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ON TRANSFORMATION LAWS FOR THETA FUNCTIONS

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ABSTRACT. We determine transformation laws for theta functions of higher degree.

1. Introduction. Siegel [**8**–**10**], for example, proves transformation laws for theta functions (depending on a single complex variable) attached to quadratic forms. If the quadratic form is indefinite, then Siegel's definition of the theta function also depends on a majorant of the quadratic form, an idea that Siegel credits to Hermite. Siegel's results have been generalized and transformation laws for theta functions of higher degree have been established. Andrianov and Maloletkin [**1**] and [**2**] use Eichler's "embedding trick" to determine transformation properties of theta series, depending on one complex $n \times n$ matrix variable, corresponding to positive definite and also indefinite quadratic forms: they use Eichler's method, see [**5**], for example, of recognizing such theta series as specializations of symplectic theta functions. Ziegler [**14**] develops a theory of holomorphic Jacobi forms of higher degree and shows that theta functions (depending on two complex matrix variables) attached to positive definite quadratic forms are examples of such forms.

The purpose of this paper is to show that Eichler's "embedding trick" can also be applied to generalize Ziegler's result. We will define $\Theta_{F,H,\zeta}(Z,W)$, a theta function of higher degree, depending on a complex $n \times n$ matrix variable Z and a complex $j \times n$ matrix variable W, attached to an indefinite quadratic form, and we will determine the behavior of $\Theta_{F,H,\zeta}(Z,W)$ under modular transformations by proceeding as in Andrianov and Maloletkin [**1**] and [**2**], see also [**7**]. Friedberg [**6**] defines a modified version of the usual symplectic theta function, $\vartheta(Z, \binom{u}{v}, w, f)$, and he proves a transformation formula for his function. We state that transformation formula in a slightly more general way and show that certain coefficients of $\Theta_{F,H,\zeta}(Z,W)$ can be regarded

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as specializations of $\vartheta(Z, \binom{u}{v}, w, f)$. As an immediate consequence, we obtain the transformation law of $\Theta_{F,H,\zeta}(Z, W)$ under modular transformations. In particular, if the signature of the quadratic form is $(j, m-j)$, then $\Theta_{F,H,\zeta}(Z, W)$ is an example of a skew-holomorphic Jacobi form in the sense of Skoruppa [**11**] if $n = j = 1$ and in the sense of Arakawa [**3**] if $n > 1$.

2. Notation and statement of the results. Let $\mathbf{A}$ be a commutative ring with unity and $M_{m,n}(\mathbf{A})$ be the set of $m \times n$ matrices with entries in $\mathbf{A}$. For any matrices $U, V \in M_{m,n}(\mathbf{A})$, set $U[V] = {}^tVUV$ where ${}^tV$ is the transpose of V. If $U \in M_{n,n}(\mathbf{A})$, let $\sigma(U)$ be the trace of U and $\det(U)$ the determinant of U.

The symplectic group,

$$\mathrm{Sp}_n(\mathbf{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid M \in M_{2n,2n}(\mathbf{R}) \text{ such that } J[M] = J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \right\}$$

where I_n is the $n \times n$ identity matrix, acts on the Siegel upper half plane

$$\mathfrak{H}^{(n)} = \left\{ Z \in M_{n,n}(\mathbf{C}) \mid Z = {}^tZ \text{ and } \mathrm{Im}\,(Z) > 0 \right\}.$$

The action of M on Z is given by

$$M \circ Z = (AZ + B)(CZ + D)^{-1}.$$

Furthermore, we set

$$\Gamma^{(n)} = \mathrm{Sp}_n(\mathbf{Z})$$

and

$$\Gamma_0^{(n)}(q) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv 0 \bmod q \right\}.$$

Let F be a symmetric, integral, invertible $m \times m$ matrix with even diagonal entries and let q be the level of F, i.e., qF^{-1} is integral and qF^{-1} has even diagonal entries. Suppose that F is of type (k, l) and let H be a majorant of F, i.e., $HF^{-1}H = F$ and ${}^tH = H > 0$. For fixed $\zeta \in M_{m,j}(\mathbf{Z})$ and for variables $Z \in \mathfrak{H}^{(n)}$ and $W \in M_{j,n}(\mathbf{C})$, we define the theta series

(1)

$$\Theta_{F,H,\zeta}(Z,W) = \sum_{N \in M_{m,n}(\mathbf{Z})} \exp\Big\{\pi i\,\sigma\Big(F[N]\,\mathrm{Re}\,(Z) + iH[N]\,\mathrm{Im}\,(Z) + 2{}^tNF\zeta W\Big)\Big\}.$$

Our main result is the following:

Theorem 1. *Let $\zeta \in M_{m,j}(\mathbf{Z})$ be such that $F\zeta = H\zeta$. For all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(q)$ and $\lambda, \mu \in M_{j,n}(\mathbf{Z})$, we have*

(2)

$$\Theta_{F,H,\zeta}\left(M \circ Z, W(CZ+D)^{-1}\right) = \phi(M,Z)\,\exp\Big\{\pi i\,\sigma\Big(F[\zeta]\left(W(CZ+D)^{-1}C{}^tW\right)\Big)\Big\}\,\Theta_{F,H,\zeta}\left(Z,W\right),$$

and

$$\Theta_{F,H,\zeta}\left(Z, W+\lambda Z+\mu\right) = \exp\Big\{-\pi i\,\sigma\Big(F[\zeta]\left(\lambda Z{}^t\lambda + 2\lambda{}^tW\right)\Big)\Big\}\,\Theta_{F,H,\zeta}\left(Z,W\right), \tag{3}$$

where

$$\phi(M,Z) = \chi_F(M)\det\left(CZ+D\right)^{k/2}\det\left(C\overline{Z}+D\right)^{l/2}, \tag{4}$$

where $\chi_F(M)$ is an eighth root of unity. More precisely, choose T integral and symmetric such that for $D^* = CT+D$, $\det D^* = \pm p$ for an odd prime p. Then

$$\begin{aligned}\phi(M,Z) = {}& \varepsilon_p^{-m}\left(\frac{2^m c^m \det F}{p}\right)\exp\left\{\pi i\,\frac{(k-l)s}{4}\right\} \\ & \times |\det(C)|^{m/2}\left\{\det\left[-i\,C^{-1}\left(CZ+D\right)\right]\right\}^{k/2} \\ & \times \left\{\det\left[i\,C^{-1}\left(C\overline{Z}+D\right)\right]\right\}^{l/2},\end{aligned}$$

where $\varepsilon_p = 1$ for $p \equiv 1 \bmod 4$, $\varepsilon_p = i$ for $p \equiv 3 \bmod 4$, $(\cdot/p)$ is the Legendre symbol, c is any diagonal element of $\left(pD^{*-1}C\right)$ with $(c,p) = 1$, *and s is the signature of $D^{*-1}C$. If C is singular, then C^{-1} is interpreted as ${}^tD\,(C^tD)^{-1}$, where $(C^tD)^{-1}$ is the Moore-Penrose generalized inverse, see Ben-Israel and Greville* [**4**]*, and the determinants are interpreted as the product of the nonzero eigenvalues. Furthermore, $|\det(C)|^{1/2}$ is positive, and $\left\{\det\left[-i\,C^{-1}\,(CZ+D)\right]\right\}^{1/2}$ and $\left\{\det\left[i\,C^{-1}\left(C\overline{Z}+D\right)\right]\right\}^{1/2}$ are given by analytic continuation from the principal value when $Z=-C^{-1}D+iY$.*

Remarks. a) Ziegler [**14**] proves the special case of Theorem 1 where F is positive definite, i.e., $F=H$ and unimodular. In this case, (4) reduces to $\phi(M,Z)=\det(CZ+D)^{m/2}$.

b) Andrianov and Maloletkin [**2**] investigate $\Theta_{F,H,\zeta}\,(Z,0)$, a function of one complex variable matrix. For that special case, they prove (2), but they determine $\phi(M,Z)$ only when m is even. In [**7**], we show how results by Stark [**12**] and Styer [**13**] can be used to compute $\phi(M,Z)$ explicitly, even when m is odd.

c) Arakawa [**3**] establishes Theorem 1 in the special case where F is unimodular and where $k=j$ and $l=m-j$. Then (4) reduces to $\phi(M,Z)=\det(CZ+D)^{j/2}\det(C\bar{Z}+D)^{(m-j)/2}$ and $\Theta_{F,H,\zeta}\,(Z,W)$ is a skew-holomorphic Jacobi form of weight $m/2$ and index $F[\zeta]/2$. Our proof of Theorem 1 gives a different demonstration of Arakawa's result.

3. Symplectic theta functions. The theta subgroup,

$$\Gamma_\vartheta^{(n)}=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma^{(n)}\mid A^tB,\ C^tD \text{ have even diagonal entries}\right\},$$

acts on the symplectic theta function,

$$\vartheta\left(Z,\begin{pmatrix}u\\ v\end{pmatrix}\right)=\sum_{m\in\mathbf{Z}^n}\exp\left\{\pi i\left(Z[m+v]-2{}^tmu-{}^tvu\right)\right\}, \tag{5}$$

where u and v are column vectors in $\mathbf{C}^n$. It is well known, see Eichler [**5**], for example, that for $M=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ in $\Gamma_\vartheta^{(n)}$,

$$\vartheta\left(M\circ Z, M\begin{pmatrix}u\\ v\end{pmatrix}\right)=\chi(M)\left[\det(CZ+D)\right]^{1/2}\vartheta\left(Z,\begin{pmatrix}u\\ v\end{pmatrix}\right) \tag{6}$$

where $\chi(M)$ is an eighth root of unity which depends upon the chosen square root of $\det(CZ+D)$, but which is otherwise independent of Z, u and v. Stark [**12**] determines $\chi(M)$ in the important special case that both C and D are nonsingular and that pD^{-1} is integral for some odd prime p. Styer [**13**] extends Stark's results and includes the case where C is singular.

We follow Friedberg [**6**] and modify the symplectic theta function (5) somewhat. For $w \in \mathbf{C}^n$, f a nonnegative integer, and Z, u and v as above, we define

$$\begin{aligned}\vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix}, w, f\right) \\ = \sum_{m\in\mathbf{Z}^n} \left({}^t w(m+v)\right)^f \exp\left\{\pi i \left(Z[m+v] - 2{}^t m u - {}^t v u\right)\right\}.\end{aligned} \tag{7}$$

Note that, for $f=0$, the theta functions in (5) and (7) coincide.

Theorem 2. *Let* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\vartheta^{(n)}$. *Then*

$$\begin{aligned}\vartheta\left(M\circ Z, M\begin{pmatrix} u \\ v \end{pmatrix},\ {}^t(CZ+D)^{-1}w, f\right) \\ = \chi(M)\left[\det(CZ+D)\right]^{1/2} \sum_{r=0}^{[f/2]} \frac{f!}{r!(f-2r)!} (\pi i)^{-r} 2^{-2r} \\ \times \left(\left((CZ+D)^{-1}C\right)[w]\right)^r \vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix}, w, f-2r\right),\end{aligned} \tag{8}$$

where $\chi(M)$ *is as in* (6).

Remark. Friedberg [**6**] considers $\vartheta\left(Z, \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), Zw, f\right)$. He proves Theorem 2, phrased slightly differently, in the special case where $\left((CZ+D)^{-1}C\right)[w] = 0$, in which case the righthand side in (8) reduces to $\chi(M)\left[\det(CZ+D)\right]^{1/2} \vartheta\left(Z, \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), w, f\right)$.

Proof. Let $w \in \mathbf{C}^n$ and replace v by $v + \xi Z^{-1} w$ in (6) and multiply both sides by $\exp\{-\xi^2 Z^{-1}[w]\}$. After some computation, see also

Friedberg [**6**], this yields

(9)

$$\sum_{m\in\mathbf{Z}^n} \exp\Big\{\pi i \Big((M\circ Z)[m+Cu+Dv] - 2\,{}^t m(Au+Bv) - {}^t(Cu+Dv)(Au+Bv) + 2\xi\,{}^t w(CZ+D)^{-1}(m+Cu+Dv)\Big)\Big\}$$
$$= \chi(M)\,[\det(CZ+D)]^{1/2} \sum_{m\in\mathbf{Z}^n} \exp\Big\{\pi i\Big(Z[m+v] - 2\,{}^t mu - {}^t vu + 2\xi\,{}^t w(m+v) + \xi^2\left((CZ+D)^{-1}C\right)[w]\Big)\Big\}.$$

Note that for $h(\xi) = \exp\{2a\,\xi + b\,\xi^2\}$, the fth derivative of h at $\xi = 0$ is

$$h^{(f)}(0) = \sum_{r=0}^{[f/2]} \frac{f!}{r!(f-2r)!}\, b^r (2a)^{f-2r}.$$

Hence, differentiating (9) f times with respect to ξ and setting $\xi = 0$ proves the theorem. □

4. Proof of Theorem 1. Now we turn to the proof of Theorem 1. It is easy to see that (3) holds for all $\lambda, \mu \in M_{j,n}(\mathbf{Z})$ and therefore it only remains to show (2).

Note that

$$\Theta_{F,H,\zeta}(Z,W) = \sum_{f\geq 0} \frac{(2\pi i)^{2f}}{(2f)!}\ \theta_{F,H,\zeta,2f}(Z,W) \tag{10}$$

where

$$\theta_{F,H,\zeta,f}(Z,W) = \sum_{N\in M_{m,n}(\mathbf{Z})} \left(\sigma({}^t NF\zeta W)\right)^f \exp\left\{\pi i\,\sigma\left(F[N]\,\mathrm{Re}\,(Z) + iH[N]\,\mathrm{Im}\,(Z)\right)\right\}.$$

We will use Eichler's "embedding trick" to prove (2). More precisely, we will regard $\theta_{F,H,\zeta,f}(Z,W)$ as a specialization of $\vartheta(Z, \binom{u}{v}, w, f)$, defined by (7), and then applying Theorem 2 will lead to (2).

Let $U \in M_{m,m}(\mathbf{C})$ and $V \in M_{n,n}(\mathbf{C})$. Then the Kronecker product of U and V is given by

$$U \otimes V = (u_{ij}V) \in M_{mn,mn}(\mathbf{C}).$$

If $T \in M_{m,n}(\mathbf{C})$, we write $T = (t_1, \dots , t_n)$, where $t_l \in M_{m,1}(\mathbf{C})$, for $l = 1, \dots , n$, are the columns of T, and we set

$$\hat{T} = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \in M_{mn,1}(\mathbf{C}).$$

For $Z = X + iY \in \mathfrak{H}^{(n)}$, set $\tilde{Z} = X \otimes F + iY \otimes H \in \mathfrak{H}^{(nm)}$. With F, ζ, and W as in Theorem 1, we set $\tilde{w} = \hat{T}$, where $T = F\zeta W$. It is easy to check, see also [**2**], that

$$\sigma({}^tNF\zeta W) = {}^t\hat{N}\tilde{w}$$

and

$$\sigma\big(F[N]\,\mathrm{Re}\,(Z) + iH[N]\,\mathrm{Im}\,(Z)\big) = \tilde{Z}[\hat{N}].$$

Hence

$$\theta_{F,H,\zeta,f}(Z, W) = \vartheta\left(\tilde{Z}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tilde{w}, f\right). \tag{11}$$

Moreover, if

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(q),$$

then

$$\tilde{M} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} = \begin{pmatrix} A \otimes I_m & B \otimes F \\ C \otimes F^{-1} & D \otimes I_m \end{pmatrix} \in \Gamma_\vartheta^{(nm)},$$

see [**2**] for details. Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(q)$. We apply (11) and Theorem 2 and after straightforward computation we find that

(12)

$$\begin{aligned}
&\theta_{F,H\zeta,f}(M \circ Z, W(CZ+D)^{-1}) \\
&\underset{(11)}{=} \vartheta\left(\tilde{M} \circ \tilde{Z}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, {}^t\left(\tilde{C}\tilde{Z} + \tilde{D}\right)^{-1} \tilde{w}, f\right) \\
&\underset{(8)}{=} \chi(\tilde{M})[\det(\tilde{C}\tilde{Z}+\tilde{D})]^{1/2} \sum_{r=0}^{[f/2]} \frac{f!}{r!(f-2r)!} (\pi i)^{-r} 2^{-2r} \\
&\qquad\qquad \times \left(((\tilde{C}\tilde{Z}+\tilde{D})^{-1}\tilde{C})[\tilde{w}]\right)^r \vartheta\left(\tilde{Z}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tilde{w}, f-2r\right) \\
&\underset{(11)}{=} \chi_F(M) \det(CZ+D)^{k/2} \det(C\overline{Z}+D)^{l/2} \sum_{r=0}^{[f/2]} \frac{f!}{r!(f-2r)!} (\pi i)^{-r} 2^{-2r} \\
&\qquad\qquad \times \sigma\Big(F[\zeta]\big(W(CZ+D)^{-1}C^tW\big)\Big)^r \theta_{F,H\zeta,f-2r}(Z,W),
\end{aligned}$$

where $\chi_F(M)$ is an eighth root of unity, which can be determined as in [**7**] using results by Stark [**12**] and Styer [**13**]. We omit the details and refer the reader to [**7**]. Using (10) then yields (2) and the proof of Theorem 1 is complete.

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