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THE C^*-ALGEBRAS OF ARBITRARY GRAPHS

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ABSTRACT. To an arbitrary directed graph we associate a row-finite directed graph whose C^*-algebra contains the C^*-algebra of the original graph as a full corner. This allows us to generalize results for C^*-algebras of row-finite graphs to C^*-algebras of arbitrary graphs: the uniqueness theorem, simplicity criteria, descriptions of the ideals and primitive ideal space, and conditions under which a graph algebra is AF and purely infinite. Our proofs require only standard Cuntz-Krieger techniques and do not rely on powerful constructs such as groupoids, Exel-Laca algebras, or Cuntz-Pimsner algebras.

1. Introduction. Since they were first introduced in 1947 [**17**], C^*-algebras have become important tools for mathematicians working in many areas. Because of the immensity of the class of all C^*-algebras, however, it has become important to identify and study special types of C^*-algebras. These special types of C^*-algebras (e.g., AF-algebras, Bunce-Deddens algebras, AH-algebras, irrational rotation algebras, group C^*-algebras, and various crossed products) have provided great insight into the behavior of more general C^*-algebras. In fact, it is fair to say that much of the development of operator algebras in the last 20 years has been based on a careful study of these special classes.

One important and very natural class of C^*-algebras comes from considering C^*-algebras generated by partial isometries. There are a variety of ways to construct these C^*-algebras, but typically any such construction will involve having the partial isometries satisfy relations that describe how their initial and final spaces are related. Furthermore, one finds that in practice it is convenient to have an object (e.g., a matrix, a graph, etc.) that summarizes these relations.

In 1977 Cuntz introduced a class of C^*-algebras that became known as Cuntz algebras [**4**]. For each $n = 2, 3, \dots, \infty$ the Cuntz algebra $\mathcal{O}_n$ is generated by n isometries satisfying certain relations. The Cuntz algebras were important in the development of C^*-algebras because

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they provided the first examples of C^*-algebras whose K-theory has torsion. In 1980 Cuntz and Krieger considered generalized versions of the Cuntz algebras [**5**]. Given an $n \times n$ matrix A with entries in $\{0, 1\}$, the Cuntz-Krieger algebra $\mathcal{O}_A$ is defined to be the C^*-algebra generated by partial isometries satisfying relations determined by A. A study of the Cuntz-Krieger algebras was made in the seminal paper [**5**] where it was shown that they arise naturally in the study of topological Markov chains. It was also shown that there are important parallels between these C^*-algebras and certain kinds of dynamical systems (e.g., shifts of finite type).

In 1982 Watatani noticed that by considering a $\{0, 1\}$-matrix as the adjacency matrix of a directed graph, one could view Cuntz-Krieger algebras as C^*-algebras associated to certain finite directed graphs [**19**]. Although Watatani published some papers using this graph approach [**10, 19**], his work went largely unnoticed. It was not until 1997 that Kumjian, Pask, Raeburn, and Renault rediscovered C^*-algebras associated to directed graphs.

This theory of C^*-algebras associated to graphs was developed in [**13, 12**] and [**2**]. In these papers the authors were able to define and work with C^*-algebras associated to finite graphs as well as C^*-algebras associated to infinite graphs that are row-finite, i.e., all vertices emit a finite number of edges. By allowing all finite graphs as well as certain infinite graphs, these graph algebras included many C^*-algebras that were not Cuntz-Krieger algebras. Furthermore, it was found that the graph not only described the relations for the generators, but also many important properties of the associated C^*-algebra could be translated into graph properties. Thus the graph provides a tool for visualizing many aspects of the associated C^*-algebra. In addition, because graph algebras consist of a wide class of C^*-algebras whose structure can be understood, other areas of C^*-algebra theory have benefited nontrivially from their study.

Despite these successes, many people were still unsatisfied with the condition of row-finiteness and wanted a theory of C^*-algebras for arbitrary graphs. This desire was further fueled by the fact that in his original paper [4] Cuntz defined a C^*-algebra $\mathcal{O}_\infty$, which seemed as though it should be the C^*-algebra associated to a graph with one vertex and a countably infinite number of edges. Despite many people's desire to extend the definition of graph algebras to arbitrary graphs, it was unclear exactly how to make sense of the defining relations in the non row-finite case. It was not until 2000 that Fowler, Laca, and Raeburn were finally able to extend the definition of graph algebras to arbitrary directed graphs [**8**]. These graph algebras now included the Cuntz algebra $\mathcal{O}_\infty$, and as expected it arises as the C^*-algebra of the graph with one vertex and infinitely many edges.

In the time since C^*-algebras associated to arbitrary graphs were defined, there have been many attempts to extend results for row-finite graph algebras to arbitrary graph algebras. However, because many of the proofs of the fundamental theorems for C^*-algebras of row-finite graphs make heavy use of the row-finiteness assumption, it has often been unclear how to proceed. In most cases where results have been generalized, the proofs have relied upon sophisticated techniques and powerful machinery such as groupoids, the Exel-Laca algebras of [**7**], and the Cuntz-Pimsner algebras of [**15**].

In this paper we describe an operation called desingularization that transforms an arbitrary graph into a row-finite graph with no sinks. It turns out that this operation preserves Morita equivalence of the associated C^*-algebra as well as the loop structure and path space of the graph. Consequently, it is a powerful tool in the analysis of graph algebras because it allows one to apply much of the machinery that has been developed for row-finite graph algebras to arbitrary graph algebras.

Desingularization was motivated by the process of "adding a tail to a sink" that is described in [**2**]. In fact, this process is actually a special case of desingularization. The difference is that now we not only add tails at sinks, but we also add (more complicated) tails at vertices that emit infinitely many edges. Consequently, we shall see that vertices that emit infinitely many edges will often behave similarly to sinks in the way that they affect the associated C^*-algebra. In fact for some of our results, such as conditions for simplicity, one can take the result for row-finite graphs and replace the word "sink" by the phrase "sink or vertex that emits infinitely many edges" to get the corresponding result for arbitrary graphs.

We begin in Section 2 with the definition of desingularization. This is our main tool for dealing with C^*-algebras associated to arbitrary graphs. It gives the reader who is comfortable with C^*-algebras of row-finite graphs a great deal of intuition into the structure of non row-finite graph algebras. This is accomplished by providing a method for easily translating questions about arbitrary graph algebras to the row-finite setting. After the definition of desingularization, we describe a correspondence between paths in the original graph and paths in the desingularization. We then show that desingularization preserves loop structure of the graph as well as Morita equivalence of the C^*-algebra. This allows us to obtain easy proofs of several known results. In particular, we prove the uniqueness theorem of [**8**] and give necessary and sufficient conditions for a graph algebra to be simple, purely infinite, and AF.

In Section 3 we describe the ideal structure of graph algebras. Here we will see that our solution is more complicated than what occurs in the row-finite case. The correspondence with saturated hereditary sets described in [**2**] no longer holds. Instead we have a correspondence of the ideals with pairs (H, S), where H is a saturated hereditary set and S is a set containing vertices that emit infinitely many edges, only finitely many of which have range outside of H.

We conclude in Section 4 with a description of the primitive ideal space of a graph algebra. Our result will again be more complicated than the corresponding result for the row-finite case, which involves maximal tails [**2**]. For arbitrary graphs we will need to account for vertices that emit infinitely many edges, and our description of the primitive ideal space will include both maximal tails and special vertices that emit infinitely many edges known as "breaking vertices".

We thank Iain Raeburn for making us aware of the related papers by Szymański [**18**] and Paterson [**14**], and we thank both Iain Raeburn and Dana Williams for their comments on the first draft of this paper. After this work was completed, it was brought to our attention that our description of the primitive ideal space in Section 4 had been obtained independently in the preprint [**1**]. Although the results in [**1**] are similar to some of our results in Section 4, one should note that the methods used in the proofs are very different. In addition, we mention that we have adopted their term "breaking vertex" to provide consistency for readers who look at both papers.

2. The desingularized graph. We closely follow the notation established in [**12**] and [**2**]. A (directed) graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ describing the source and range of each edge. We let E^* denote the set of finite paths in E, and we let E^∞ denote the set of infinite paths. The maps r, s extend to E^* in the obvious way and s extends to E^∞.

A vertex v is called a *sink* if $|s^{-1}(v)| = 0$, and v is called an *infinite-emitter* if $|s^{-1}(v)| = \infty$. If v is either a sink or an infinite-emitter, we call it a *singular vertex*. A graph E is said to be *row-finite* if it has no infinite-emitters.

Given any graph (not necessarily row-finite), a *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and partial isometries $\{s_e \mid e \in E^1\}$ with orthogonal ranges satisfying the *Cuntz-Krieger relations*:

1. $s_e^* s_e = p_{r(e)}$ for every $e \in E^1$;
2. $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$;
3. $p_v = \sum_{\{e \mid s(e)=v\}} s_e s_e^*$ for every $v \in E^0$ that is not a singular vertex.

The graph algebra $C^*(E)$ is defined to be the C^*-algebra generated by a universal Cuntz-Krieger E-family. For the existence of such a C^*-algebra, one can either modify the proofs in [**11**, Theorem 2.1] or [**12**, Theorem 1.2], or one can appeal to more general constructions such as [**3**] or [**15**].

Given a graph E we shall construct a graph F, called a *desingularization of E*, with the property that F has no singular vertices and $C^*(E)$ is isomorphic to a full corner of $C^*(F)$. Loosely speaking, we will build F from E by replacing every singular vertex v_0 in E with its own infinite path, and then redistributing the edges of $s^{-1}(v_0)$ along the vertices of the infinite path. Note that if v_0 happens to be a sink, then $|s^{-1}(v_0)| = 0$ and there are no edges to redistribute. In that case our procedure will coincide with the process of adding an infinite tail to a sink described in [**2**].

Definition 2.1. Let E be a graph with a singular vertex v_0. We *add a tail* to v_0 by performing the following procedure. If v_0 is a sink, we add a graph of the form

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots \tag{2.1}$$

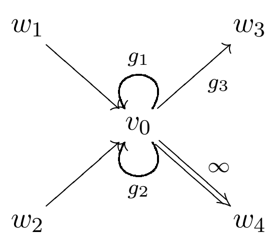
as described in [**2**]. If v_0 is an infinite emitter we first list the edges $g_1, g_2, g_3, \ldots$ of $s^{-1}(v_0)$. Then we add a graph of the form shown in (2.1), remove the edges in $s^{-1}(v_0)$, and for every $g_j \in s^{-1}(v_0)$ we draw an edge f_j from v_{j-1} to $r(g_j)$.

For any j we shall also define α^j to be the path $\alpha^j := e_1 e_2 \cdots e_{j-1} f_j$ in F.

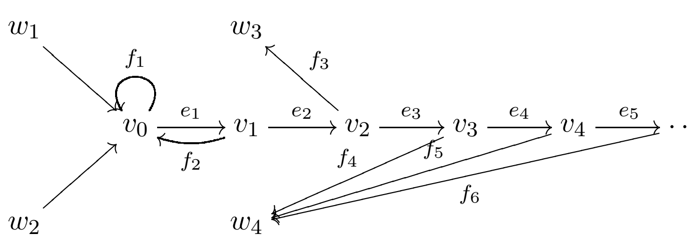
Note that different orderings of the edges of $s^{-1}(v_0)$ may give rise to nonisomorphic graphs via the above procedure.

Definition 2.2. If E is a directed graph, a *desingularization of* E is a graph F obtained by adding a tail at every singular vertex of E.

Example 2.3. Suppose we have a graph E containing this fragment:



where the double arrow labeled ∞ denotes a countably infinite number of edges from v_0 to w_4. Let us label the edges from v_0 to w_4 as $\{g_4, g_5, g_6, \ldots\}$. Then a desingularization of E is given by the following graph F.



Example 2.4. If E is the $\mathcal{O}_\infty$ graph (one vertex with infinitely many loops), a desingularization F looks like this:

$$\begin{tikzcd} \cdot \arrow[loop left] \arrow[r] & \cdot \arrow[l, bend left] \arrow[r] & \cdot \arrow[ll, bend left] \arrow[r] & \cdot \arrow[lll, bend left] \arrow[r] & \cdot \arrow[llll, bend left] \arrow[r] & \cdots \arrow[lllll, bend left] \end{tikzcd}$$

Example 2.5. The following graph was mentioned in [**8**, Remark 11]:

$$\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow v_0 \overset{\infty}{\Longrightarrow} \cdot \longrightarrow \cdot \longrightarrow \cdots$$

A desingularization of it is:

$$\begin{tikzcd} \cdots \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & v_0 \arrow[r] \arrow[d] & \cdot \arrow[r] & \cdot \arrow[r] & \cdots \\ & & & v_1 \arrow[ur] \arrow[d] & & & \\ & & & v_2 \arrow[uur] \arrow[d] & & & \\ & & & v_3 \arrow[uuur] \arrow[d] & & & \\ & & & \vdots \arrow[uuuur] & & & \end{tikzcd}$$

It is crucial that desingularizing a graph preserves connectivity, path space, and loop structure in the appropriate senses, and this will turn out to be the case. We make these ideas precise with the next three lemmas: Lemma 2.6 describes how the path spaces of E and F are related, Lemma 2.7 shows that desingularization preserves loop structure, and Lemma 2.8 describes the relationship between cofinality of a graph and cofinality of its desingularization.

We first review some notation. If E is a directed graph and $S_1, S_2 \subseteq E^0$ we say S_1 *connects to* S_2, denoted $S_1 \geq S_2$, if for every $v \in S_1$ there exists $w \in S_2$ and $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. Frequently one or both of the S_is will contain a single vertex v, in which case we write v rather than $\{v\}$. If λ is a finite or infinite path in E, we write $S \geq \lambda$ to mean $S \geq \{s(\lambda_i)\}_{i=1}^{|\lambda|}$. Finally, a graph E is said to be *cofinal* if for every infinite path λ we have $E^0 \geq \lambda$.

Lemma 2.6. *Let E be a graph and let F be a desingularization of E.*

(a) *There are bijective maps*

$$\begin{aligned} \phi : E^* &\longrightarrow \{\beta \in F^* \mid s(\beta), r(\beta) \in E^0\} \\ \phi_\infty : E^\infty \cup &\{\alpha \in E^* \mid r(\alpha) \textit{ is a singular vertex}\} \\ &\longrightarrow \{\lambda \in F^\infty \mid s(\lambda) \in E^0\}. \end{aligned}$$

The map ϕ preserves source and range, and hence ϕ preserves loops, and the map ϕ_∞ preserves source.

(b) *The map ϕ_∞ preserves $\geq$ in the following sense. For every $v \in E^0$ and $\lambda \in E^\infty \cup \{\alpha \in E^* \mid r(\alpha)$ is a singular vertex$\}$, we have $v \geq \lambda$ in E if and only if $v \geq \phi_\infty(\lambda)$ in F.*

Proof. First define a map $\phi' : E^1 \to F^*$. If $e \in E^1$, then e will have one of two forms: either $s(e)$ is not a singular vertex, in which case $e \in F^1$, or else $s(e)$ is a singular vertex, in which case $e = g_j$ for some j. We define ϕ' by

$$\phi'(e) = \begin{cases} e & \text{if } s(e) \text{ is not singular;} \\ \alpha^j & \text{if } e = g_j \text{ for some } j, \end{cases}$$

where $\alpha^j := e_1 \cdots e_{j-1} f_j$ is the path described in Definition 2.1. Since ϕ' preserves source and range, it extends to a map on the finite path space E^*. In particular, for $\alpha = \alpha_1 \cdots \alpha_n \in E^*$ define $\phi(\alpha) = \phi'(\alpha_1)\phi'(\alpha_2) \ldots \phi'(\alpha_{|\alpha|})$. It is easy to check that ϕ is injective, that it preserves source and range, and that it is onto the set $\{\beta \in F^* \mid s(\beta), r(\beta) \in E^0\}$. We define ϕ_∞ similarly. In particular, if $\lambda = \lambda_1\lambda_2 \cdots \in E^\infty$, define $\phi_\infty(\lambda) = \phi'(\lambda_1)\phi'(\lambda_2) \cdots$. If α is a finite path whose range is a singular vertex v_0, we define $\phi_\infty(\alpha) = \phi(\alpha)e_1e_2 \cdots$, where $e_1e_2, \ldots$ is the tail in F added to v_0.

To show that ϕ_∞ is a bijection, we construct an inverse $\psi_\infty : \{\lambda \in F^\infty \mid s(\lambda) \in E^0\} \to E^\infty \cup \{\alpha \in E^* \mid r(\alpha) \text{ is a singular vertex}\}$. Notice that every $\lambda \in F^\infty$ either returns to E infinitely often or it ends up in one of the added infinite tails. More precisely, λ has one of two forms: either $\lambda = a^1 a^2 \cdots$ or $\lambda = a^1 a^2 \cdots a^n e_1 e_2 e_3 \cdots$, where each a^k is either an edge of E or an α^j. We define ψ' by

$$\psi'(a^k) := \begin{cases} a^k & \text{if } a^k \in E^1; \\ g_j & \text{if } a^k = \alpha^j \text{ for some } j, \end{cases}$$

and we define

$$\psi_\infty(\lambda) := \begin{cases} \psi'(a^1)\psi'(a^2)\cdots & \text{if } \lambda = a^1 a^2 \cdots; \\ \psi'(a^1)\cdots\psi'(a^n) & \text{if } \lambda = a^1 \cdots a^n e_1 e_2 \cdots. \end{cases}$$

It is easy to check that ϕ_∞ and ψ_∞ are inverses, and we have established (a).

To prove (b), let $\lambda \in E^\infty \cup \{\alpha \in E^* \mid r(\alpha) \text{ is a singular vertex}\}$ and $v \geq \lambda$ in E. Then there exists a finite path α in E such that $s(\alpha) = v$ and $r(\alpha) = w$ for some $w \in E^0$ lying on the path λ. Note that the vertices of E that are on the path $\phi_\infty(\lambda)$ are exactly the same as the vertices on the path λ. Hence w must also be a vertex on the path $\phi_\infty(\lambda)$. Now, because ϕ preserves source and range, $\phi(\alpha)$ is a path that starts at v and ends at w, which is a vertex on $\phi_\infty(\lambda)$. Thus $v \geq \phi_\infty(\lambda)$.

For the converse let $\lambda \in E^\infty \cup \{\alpha \in E^* \mid r(\alpha) \text{ is a singular vertex}\}$ and $v \in E^0$, and suppose that $v \geq \phi_\infty(\lambda)$ in F. Then there exists a finite path β in F with $s(\beta) = v$ and $r(\beta) = w$ for some vertex w on the path $\phi_\infty(\lambda)$. Notice that if $r(\beta)$ is a vertex on one of the added infinite tails, then $\phi_\infty(\lambda)$ must have passed through v_0, and so must have β. Thus we may assume $r(\beta) \in E^0 \subseteq F^0$. Now β is a finite path in F that starts and ends in E^0, so it can be pulled back to a path $\phi^{-1}(\beta) \in E^*$ with source v and range $r(\beta)$. Since $r(\beta)$ lies on the path $\phi_\infty(\lambda)$, it lies on the path λ, and thus $\phi^{-1}(\beta)$ is a path from v to some vertex of λ. Hence $v \geq \lambda$ in E. □

A *loop* in a graph E is a finite path $\alpha = \alpha_1 \alpha_2 \cdots \alpha_{|\alpha|}$ with $s(\alpha) = r(\alpha)$. The vertex $s(\alpha) = r(\alpha)$ is called the *base point* of the loop. A loop is said to be *simple* if $s(\alpha_i) = s(\alpha_1)$ implies $i = 1$. Therefore a simple loop is one that does not return to its base point more than once. An *exit* for a loop α is an edge f such that $s(f) = s(\alpha_i)$ for some

i, and $f \neq \alpha_i$. A graph E is said to satisfy *Condition*(L) if every loop has an exit and E is said to satisfy *Condition* (K) if no vertex in E is the base point of exactly one simple loop.

Lemma 2.7. *Let E be a graph and let F be a desingularization of E. Then*

(a) *E satisfies Condition* (L) *if and only if F satisfies Condition* (L).

(b) *E satisfies Condition* (K) *if and only if F satisfies Condition* (K).

Proof. If α is a loop in E with no exits, then all the vertices on α emit exactly one edge. Hence none of these vertices are singular vertices, and $\phi(\alpha)$ is a loop in F with no exits. If α is a loop in F with no exits, then we claim that none of the singular vertices of E can appear in the loop. To see this, note that if v_0 is a sink in E, then it cannot be a part of a loop in F; and if v_0 is an infinite-emitter in E, then v_0 is the source of two edges, which would necessarily create an exit for any loop. Since none of the singular vertices of E appear in α, it follows that $\phi^{-1}(\alpha)$ is a loop in E with no exits. This establishes part (a).

Now suppose $v \in E^0$ is the base of exactly one simple loop α in E. Then $\phi(\alpha)$ is a simple loop in F. If there were another simple loop β in F based at v, then $\phi^{-1}(\beta)$ would be simple loop in E based at v that is different from α. Thus if F satisfies Condition (K), then E satisfies Condition (K).

Now suppose E satisfies Condition (K). Let $v \in F^0$ be the base of a simple loop α in F. If $v \in E^0$, then $\phi^{-1}(\alpha)$ is a simple loop in E based at v. Since E satisfies Condition (K), there is a simple loop β in E different from $\phi^{-1}(\alpha)$. Certainly, $\phi(\beta)$ is a simple loop in F and, because ϕ is injective, $\phi(\beta)$ must be different from α.

Now suppose v is on an added infinite tail; that is, $v = v_n$ for some $n \geq 1$. Then α must have the form $\alpha' e_1 e_2 \cdots e_n$ for some $\alpha' \in F^*$. Now, $e_1 e_2 \cdots e_n \alpha'$ is a simple loop in F based at v_0 and hence $\phi^{-1}(e_1 \cdots e_n \alpha')$ is a simple loop in E based at v_0. Since E satisfies Condition (K), there must be another simple loop β in E based at v_0. Now $\phi(\beta)$ will be a simple loop in F based at v_0. If v_n is not a vertex on $\phi(\beta)$, then $\alpha' \phi(\beta) e_1 \cdots e_n$ will be another simple loop based at v_n that is different from α. On the other hand, if v_n is a vertex of $\phi(\beta)$, then $\phi(\beta)$ has the form $e_1 \cdots e_n \beta'$, where $\beta \in F^*$. Since $\phi(\beta)$ is a simple loop based at v_0, we know that $s(\beta_i) \neq v_0$ for $1 \leq i \leq |\beta'|$. Hence v_n is not a vertex on the path β'. Therefore $\beta' e_1 \cdots e_n$ is a simple loop based at v_n. Furthermore, it is different from the loop $\alpha = \alpha' e_1 \cdots e_n$, because if they were equal then we would have $\alpha' = \beta'$, which contradicts the fact that $\alpha = \alpha' e_1 \cdots e_n$ and $\phi(\beta) = \beta' e_1 \cdots e_n$ are distinct. Thus F satisfies Condition (K). $\square$

Lemma 2.8. *Let E be a graph and let F be a desingularization of E. Then the following are equivalent*:

(1) *F is cofinal*;

(2) *E is cofinal and for every singular vertex $v_0 \in E^0$ we have $E^0 \geq v_0$.*

Proof. Assume F is cofinal and fix $v \in E^0$. Suppose $\lambda \in E^\infty$. Because F is cofinal, $v \geq \phi_\infty(\lambda)$ in F. Thus by Lemma 2.6(b), $v \geq \lambda$ in E. Now let $v_0 \in E^0$ be any singular vertex. Then $\phi_\infty(v_0)$ is the infinite tail $e_1 e_2 \cdots$ added to v_0. By cofinality of F, v connects to $e_1 e_2 \cdots$, and since any path that connects to $e_1 e_2 \cdots$ connects to v_0, we know that there is a path $\alpha \in F^*$ from v to v_0. But then $\phi^{-1}(\alpha)$ is a path from v to v_0 in E. Hence $E^0 \geq v_0$.

Now assume E is cofinal and for every singular vertex v_0 we have $E^0 \geq v_0$. If E has a sink v_0, then since E is cofinal it follows that $E^\infty = \varnothing$. Furthermore, since $E^0 \geq v_0$ it must be the case that v_0 is the only sink in E. Hence F is obtained from E by adding a single tail at v_0. Now if $\lambda \in F^\infty$, then since $E^\infty = \varnothing$ we must have that λ eventually ends up in the tail. If $w \in F^0$, then either w is in the tail or $w \in E^0$. Since $E^0 \geq v_0$ this implies that in either case $w \geq \lambda$. Hence F is cofinal.

Now assume that E has no sinks. Let $\lambda \in F^\infty$ and $v \in F^0$. We must show that $v \geq \lambda$ in F. We will first show that it suffices to prove this for the case when $s(\lambda) \in E^0$ and $v \in E^0$. If $v = v_n$, a vertex in one of the added infinite tails, then because E has no sinks, v_n must be the source of some edge f_j with $r(f_j) \in E^0$ and we see that $r(f_j) \geq \lambda$ in F implies $v_n \geq \lambda$ in F. Likewise, if $s(\lambda) = v_n$, a vertex in the infinite tail added to v_0, then $v \geq e_1 e_2 \cdots e_n \lambda$ in F implies $v \geq \lambda$ in F. Thus we may replace λ by $e_1 e_2 \cdots e_n \lambda$. Hence we may assume that $s(\lambda) \in E^0$ and $v \in E^0$.

Since λ is a finite path in F whose source is in E^0, Lemma 2.6(a) implies that $\lambda = \phi_\infty(\mu)$, where μ is either an infinite path in E or a finite path in E ending at a singular vertex. If μ is an infinite path, then cofinality of E implies that $v \geq \mu$ and Lemma 2.6(b) implies that $v \geq \phi_\infty(\mu) = \lambda$. If μ is a finite path ending at a singular vertex, then $v \geq \mu$ by assumption and so $v \geq \phi_\infty(\mu) = \lambda$. Thus F is cofinal. $\square$

The next two lemmas will be used to prove Theorem 2.11, which states that $C^*(E)$ is isomorphic to a full corner of $C^*(F)$. Lemma 2.9 says, roughly speaking, that a Cuntz-Krieger F-family contains a Cuntz-Krieger E-family; and Lemma 2.10 says that we can extend a Cuntz-Krieger E-family to obtain a Cuntz-Krieger F-family.

Lemma 2.9. *Suppose E is a graph and let F be a desingularization of E. If $\{T_e, Q_v\}$ is a Cuntz-Krieger F-family, then there exists a Cuntz-Krieger E-family in $C^*(\{T_e, Q_v\})$.*

Proof. For every vertex v in E, define $P_v := Q_v$. For every edge e in E with $s(e)$ not a singular vertex, define $S_e := T_e$. If e is an edge in E with $s(e) = v_0$ a singular vertex, then $e = g_j$ for some j, and we define $S_e := T_{\alpha^j}$. The fact that $\{S_e, P_v \mid e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family follows immediately from the fact that $\{T_e, Q_v \mid e \in F^1, v \in F^0\}$ is a Cuntz-Krieger F-family. $\square$

Lemma 2.10. *Let E be a graph and let F be a desingularization of E. For every Cuntz-Krieger E-family $\{S_e, P_v \mid e \in E^1, v \in E^0\}$ on a Hilbert space $\mathcal{H}_E$, there exists a Hilbert space $\mathcal{H}_F = \mathcal{H}_E \oplus \mathcal{H}_T$ and a Cuntz-Krieger F-family $\{T_e, Q_v \mid e \in F^1, v \in F^0\}$ on $\mathcal{H}_F$ satisfying:*

- *$P_v = Q_v$ for every $v \in E^0$;*
- *$S_e = T_e$ for every $e \in E^1$ such that $s(e)$ is not a singular vertex;*
- *$S_e = T_{\alpha^j}$ for every $e = g_j \in E^1$ such that $s(e)$ is a singular vertex;*
- *$\sum_{v \notin E^0} Q_v$ is the projection onto $\mathcal{H}_T$.*

Proof. We prove the case where E has just one singular vertex v_0. If v_0 is a sink, then the result follows from [**2**, Lemma 1.2]. Therefore let us assume that v_0 is an infinite-emitter. Given a Cuntz-Krieger E-family $\{S_e, P_v\}$ we define $R_0 := 0$ and $R_n := \sum_{j=1}^n S_{g_j} S_{g_j}^*$ for each positive integer n. Note that the R_ns are projections because the S_{g_j}s have orthogonal ranges. Furthermore, $R_n \leq R_{n+1} < P_{v_0}$ for every n.

Now for every integer $n \geq 1$ define $\mathcal{H}_n := (P_{v_0} - R_n)\mathcal{H}_E$ and set

$$\mathcal{H}_F := \mathcal{H}_E \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n.$$

For every $v \in E^0$ define $Q_v = P_v$ acting on the $\mathcal{H}_E$ component of $\mathcal{H}_F$ and zero elsewhere. That is, $Q_v(\xi_E, \xi_1, \xi_2, \dots) = (P_v \xi_E, 0, 0, \dots)$. Similarly, for every $e \in E^1$ with $s(e) \neq v_0$ define $T_e = S_e$ on the $\mathcal{H}_E$ component. For each vertex v_n on the infinite tail define Q_{v_n} to be the projection onto $\mathcal{H}_n$. That is, $Q_{v_n}(\xi_E, \xi_1, \dots, \xi_n, \dots) = (0, 0, \dots, \xi_n, 0, \dots)$. Now note that because the R_ns are nondecreasing, $\mathcal{H}_n \subseteq \mathcal{H}_{n-1}$ for each n. Thus for each edge of the form e_n we can define T_{e_n} to be the inclusion of $\mathcal{H}_n$ into $\mathcal{H}_{n-1}$ (where $\mathcal{H}_0$ is taken to mean $P_{v_0}\mathcal{H}_E$). More precisely,

$$T_{e_n}(\xi_E, \xi_1, \xi_2, \dots) = (0, 0, \dots, 0, \xi_n, 0, \dots),$$

where the ξ_n is in the $\mathcal{H}_{n-1}$ component.

Finally, for each edge g_j and for each $\xi \in \mathcal{H}_E$ we have $S_{g_j}\xi \in \mathcal{H}_{j-1}$. To see this recall that $\mathcal{H}_{j-1} = (P_{v_0} - R_{j-1})\mathcal{H}_E$, and thus $(P_{v_0} - R_{j-1})S_{g_j}\xi = S_{g_j}\xi$. Therefore we can define T_{f_j} by

$$T_{f_j}(\xi_E, \xi_1, \xi_2, \dots) = (0, \dots, 0, S_{g_j}\xi_E, 0, \dots),$$

where the nonzero term appears in the $\mathcal{H}_{j-1}$ component.

We will now check that the collection $\{T_e, Q_v\}$ is a Cuntz-Krieger F-family. It follows immediately from definitions and the Cuntz-Krieger relations on E that $T_e^* T_e = Q_{r(e)}$ for every e that is not of the form f_j or e_n, and that $Q_v = \sum_{s(e)=v} T_e T_e^*$ for every v not on the infinite tail. Furthermore, it is easy to check using the definitions that the Q_vs are mutually orthogonal and that $T_{e_n}^* T_{e_n} = Q_{r(e_n)}$ for every edge e_n

on the infinite tail. Now note that for every f_j,

$$\begin{aligned}T^*_{f_j}T_{f_j}(\xi_E,\xi_1,\xi_2,\dots) &= T^*_{f_j}(0,\dots,0,S_{g_j}\xi_E,0,\dots)\\ &= (S^*_{g_j}S_{g_j}\xi_E,0,0,\dots)\\ &= (P_{r(e_j)}\xi_E,0,0,\dots)\\ &= Q_{r(e_j)}(\xi_E,\xi_1,\xi_2,\dots).\end{aligned}$$

Finally, let v_n be a vertex on the infinite tail. The edges emanating from v_n are e_{n+1} and f_{n+1}, and we have

$$T_{e_{n+1}}T^*_{e_{n+1}}(\xi_E,\xi_1,\dots) = (0,\dots,0,(P_{v_0}-R_{n+1})\xi_n,0,\dots),$$

where the nonzero term is in the $\mathcal{H}_n$ component. Also

$$T_{f_{n+1}}T^*_{f_{n+1}}(\xi_E,\xi_1,\dots) = (0,\dots,0,S_{g_{n+1}}S^*_{g_{n+1}}\xi_n,0,\dots),$$

where the nonzero term is again in the $\mathcal{H}_n$ component. We then have the following:

$$\begin{aligned}&(T_{e_{n+1}}T^*_{e_{n+1}}+T_{f_{n+1}}T^*_{f_{n+1}})(\xi_E,\xi_1,\dots)\\ &\qquad= (0,\dots,0,(P_{v_0}-R_{n+1}+S_{g_{n+1}}S^*_{g_{n+1}})\xi_n,0,\dots)\\ &\qquad= (0,\dots,0,(P_{v_0}-R_n)\xi_n,0,\dots)\\ &\qquad= (0,\dots,0,\xi_n,0,\dots)\\ &\qquad= Q_{v_n}(\xi_E,\xi_1,\dots).\end{aligned}$$

Thus $\sum_{\{e:s(e)=v_n\}} T_eT^*_e = T_{e_{n+1}}T^*_{e_{n+1}} + T_{f_{n+1}}T^*_{f_{n+1}} = Q_{v_n} = Q_{r(e_n)}$ and we have established that $\{T_e,Q_v\}$ is a Cuntz-Krieger F-family. It is easy to verify that the bulleted points in the statement of the lemma are satisfied. □

Theorem 2.11. *Let E be a graph and let F be a desingularization of E. Then $C^*(E)$ is isomorphic to a full corner of $C^*(F)$. Consequently, $C^*(E)$ and $C^*(F)$ are Morita equivalent.*

Proof. Again for simplicity we assume that E has only one singular vertex v_0. Let $\{t_e,q_v\,|\,e\in F^1, v\in F^0\}$ denote the canonical set of generators for $C^*(F)$ and let $\{s_e, p_v \,|\, e \in E^1, v \in E^0\}$ denote the Cuntz-Krieger E-family in $C^*(F)$ constructed in Lemma 2.9. Define $B := C^*(\{s_e, p_v\})$ and $p := \sum_{v \in E^0} q_v$. To prove the proposition, we will show that $C^*(E) \cong B \cong pC^*(F)p$ is a full corner in $C^*(F)$.

Since B is generated by a Cuntz-Krieger E-family, in order to show that $B \cong C^*(E)$ it suffices to prove that B satisfies the universal property of $C^*(E)$. Let $\{S_e, P_v \,|\, e \in E^1, v \in E^0\}$ be a Cuntz-Krieger E-family on a Hilbert space $\mathcal{H}_E$. Then by Lemma 2.10 we can construct a Hilbert space $\mathcal{H}_F$ and a Cuntz-Krieger F-family $\{T_e, Q_v \,|\, e \in F^1, v \in F^0\}$ on $\mathcal{H}_F$ such that $Q_v = P_v$ for every $v \in E^0$, $T_e = S_e$ for every $e \in F^1$ with $s(e) \neq v_0$, and $S_{g_j} = T_{\alpha^j}$ for every edge g_j in E whose source is v_0. Now by the universal property of $C^*(F)$, we have a homomorphism π from $C^*(F)$ onto $C^*(\{T_e, Q_v \,|\, e \in F^1, v \in F^0\})$ that takes t_e to T_e and q_v to Q_v.

For any $v \in E^0$ we have $p_v = q_v$, so $\pi(p_v) = Q_v = P_v$. Let $e \in E^1$. If $s(e) \neq v_0$, then $s_e = t_e$ and $\pi(s_e) = T_e = S_e$. Finally, if $s(e) = v_0$ then $e = g_j$ for some j, and $s_e = t_{\alpha^j}$ so that $\pi(s_{g_j}) = T_{\alpha^j} = S_{g_j}$. Thus $\pi|_B$ is a representation of B on $\mathcal{H}_E$ that takes generators of B to the corresponding elements of the given Cuntz-Krieger E-family. Therefore B satisfies the universal property of $C^*(E)$ and $C^*(E) \cong B$.

We now show that $B \cong pC^*(F)p$. Just as in [**2**, Lemma 1.2(c)], we have that $\sum_{v \in E^0} q_v$ converges strictly in $M(C^*(F))$ to a projection p and that for any $\mu, \nu \in F^*$ with $r(\mu) = r(\nu)$,

$$pt_\mu t_\nu^* p = \begin{cases} t_\mu t_\nu^* & \text{if } s(\mu), s(\nu) \in E^0; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the generators of B are contained in $pC^*(F)p$ and $B \subseteq pC^*(F)p$. To show the reverse inclusion, let μ and ν be finite paths in F with $r(\mu) = r(\nu)$. We need to show that $pt_\mu t_\nu^* p \in B$. If either μ or ν does not start in E^0, then $pt_\mu t_\nu^* p = 0$ by the above formula. Hence we may as well assume that both μ and ν start in E^0. Now, if $r(\mu) = r(\nu) \in E^0$ as well, there will exist unique $\mu', \nu' \in E^0$ with $\phi(\mu') = \mu$ and $\phi(\nu') = \nu$. In this case, $t_\mu = s_{\mu'}$ and $t_\nu = s_{\nu'}$, so

$$pt_\mu t_\nu^* p = t_\mu t_\nu^* = s_{\mu'} s_{\nu'}^* \in B.$$

On the other hand, if $r(\mu) = r(\nu) \notin E^0$ then $r(\mu) = r(\nu) = v_n$ for some n. We shall prove that $pt_\mu t_\nu^* p \in B$ by induction on n. Suppose that

$pt_{\mu'}t_{\nu'}^*p \in B$ for any paths μ' and ν' with $r(\mu') = r(\nu') = v_{n-1}$. Then if $r(\mu) = r(\nu) = v_n$ we shall write $\mu = \mu' e_n, \nu = \nu' e_n$ for finite paths μ' and ν' with $r(\mu') = r(\nu') = v_{n-1}$. Now there are precisely two edges, e_n and f_n with source v_{n-1}. Thus

$$\begin{aligned} pt_\mu t_\nu^* p &= pt_{\mu'} t_{e_n} t_{e_n}^* t_{\nu'}^* p \\ &= pt_{\mu'}(q_{v_{n-1}} - t_{f_n} t_{f_n}^*) t_{\nu'}^* p \\ &= pt_{\mu'} t_{\nu'}^* p - pt_{\mu' f_n} t_{\nu' f_n}^* p, \end{aligned}$$

which is in B. Hence $pC^*(F)p \subseteq B$. Finally, we note that $pC^*(F)p$ is full by an argument identical to the one given in [**2**, Lemma 1.2(c)]. □

Theorem 2.11 allows us to get easy proofs of several known results by passing to a desingularization and using the corresponding result for row-finite graphs.

Corollary 2.12. *Suppose E is a graph in which every loop has an exit, and that $\{S_e, P_v\}$ and $\{T_e, Q_v\}$ are two Cuntz-Krieger E-families in which all the projections P_v and Q_v are non-zero. Then there is an isomorphism $\phi : C^*(\{S_e, P_v\}) \to C^*(\{T_e, Q_v\})$ such that $\phi(S_e) = T_e$ for all $e \in E^1$ and $\phi(P_v) = Q_v$ for all $v \in E^0$.*

Proof. Let F be a desingularization of E. Use Lemma 2.10 to construct F-families from the given E-families. Then apply [**2**, Theorem 3.1] to get an isomorphism between the C^*-algebras generated by the F-families that will restrict to an isomorphism between $C^*(\{S_e, P_v\})$ and $C^*(\{T_e, Q_v\})$. □

Corollary 2.13. *Let E be a graph. Then $C^*(E)$ is an AF-algebra if and only if E has no loops.*

Proof. This follows from [**12**, Theorem 2.4] and the fact that the class of AF-algebras is closed under stable isomorphism (see [**6**, Theorem 9.4]). □

Corollary 2.14. *Let E be a graph. Then $C^*(E)$ is purely infinite if and only if every vertex in E connects to a loop and every loop in E has an exit.*

Proof. By [**2**, Proposition 5.3] and the fact that pure infiniteness is preserved by passing to corners, every vertex connects to a loop and every loop has an exit implies pure infiniteness. For the converse we note that the proof given in [**12**, Theorem 3.9] works for arbitrary graphs. □

The following result generalizes [**8**, Theorem 3] and [**9**, Corollary 4.5] and it was proven independently in [**18**] and [**14**].

Corollary 2.15. *Let E be a graph. Then $C^*(E)$ is simple if and only if*

(1) *every loop in E has an exit*;

(2) *E is cofinal*;

(3) *for every singular vertex $v_0 \in E^0$, $E^0 \geq v_0$.*

Proof. Letting F denote a desingularization of E, we have

$$\begin{aligned} C^*(E) \text{ is simple} &\iff C^*(F) \text{ is simple (by Theorem 2.11)} \\ &\iff F \text{ is cofinal and every loop in } F \text{ has an exit} \\ &\qquad\qquad\qquad\qquad \text{(by [\textbf{2}, Proposition 5.1])} \\ &\iff E \text{ satisfies (1),(2), and (3)} \\ &\qquad\qquad\qquad\qquad \text{(by Lemmas 2.7 and 2.8).} \qquad \square \end{aligned}$$

Remark 2.16. We see from the results above that the dichotomy described in [**2**, Remark 5.6] holds for arbitrary graphs: If $C^*(E)$ is simple, then it is either AF or purely infinite. For if E has no loops then Corollary 2.13 shows that $C^*(E)$ is AF. If E does have loops, then Corollary 2.15 says that they all have exits and that E is cofinal; thus, every vertex connects to every loop and Corollary 2.14 applies.

3. Ideal structure. Let E be a directed graph. A set $H \subseteq E^0$ is *hereditary* if whenever $v \in H$ and $v \geq w$, then $w \in H$. A hereditary set H is called *saturated* if every vertex that is not a singular vertex and that feeds only into H is itself in H, that is, if

$$v \text{ not singular and } \{r(e) \mid s(e) = v\} \subseteq H \text{ implies } v \in H.$$

If E is row-finite this definition reduces to the one given in [**2**]. It was shown in [**2**, Theorem 4.4] that if E is row-finite and satisfies Condition (K), then every saturated hereditary subset H of E^0 gives rise to exactly one ideal $I_H :=$ the ideal generated by $\{p_v \mid v \in H\}$ in $C^*(E)$. If E is a graph that is not row-finite, it is easy to check that with the above definition of saturated [**2**, Lemma 4.2] and [**2**, Lemma 4.3] still hold. Consequently, $H \mapsto I_H$ is still injective, just as in the proof of [**2**, Theorem 4.1]. However, it is no longer true that this map is surjective; that is, there may exist ideals in $C^*(E)$ that are not of the form I_H for some saturated hereditary set H. The reason the proof for row-finite graphs no longer works is that if I is an ideal, then $\{s_e + I, p_v + I\}$ will not necessarily be a Cuntz-Krieger $E \setminus H$-family for the graph $E \setminus H$ defined in [**2**, Theorem 4.1]. It turns out that to describe an arbitrary ideal in $C^*(E)$ we need a saturated hereditary subset and one other descriptor. Loosely speaking, this descriptor tells us how close $\{s_e + I, p_v + I\}$ is to being a Cuntz-Krieger $E \setminus H$-family.

Given a saturated hereditary subset $H \subseteq E^0$, define

$$B_H := \{v \in E^0 \mid v \text{ is an infinite-emitter and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

Therefore B_H is the set of infinite-emitters that point to only a finite number of vertices not in H. Since H is hereditary, B_H is disjoint from H. Now fix a saturated hereditary subset H and let S be any subset of B_H. Let $\{s_e, p_v\}$ be the canonical generating Cuntz-Krieger E-family and define

$$I_{(H,S)} := \text{the ideal in } C^*(E) \text{ generated by } \{p_v \mid v \in H\} \cup \{p^H_{v_0} \mid v_0 \in S\},$$

where

$$p^H_{v_0} := p_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e)\notin H}} s_e s_e^*.$$

Note that the definition of B_H ensures that the sum on the right is finite.

Our goal is to show that the correspondence $(H, S) \mapsto I_{(H,S)}$ is a lattice isomorphism, so we must describe the lattice structure on

$$\{(H, S) \mid H \text{ is a saturated hereditary subset of } E^0 \text{ and } S \subseteq B_H\}.$$

We say $(H, S) \leq (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$. With this definition, the reader who is willing to spend a few minutes can check using nothing more than basic set theory that the following equations define a greatest lower bound and least upper bound:

$$(H_1, S_1) \wedge (H_2, S_2) := \big((H_1 \cap H_2), (H_1 \cup H_2 \cup S_1 \cup S_2) \cap B_{H_1 \cap H_2}\big)$$
$$(H_1, S_1) \vee (H_2, S_2) := \left(\bigcup_{n=0}^{\infty} X_n \ , \ (S_1 \cup S_2) \cap B_{\bigcup_{n=0}^{\infty} X_n} \right)$$

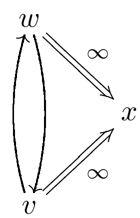
where X_n is defined recursively as $X_0 := H_1 \cup H_2$ and $X_{n+1} := X_n \cup \{v \in E^0 \mid 0 < |s^{-1}(v)| < \infty \text{ and } \{r(e) \mid s(e) = v\} \subseteq X_n\} \cup \{v \in E^0 \mid v \in S_1 \cup S_2 \text{ and } \{r(e) \mid s(e) = v\} \subseteq X_n\}$. The reason for this strange definition of the X_ns is the following: If Y_0 is a hereditary subset, then the saturation of Y_0 may be defined as the increasing union of $Y_{n+1} := Y_n \cup \{v \in E^0 \mid 0 < |s^{-1}(v)| < \infty \text{ and } \{r(e) \mid s(e) = v\} \subseteq Y_n\}$. In the X_ns above we need not only these elements, but also at each stage we must include the infinite emitters in $S_1 \cup S_2$ that only feed into X_n.

We now describe a correspondence between pairs (H, S) as above and saturated hereditary subsets of vertices in a desingularization of E. Suppose that E is a graph and let F be a desingularization of E. Also let H be a saturated hereditary subset of E^0 and let $S \subseteq B_H$. We define a saturated hereditary subset $H_S \subseteq F^0$. First set $\tilde{H} := H \cup \{v_n \in F^0 \mid v_n \text{ is on a tail added to a vertex in } H\}$. Now for each $v_0 \in S$ let N_{v_0} be the smallest nonnegative integer such that $r(e_j) \in H$ for all $j \geq N_{v_0}$. The number N_{v_0} exists since $v_0 \in B_H$ implies that there must be a vertex on the tail added to v_0 beyond which each vertex points only to the next vertex on the tail and into H. Define $T_{v_0} := \{v_n \mid v_n \text{ is on the infinite tail added to } v_0 \text{ and } n \geq N_{v_0}\}$ and define

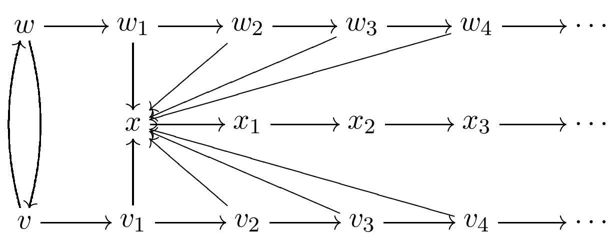
$$H_S := \tilde{H} \cup \textstyle\bigcup_{v_0 \in S} T_{v_0}.$$

Note that for $v_0 \in B_H$ we have $v_0 \notin H_S$. Furthermore, the tail attached to v_0 will eventually be inside H_S if and only if $v_0 \in S$. It is easy to check that H_S is hereditary, and choosing N_{v_0} to be minimal ensures that H_S is saturated.

Example 3.1. Suppose E is the following graph:



A desingularization F is given by



The only saturated hereditary (proper) subset in E is the set $H = \{x\}$. In this case $B_H = \{v, w\}$. There are four subsets of B_H and there are four saturated hereditary (proper) subsets in the desingularization. In particular, if $S = \varnothing$, then H_S consists of only the tail added to x; if S contains w, then H_S also includes $\{w_2, w_3, \dots\}$; and if S contains v, then H_S also includes $\{v_2, v_3, \dots\}$.

The proof of the following lemma is straightforward.

Lemma 3.2. *Let E be a graph and let F be a desingularization of E. The map $(H, S) \mapsto H_S$ is an isomorphism from the lattice*

$$\{(H, S) \mid H \text{ is a saturated hereditary subset of } E^0 \text{ and } S \subseteq B_H\}$$

onto the lattice of saturated hereditary subsets of F.

Suppose E is a graph that satisfies Condition (K) and F is a desingularization of E. Because $C^*(E)$ is isomorphic to the full corner $pC^*(F)p$, we have that $C^*(E)$ and $C^*(F)$ are Morita equivalent via the imprimitivity bimodule $pC^*(F)$. It then follows from [**16**, Proposition 3.24] that the Rieffel correspondence between ideals in $C^*(F)$ and ideals in $C^*(E)$ is given by the map $I \mapsto pIp$.

Proposition 3.3. *Let E be a graph satisfying Condition* (K)*, and let F be a desingularization of E. Let H be a saturated hereditary subset of E^0 and let $S \subseteq B_H$. If $\{t_e, q_v\}$ is a generating Cuntz-Krieger F-family and $p = \sum_{v \in E^0} q_v$, then $pI_{H_S}p = I_{(H,S)}$.*

Proof. That $pI_{H_S}p \subseteq I_{(H,S)}$ is immediate from (2.2). We show the reverse inclusion by showing that the generators of $I_{(H,S)}$ are in $pI_{H_S}p$. Letting $\{s_e, p_v\}$ denote the Cuntz-Krieger E-family defined in the proof of Lemma 2.9, the generators for $I_{(H,S)}$ are $\{p_v \mid v \in H\} \cup \{p^H_{v_0} \mid v_0 \in S\}$. Clearly for $v \in H$, we have $p_v = q_v = pq_vp \in pI_{H_S}p$, so all that remains to show is that for every $v_0 \in S$ we have $p^H_{v_0} \in pI_{H_S}p$.

Let $v_0 \in S$ and $n := N_{v_0}$. Then

$$\begin{aligned} q_{v_0} &= t_{e_1}t^*_{e_1} + t_{f_1}t^*_{f_1} \\ &= t_{e_1}q_{v_1}t^*_{e_1} + t_{f_1}t^*_{f_1} \\ &= t_{e_1}(t_{e_2}t^*_{e_2} + t_{f_2}t^*_{f_2})t^*_{e_1} + t_{f_1}t^*_{f_1} \\ &= t_{e_1e_2}q_{v_2}t^*_{e_1e_2} + t_{e_1f_2}t^*_{e_1f_2} + t_{f_1}t^*_{f_1} \\ &\quad\vdots \\ &= t_{e_1\cdots e_n}t^*_{e_1\cdots e_n} + \sum_{j=1}^{n} t_{\alpha^j}t^*_{\alpha^j} \end{aligned}$$

Now since $r(e_n) = v_n \in H_S$ we see that $q_{v_n} \in I_{H_S}$ and hence $t_{e_n} = t_{e_n}t^*_{e_n}t_{e_n} = t_{e_n}q_{v_n} \in I_{H_S}$. Consequently, $t_{e_1\cdots e_n}t^*_{e_1\cdots e_n} \in I_{H_S}$. Similarly, whenever $r(\alpha^j) \in H$, then $t_{\alpha^j}t^*_{\alpha^j} \in I_{H_S}$. Now, by definition, every α^j with $r(\alpha^j) \notin H$ has $j < n$. Therefore the above equation shows us that

$$\begin{aligned} p^H_{v_0} &= p_{v_0} - \sum_{\substack{s(g_j)=v_0 \\ r(g_j)\notin H}} s_{g_j} s^*_{g_j} \\ &= q_{v_0} - \sum_{\substack{s(\alpha^j)=v_0 \\ r(\alpha^j)\notin H}} t_{\alpha^j} t^*_{\alpha^j} \\ &= \sum_{\substack{r(\alpha^j)\in H \\ j<n}} t_{\alpha^j} t^*_{\alpha^j} + t_{e_1\cdots e_n} t^*_{e_1\cdots e_n} \end{aligned}$$

which is an element of I_{H_S} by the previous paragraph. Hence $I_{H_S} \subseteq I_{H,S}$. □

Corollary 3.4. *Let E be a graph satisfying Condition* (K)*, and let F be a desingularization of E. If H is a saturated hereditary subset of E^0 and $S \subseteq B_H$, then $I_{(H,S)}$ is a primitive ideal in $C^*(E)$ if and only if I_{H_S} is a primitive ideal in $C^*(F)$.*

We now have the following:

$$\begin{tikzcd} \begin{array}{c}\{(H,S) \mid H \text{ saturated, hereditary} \\ \text{in } E \text{ and } S \subseteq B_H\}\end{array} \arrow[r, dotted] \arrow[d] & \text{ideals in } C^*(E) \\ \begin{array}{c}\text{saturated, hereditary} \\ \text{subsets of } F\end{array} \arrow[r] & \text{ideals in } C^*(F). \arrow[u] \end{tikzcd}$$

The map on the left is $(H,S) \mapsto H_S$, which is a lattice isomorphism by Lemma 3.2. The lattice isomorphism $H \mapsto I_H$ across the bottom comes from [**2**, Theorem 4.4]. The map on the right is $I_{H_S} \mapsto I_{(H,S)}$ and is an isomorphism because it agrees with the Rieffel correspondence, Proposition 3.3. Composing the three yields the following:

Theorem 3.5. *Let E be a graph that satisfies Condition* (K)*. Then the map $(H,S) \mapsto I_{(H,S)}$ is a lattice isomorphism from the lattice*

$$\{(H,S) \mid H \textit{ is a saturated hereditary subset of } E^0 \textit{ and } S \subseteq B_H\}$$

onto the lattice of ideals in $C^*(E)$.

4. Primitive ideal space. The following definition generalizes that in [**2**, Proposition 6.1].

Definition 4.1. Let E be a graph. A nonempty subset $\gamma \subseteq E^0$ is called a *maximal tail* if it satisfies the following conditions:

(a) for every $w_1, w_2 \in \gamma$ there exists $z \in \gamma$ such that $w_1 \geq z$ and $w_2 \geq z$;

(b) for every $v \in \gamma$ that is not a singular vertex, there exists an edge e with $s(e) = v$ and $r(e) \in \gamma$;

(c) $v \geq w$ and $w \in \gamma$ imply $v \in \gamma$.

Given a graph E we denote by Λ_E the set of all maximal tails in E. Note that if v_0 is a sink, then the set $\lambda_{v_0} := \{v \in E^0 \mid v \geq v_0\}$ is a maximal tail according to Definition 4.1 but was not considered to be a maximal tail in [**2**, Section 6]. In addition, when v_0 is an infinite-emitter $\lambda_{v_0} := \{v \in E^0 \mid v \geq v_0\}$ is a maximal tail.

Definition 4.2. If E is a graph, then a *breaking vertex* is an element $v \in E^0$ such that $|s^{-1}(v)| = \infty$ and $0 < |\{e \in E^1 \mid s(e) = v \text{ and } r(e) \geq v\}| < \infty$. We denote the set of breaking vertices of E by $BV(E)$.

Remark 4.3. Notice that if H is a hereditary subset in a graph E and $v_0 \in B_H$, then v_0 is a breaking vertex if and only if there exists an edge $e \in E^1$ with $s(e) = v_0$ and $r(e) \geq v_0$. Also note that if H is a saturated hereditary subset in a graph E and $E^0 \setminus H = \lambda_{v_0}$ for some singular vertex v_0, then $v_0 \in B_H$ if and only if v_0 is a breaking vertex.

We let $\Xi_E := \Lambda_E \cup BV(E)$ denote the disjoint union of the maximal tails and the breaking vertices. We shall see that the elements of Ξ_E correspond to the primitive ideals in $C^*(E)$.

Lemma 4.4. *If E is a graph and γ is a maximal tail in E, then $\gamma = \{v \in E^0 \mid v \geq \alpha\}$ for some $\alpha \in E^\infty \cup \{\alpha \in E^* \mid r(\alpha) \text{ is a singular vertex}\}$.*

Proof. It is straightforward to see that if $\alpha \in E^\infty \cup \{\alpha \in E^* \mid r(\alpha) \text{ is a singular vertex}\}$, then $\{v \in E^0 \mid v \geq \alpha\}$ is a maximal tail [**2**, Remark 6.4].

Conversely, suppose that γ is a maximal tail. We shall create a path in E inductively. Begin with an element $w \in \gamma$. If there exists an element $w' \in \gamma$ for which $w' \ngeq w$, then we may use property (a) of maximal tails to choose a path β^1 with $s(\beta^1) = w$ and $w' \geq r(\beta^1)$. Now having chosen β^i, we do one of two things: if $w' \geq r(\beta^i)$ for all $w' \in \gamma$, we stop. If there exists $w' \in \gamma$ such that $w' \ngeq r(\beta^i)$, then we choose a path β^{i+1} with $s(\beta^{i+1}) = r(\beta^i)$ and $w' \geq r(\beta^{i+1})$. We then continue in this manner to produce a path $\beta := \beta^1\beta^2\cdots$, which may be either finite or infinite. Note that since γ has either a finite or countable number of elements, we may choose β in such a way that $w \geq \beta$ for all $w \in \gamma$.

Now if β is an infinite path we define $\alpha := \beta$. On the other hand, if β is a finite path then one of two things must occur. Either $r(\beta)$ is a singular vertex or there is an edge $e_1 \in E^1$ with $s(e_1) = r(\beta)$ and $r(e_1) \in \gamma$. Continuing in this way, we see that having chosen e_i, either $r(e)$ is a singular vertex or there exists $e_{i+1} \in E^1$ with $s(e_{i+1}) = r(e_i)$ and $r(e_{i+1}) \in \gamma$. Using this process we may extend β to a path $\alpha := \beta e_1 e_2 \cdots$ that is either infinite or is finite and ends at a singular vertex.

Now since every vertex on α is an element of γ we certainly have $\{v \in F^0 \mid v \geq \alpha\} \subseteq \gamma$. Also, for every element $v \in \gamma$ there exists an i such that $v \geq r(\beta^i) \geq \alpha$ so we have $\gamma \subseteq \{v \in F^0 \mid v \geq \alpha\}$. □

Theorem 4.5. *Let E be a graph. An ideal I in $C^*(E)$ is a primitive ideal if and only if one of the following two statements holds*:

1. $I = I_{(H,S)}$, *where $E^0 \setminus H$ is a maximal tail and $S = B_H$; or*

2. $I = I_{(H,S)}$, *where $E^0 \setminus H = \lambda_{v_0}$ for some breaking vertex v_0 and $S = B_H \setminus \{v_0\}$.*

Proof. It follows from Theorem 3.5 that any ideal in $C^*(E)$ has the form $I_{(H,S)}$ for some saturated hereditary set $H \subseteq E^0$ and some $S \subseteq B_H$. Let F be a desingularization of E. It follows from Corollary 3.4 that $I_{(H,S)}$ is primitive if and only if I_{H_S} is primitive.

Now suppose that $I_{(H,S)}$, and hence I_{H_S}, is primitive. It follows from [**2**, Proposition 6.1] that $F^0 \setminus H_S$ is a maximal tail in F. Thus by Lemma 4.4 we have $F^0 = \{w \in F^0 \,|\, w \geq \alpha\}$ for some $\alpha \in F^\infty$. Now $\phi_\infty^{-1}(\alpha)$ is either an infinite path in E or a finite path in E ending at a singular vertex. In either case $\gamma := \{w \in E^0 \,|\, w \geq \phi_\infty^{-1}(\alpha)\}$ is a maximal tail in E. Furthermore,

$$\begin{aligned} v \in E^0 \setminus H \iff v \notin H \iff v \notin H_S \iff v \geq \alpha \text{ in } F \\ \iff v \geq \phi_\infty^{-1}(\alpha) \text{ in } E \iff v \in \gamma. \end{aligned}$$

Therefore $E^0 \setminus H = \gamma$ is a maximal tail.

Now if $S = B_H$, then we are in the case described in part (1) of the theorem and the claim holds. Let us therefore suppose that there exists $v_0 \in B_H \setminus S$. If we define $T_{v_0} := \{v_0, v_1, v_2, \dots\}$ to be the vertices on the tail added to v_0, then we see that $v_0 \notin S$ implies that $T_{v_0} \subseteq F^0 \setminus H_S = \{w \in F^0 \,|\, w \geq \alpha\}$. Now for each vertex v_i with $i \geq N_{v_0}$ there are two edges, e_{i+1} and f_{i+1}, with source v_i. Since $r(f_{i+1}) \in H_S$ and $r(e_{i+1}) = v_{i+1}$, it must be the case that α has the form $\alpha = \alpha' e_1 e_2 e_3 \cdots$ for some finite path α' in F. Consequently, $\phi_\infty^{-1}(\alpha)$ is a finite path in E ending at v_0, and $\gamma = \lambda_{v_0}$. Now let $X := \{e \in E^1 \,|\, s(e) = v_0 \text{ and } r(e) \geq v_0\}$. Note that if $s(e) = v_0$ and $r(e) \geq v_0$, then $r(e) \notin H$ since H is hereditary. Because $v_0 \in B_H$ it follows that we must have $|X| < \infty$. Furthermore, since $v_0 \in B_H$ there exists $e \in E^1$ with $s(e) = v_0$ and $r(e) \notin H$. But then $r(e) \in \gamma$ and $r(e) \geq \phi_\infty^{-1}(\alpha)$ and hence $r(e) \geq v_0$. Thus $|X| > 0$, and by definition v_0 is a breaking vertex. All that remains is to show that $S = B_H \setminus \{v_0\}$. Let us suppose that $w_0 \in B_H$. If $w_0 \notin S$, then $T_{w_0} \subseteq F^0 \setminus H_S = \{w \in F^0 \,|\, w \geq \alpha\}$. But because the w_is for $i \geq N_{w_0}$ can only reach elements of H and T_{w_0}, the only way to have $w_i \geq \alpha = \alpha' e_1 e_2 \cdots$ for all i is if we have $w_0 = v_0$. Hence v_0 is the only element of $B_H \setminus S$ and $S = B_H \setminus \{v_0\}$. Thus we have established all of the claims in part (2).

For the converse let $E^0 \setminus H$ be a maximal tail. Consider the following two cases.

Case I. $S = B_H$. We shall show that $F^0 \setminus H_S$ is a maximal tail in F. Since H_S is a saturated hereditary subset of F^0, the set $F^0 \setminus H_S$ certainly satisfies (b) and (c) in the definition of maximal tail. We shall prove that (a) also holds. Let $w_1, w_2 \in F^0 \setminus H_S$. If it is the case that $w_1, w_2 \in E^0$, then we must also have $w_1, w_2 \in E^0 \setminus H$, and hence there exists $z \in E^0 \setminus H$ such that $w_1 \geq z$ and $w_2 \geq z$ in E. But then $z \in F^0 \setminus H_S$ and $w_1 \geq z$ and $w_2 \geq z$ in F.

On the other hand, if one of the w_is is not in E^0, then it must be on an infinite tail T_{v_0}. Because $w_i \notin H_S$ and $S = B_H$, we must have $w_i \geq z$ for some $z \in E^0 \setminus H$. Thus we can replace w_i with z and reduce to the case when $w_i \in E^0$.

Hence $F^0 \setminus H_S$ also satisfies (a) and is a maximal tail. Consequently, I_{H_S} is a primitive ideal by [**2**, Proposition 6.1], and $I_{(H,S)}$ is a primitive ideal by Corollary 3.4.

Case II. $E^0 \setminus H = \lambda_{v_0}$ for some breaking vertex v_0 and $S = B_H \setminus \{v_0\}$. As in Case I, it suffices to show that $F^0 \setminus H_S$ satisfies (a) in the definition of maximal tail. To see this, let $w \in F^0 \setminus H_S$. If $w \in E^0$, then we must have $w \in E^0 \setminus H = \lambda_{v_0}$ and $w \geq v_0$. If $w \notin E^0$, then w must be on one of the added tails in F. Since $S = B_H \setminus \{v_0\}$ we must have that w is an element on $T_{v_0} = \{v_0, v_1, v_2, \dots\}$. In either case we see that w can reach an element of T_{v_0} in F. Consequently, $F^0 \setminus H_S \geq T_{v_0}$ and $F^0 \setminus H_S$ clearly satisfies (a). □

Definition 4.6. Let E be a graph that satisfies Condition (K). We define a map $\phi_E : \Xi_E \to \operatorname{Prim} C^*(E)$ as follows. For $\gamma \in \Lambda_E$ let $H(\gamma) := E^0 \setminus \gamma$ and define $\phi_E(\gamma) := I_{(H(\gamma), B_{H(\gamma)})}$. For $v_0 \in BV(E)$ we define $\phi_E(v_0) := I_{(H(\lambda_{v_0}), B_{H(\lambda_{v_0})} \setminus \{v_0\})}$. The previous theorem shows that ϕ_E is a bijection.

We now wish to define a topology on Ξ_E that will make ϕ_E a homeomorphism. As usual our strategy will be to translate the problem to a desingularized graph and make use of the corresponding results in [**2**]. In particular, if E is any graph and F is a desingularization of E, then we have the following picture:

$$\begin{tikzcd}
\Xi_E \arrow[r, dotted, "h"] \arrow[d, "\phi_E"'] & \Xi_F \arrow[d, "\phi_F"] \\
\operatorname{Prim} C^*(E) \arrow[r, "\psi"] & \operatorname{Prim} C^*(F),
\end{tikzcd}$$

where ψ is the Rieffel correspondence restricted to the primitive ideal space. If we use the topology on $\Xi_F = \Lambda_F$ defined in [**2**, Theorem 6.3], then ϕ_F is a homeomorphism. To define a topology on Ξ_E that makes ϕ_E a homeomorphism we will simply use the composition $h := \phi_F^{-1} \circ \psi \circ \phi_E$ to pull the topology on Ξ_F back to a topology on Ξ_E. We start with a proposition that describes the map h.

Proposition 4.7. *Let E be a graph satisfying Condition* (K) *and let F be a desingularization of E.*

1. *If $\alpha \in E^\infty \cup \{\alpha \in E^* \mid r(\alpha)$ is a singular vertex$\}$ and $\gamma = \{v \in E^0 \mid v \geq \alpha\} \in \Lambda_E$, then $h(\gamma) = \{v \in F^0 \mid v \geq \phi_\infty(\alpha)\}$.*

2. *If v_0 is a breaking vertex, then $h(v_0) = \{v \in F^0 \mid v \geq e_1 e_2 \cdots\}$, where $e_1 e_2 \cdots$ is the path on the tail added to v_0.*

Proof. To prove part (1), let $H := E^0 \setminus \gamma$ and $S := B_H$. Then using Proposition 3.3 we have $h(\gamma) = \phi_F^{-1} \circ \psi \circ \phi_E(\gamma) = \phi_F^{-1} \circ \psi(I_{(H,S)}) = \phi_F^{-1}(I_{H_S}) = F^0 \setminus H_S$. We shall show that $F^0 \setminus H_S = \{v \in F^0 \mid v \geq \phi_\infty(\alpha)\}$. To begin, if $v \in E^0$ then

$$\begin{aligned} v \in F^0 \setminus H_S &\iff v \in E^0 \setminus H \iff v \in \gamma \\ &\iff v \geq \alpha \text{ in } E \iff v \geq \phi_\infty(\alpha) \text{ in } F. \end{aligned}$$

where the last step follows from Lemma 2.6. On the other hand, suppose $v \in F^0 \setminus E^0$. Then since $S = B_H$ every vertex $v \in F^0 \setminus H_S$ must connect to some vertex $w \in E^0 \setminus H$. So we may replace v with w and repeat the above argument. Thus we have proven (1).

For part (2), let v_0 be a breaking vertex and set $\lambda_{v_0} := \{w \in E^0 \mid w \geq v_0\}$ and $S := B_{(E^0 \setminus \lambda_{v_0})} \setminus \{v_0\}$. Then $h(v_0) = \phi_F^{-1} \circ \psi \circ \phi_E(v_0) = \phi_F^{-1} \circ \psi(I_{(H,S)}) = \phi_F^{-1}(I_{H_S}) = F^0 \setminus H_S$. An argument similar to the one above shows that $F^0 \setminus H_S = \{v \in F^0 \mid v \geq e_1 e_2 \cdots\}$. □

Definition 4.8. Let E be a graph and let $S \subseteq E^0$. If γ is a maximal tail, then we write $\gamma \to S$ if $\gamma \geq S$. If v_0 is a breaking vertex in E, then we write $v_0 \to S$ if the set $\{e \in E^0 \mid s(e) = v_0, r(e) \geq S\}$ contains infinitely many elements.

Lemma 4.9. *Let $\delta \in \Xi_E$ and let $P \subseteq \Xi_E$. Then $\delta \to \bigcup_{\lambda\in P} \lambda$ in E if and only if $h(\delta) \geq \bigcup_{\lambda\in P} h(\lambda)$ in F.*

Proof. If δ is a maximal tail, then from Lemma 4.4 we have $\delta = \{v \in E^0 \,|\, v \geq \alpha\}$ for some $\alpha \in E^\infty \cup \{\alpha \in E^* \,|\, r(\alpha) \text{ is a singular vertex}\}$. Similarly, for each $\lambda \in P \cap \Lambda_E$ we may write $\lambda = \{v \in E^0 \,|\, v \geq \alpha^\lambda\}$ for some $\alpha^\lambda \in E^\infty \cup \{\alpha \in E^* \,|\, r(\alpha) \text{ is a singular vertex}\}$. Now

$$\begin{aligned}
&\delta \longrightarrow \bigcup_{\lambda\in P} \lambda \\
&\Longleftrightarrow \alpha \geq \bigcup_{\lambda\in P\cap\Lambda_E} \{r(\alpha_i^\lambda)\}_{i=1}^{|\alpha^\lambda|} \cup \bigcup_{v_0\in P\cap BV(E)} v_0 \\
&\Longleftrightarrow \phi_\infty(\alpha) \geq \bigcup_{\lambda\in P\cap\Lambda_E} \{r(\phi_\infty(\alpha^\lambda)_i)\}_{i=1}^{|\alpha^\lambda|} \cup \bigcup_{v_0\in P\cap BV(E)} \phi_\infty(v_0) \\
&\Longleftrightarrow \{v \in F^0 \,|\, v \geq \phi_\infty(\alpha)\} \\
&\qquad \geq \bigcup_{\lambda\in P\cap\Lambda_E} \{v \in F^0 \,|\, v \geq \phi_\infty(\alpha_i^\lambda)\} \\
&\qquad\qquad \cup \bigcup_{v_0\in P\cap BV(E)} \{v \in F^0 \,|\, v \geq e_1^{v_0} e_2^{v_0} \cdots\} \\
&\Longleftrightarrow h(\delta) \geq \bigcup_{\lambda\in P} h(\lambda)
\end{aligned}$$

So the claim holds when δ is a maximal tail.

Now let us consider the case when $\delta = v_0$ is a breaking vertex. It follows from Lemma 4.7 that $h(v_0) = \{v \in F^0 \,|\, v \geq e_1e_2\cdots\}$, where $e_1e_2\cdots$ is the path on the tail added to v_0. Now suppose that $v_0 \to \cup_{\lambda\in P}\lambda$. Fix $v \in h(\delta)$. Note that either $v \geq v_0$ in F or v is on the infinite tail added to v_0 in F. Because $v_0 \to \cup_{\lambda\in P}\lambda$, there are infinitely many edges in E from v_0 to vertices that connect to $\cup_{\lambda\in P}\lambda$. Thus no matter how far out on the tail v happens to be, there must be an edge in F whose source is a vertex further out on the tail than v and whose range is a vertex that connects to a vertex $w \in \lambda$ for some $\lambda \in P$. Since $w \in \lambda$ we must have $w \in h(\lambda)$ and thus $v \geq \cup_{\lambda\in P}h(\lambda)$.

Now assume that $h(v_0) \geq \cup_{\lambda\in P}h(\lambda)$. Then every vertex on the infinite tail attached to v_0 connects to a vertex in $\cup_{\lambda\in P}h(\lambda)$. In fact it is true that every vertex on the infinite tail attached to v_0 connects to a vertex in $\cup_{\lambda \in P} h(\lambda) \cap E^0$, which implies that every vertex on the infinite tail connects to a vertex in $\cup_{\lambda \in P} \lambda$. But this implies that there must be infinitely many edges from v_0 to vertices that connect to $\cup_{\lambda \in P} \lambda$. Thus $v_0 \to \cup_{\lambda \in P} \lambda$. $\square$

Theorem 4.10. *Let E be a graph satisfying Condition* (K)*. Then there is a topology on Ξ_E such that for $S \subseteq \Xi_E$,*

$$\overline{S} := \Big\{ \delta \in \Xi_E \mid \delta \to \bigcup_{\lambda \in S} \lambda \Big\},$$

and the map ϕ_E given in Definition 4.6 *is a homeomorphism from Ξ_E onto* $\operatorname{Prim} C^*(E)$*.*

Proof. Since h is a bijection, we may use h to pull the topology defined on $\Xi_F = \Lambda_F$ in [**2**, Theorem 6.3] back to a topology on Ξ_E. Specifically, if $S \subseteq \Xi_E$ then $S = h^{-1}(P)$ for some $P \subseteq \Xi_F$, and we define $\overline{S} := h^{-1}(\overline{P})$. But from Lemma 4.9 we see that this is equivalent to defining $\overline{S} = \{\delta \in \Xi_E \mid \delta \to \bigcup_{\lambda \in S} \lambda\}$. Now with this topology h, and consequently ϕ_E, is a homeomorphism. $\square$

5. Concluding remarks. When we defined a desingularization of a graph in Section 2, for each singular vertex v_0 we chose an ordering of the edges $s^{-1}(v_0)$ and then redistributed these edges along the added tail in such a way that every vertex on the tail was the source of exactly one of these edges. Another way we could have defined a desingularization would be to instead redistribute a finite number of edges to each vertex on the added tail. Thus if v_0 is a singular vertex, we could choose a partition of $s^{-1}(v_0)$ into a countable collection $S_0^{v_0}, S_1^{v_0}, S_2^{v_0}, \dots$ of finite (or empty) disjoint sets. Having done this, we add a tail to E by first adding a graph of the form

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots$$

We then remove the edges in $s^{-1}(v_0)$ and for each i and each $g \in S_i^{v_0}$ we draw an edge from v_i to $r(g)$. More formally, if the elements of $S_i^{v_0}$ are listed as $\{g_i^1, g_i^2, \dots, g_i^{m_i}\}$ we define $F^0 := E^0 \cup \{v_i\}_{i=1}^{\infty}$, $F^1 := (E^1 \setminus s^{-1}(v_0)) \cup \{e_i\}_{i=1}^{\infty} \cup \{f_i^j \mid 1 \leq i \leq \infty \text{ and } 1 \leq j \leq m_i\}$, and extend r and s by $s(e_i) = v_{i-1}$, $r(e_i) = v_i$, $s(f_i^j) = v_i$, and $r(f_i^j) = r(g_i^j)$.

If we add tails in this manner, then we can define a desingularization of E to be the graph F formed by adding a tail to each singular vertex in E. Here a choice of partition $S_0^{v_0}, S_1^{v_0}, S_2^{v_0}, \ldots$ must be made for each singular vertex, and different choices will sometimes produce nonisomorphic graphs.

With this slightly more general definition of desingularization, all of the results of this paper still hold and the proofs of those results remain essentially the same. We avoided using this broader definition only because the partitioning and the use of double subscripts in the f_i^js creates very cumbersome notation, and we were afraid that this would obscure the main points of this article. However, we conclude by mentioning this more general method of desingularization because we believe that in practice there may be situations in which it is convenient to use. For example, if H is a saturated hereditary subset of E^0, then for each $v_0 \in B_H$ one may wish to choose a partition of $s^{-1}(v)$ with $S_0^{v_0} := \{e \in E^1 \mid s(e) = v_0 \text{ and } r(e) \notin H\}$. Then a desingularization created using this partition will have the property that every vertex on a tail added to v_0 will point only to the next vertex on the tail and elements of H.

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