1970), 319–323.

8. I. Niven, H.S. Zuckerman and H.L. Montgomery, *An introduction to the theory of numbers*, John Wiley, New York, 1991.

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