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COMPACT MULTIPLICATION OPERATORS ON WEIGHTED SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. In this note we characterize the compact multiplication operators M_π on the weighted locally convex spaces $CV_0(X, E)$ of vector-valued continuous functions induced by the operator-valued mappings $\pi : X \to B(E)$.

0. Introduction. Let $L(X, E)$ be a vector space of functions from a nonempty set X to a vector space E over the field $\mathbf{C}$ or $\mathbf{R}$. Let $T(E)$ be a set of linear transformations from E to itself. If $\phi : X \to X$ and $\pi : X \to T(E)$ are mappings such that the weighted composite function $\pi.f \circ \phi$ belongs to $L(X, E)$, whenever $f \in L(X, E)$, then the mapping taking f to $\pi.f \circ \phi$ is a linear transformation on $L(X, E)$ and we denote it by $W_{\pi,\phi}$. In case $L(X, E)$ is a topological vector space and the mapping $W_{\pi,\phi}$ is continuous, we call $W_{\pi,\phi}$ the weighted composition operator on $L(X, E)$ induced by the symbol (π, ϕ). In case ϕ is the identity map, we call it the multiplication operator induced by π and we denote it by M_π. For details on these operators we refer to [**11**].

The compact weighted composition operators on spaces of continuous functions have been studied extensively by many authors like Kamowitz [**5**], Feldman [**3**], Singh and Summers [**13**], Jamison and Rajagopalan [**4**], Takagi [**14**], Chan [**2**] and Singh and Manhas [**12**]. As we know, the class of weighted composition operators include the class of multiplication operators and the class of composition operators. One natural question arises: is it possible to get the behavior of the compact multiplication operators from the study of the compact weighted composition operators? In general, it is not possible since the conditions obtained earlier for a weighted composition operator to be compact is not satisfied by the identity map ϕ. So it motivates us to

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look for a separate study of the compact multiplication operators on the weighted spaces of vector-valued continuous functions which include many nice concrete spaces of continuous functions. Our main theorem makes sure that there are nonzero compact multiplication operators on these spaces of continuous functions whereas it is not the case with L^p-spaces. In [**9**], Singh and Kumar have shown that zero operator is the only compact multiplication operator on L^p-spaces (with nonatomic measure). Moreover, in [**15**], Takagi has proved that there is no nonzero compact weighted composition operator on L^p-spaces (with nonatomic measure).

1. Preliminaries. Let $C(X,E)$ be the vector space of all continuous functions from a completely regular Hausdorff space X to a nonzero locally convex Hausdorff space E. Let $\mathrm{cs}\,(E)$ be the set of all continuous semi-norms on E, and let $(B(E),u)$ be the locally convex space of all continuous linear operators on E, where u denotes the topology of uniform convergence on bounded subsets of E. Let V be a directed upward set of nonnegative upper semi-continuous functions on X. Each element of V is called a 'weight' on X. The weighted spaces of vector-valued continuous functions associated with the system of weights V are introduced as follows:

$$CV_0(X,E)=\{f\in C(X,E): vf \text{ vanishes at infinity on } X \text{ for each } v\in V\},$$

and

$$CV_b(X,E)=\{f\in C(X,E): vf(X) \text{ is bounded in } E \text{ for each } v\in V\}.$$

Let $v\in V$ and $p\in \mathrm{cs}\,(E)$. For $f\in C(X,E)$, if we define

$$\|f\|_{v,p}=\sup\{v(x)p(f(x)):x\in X\},$$

then $\|\quad\|_{v,p}$ can be regarded as a semi-norm on either $CV_b(X,E)$ or $CV_0(X,E)$, and the family $\{\|\quad\|_{v,p}: v\in V, p\in \mathrm{cs}\,(E)\}$ of semi-norms defines a Hausdorff locally convex topology on each of these spaces. With this topology the vector spaces $CV_0(X,E)$ and $CV_b(X,E)$ are called the weighted locally convex spaces of vector-valued continuous functions. They have a basis of closed absolutely convex neighborhoods of the origin of the form

$$B_{v,p}=\{f\in CV_b(X,E) \text{ resp. } CV_0(X,E):\|f\|_{v,p}\le 1\}.$$

For details on these spaces, we refer to Bierstedt [**1**], Nachbin [**6**] and Prolla [**7**].

2. Compact multiplication operators. At the outset, we shall record certain definitions and results which are needed to establish the desired results.

An operator $A \in B(E)$ is said to be compact if it maps bounded subsets of E into relatively compact subsets of E. A completely regular Hausdorff space X is called a $K_{\mathbf{R}}$-space, if a function $f : X \to \mathbf{R}$ is continuous if and only if $f|_K$ is continuous for each compact subset K of X. Clearly all locally compact or metrizable spaces are $K_{\mathbf{R}}$-spaces. A system of weights V is said to satisfy condition $(*)$ if for each compact subset K of X, there exists $v \in V$ such that $\inf\{v(x) : x \in K\} > 0$ (which implies that the topology of $CV_0(X, E)$ is stronger than uniform convergence on compact subsets of X). A completely regular Hausdorff space X is said to be a $V_{\mathbf{R}}$-space with respect to a given system V of weights on X if a function $f : X \to \mathbf{R}$ is necessarily continuous whenever, for each $v \in V$, the restriction of f to $\{x \in X : v(x) \geq 1\}$ is continuous. Moreover, if V satisfies the condition $(*)$, then any $K_{\mathbf{R}}$-space X is a-fortiori a $V_{\mathbf{R}}$-space. A subset $H \subseteq C(X, E)$ is called equicontinuous at $x_0 \in X$ if, for every neighborhood N of zero in E, there exists a neighborhood G of x_0 in X such that $f(x) - f(x_0) \in N$, for all $f \in H$ and $x \in G$. If H is equicontinuous at every point of X, we say that H is equicontinuous on X. The assumption that, for each $x \in X$, there exists $f \in CV_0(X)$ such that $f(x) \neq 0$ will be in force throughout this section for completely regular Hausdorff spaces X. The following compactness criterion can be found in [**8**].

Theorem 2.1. *Let X be a completely regular Hausdorff $V_{\mathbf{R}}$-space, and let E be a quasi-complete locally convex Hausdorff space. Then a subset $H \subseteq CV_0(X, E)$ is relatively compact if and only if*

(a) *H is equicontinuous*;

(b) *$H(x) = \{f(x) : f \in H\}$ is relatively compact in E for each $x \in X$; and*

(c) *vH vanishes at infinity on X for each $v \in V$, i.e., given $v \in V$, $p \in cs(E)$ and $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that $v(x)p(f(x)) < \varepsilon$ for every $x \in X \setminus K$ and $f \in H$.*

Corollary 2.2. *Let V be a system of constant weights on X. Let X be a locally compact Hausdorff space, and let E be a quasi-complete locally convex Hausdorff space. Then a subset $H \subseteq CV_0(X,E)$ is relatively compact if and only if*

(a′) *H is equicontinuous*;

(b′) *$H(x)$ is relatively compact in E for each $x \in X$; and*

(c′) *H vanishes at infinity on X, i.e., given $p \in cs(E)$, and $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that $p(f(x)) < \varepsilon$, for every $x \in X \setminus K$ and $f \in H$.*

Theorem 2.3. *Let X be a locally compact Hausdorff space, and let E be a Banach space. Let V be a system of weights on X with condition $(*)$. Let $\pi : X \to B(E)$ be an operator-valued mapping. Then M_π is a compact multiplication operator on $CV_0(X,E)$ if and only if*

(i) *$\pi : X \to B(E)$ is continuous in the uniform operator topology*;

(ii) *for every $x \in X$, π_x is a compact operator on E*;

(iii) *$\pi : X \to B(E)$ vanishes at infinity uniformly, i.e., for every $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that $\|\pi_x\| < \varepsilon$, for every $x \in X \setminus K$*;

(iv) *for every bounded set $F \subseteq CV_0(X,E)$, $x_0 \in X$ and $\varepsilon > 0$, there exists a neighborhood G of x_0 such that*

$$\|\pi_{x_0}(f(x) - f(x_0))\| < \varepsilon, \quad \textit{for every } x \in G \textit{ and } f \in F.$$

Proof. Firstly, we show that these conditions are necessary for M_π to be a compact multiplication operator on $CV_0(X,E)$. To establish condition (i), we fix $x_0 \in X$ and $\varepsilon > 0$. Consider the set $B = \{y \in E : \|y\| \leq 1\}$. Let K_0 be a neighborhood of x_0 in X such that $\overline{K}_0$ is compact. According to [**6**, p. 69], there exists $h \in CV_0(X)$ such that $h(\overline{K}_0) = 1$. For each $y \in B$, we define the function $g_y : X \to E$ as $g_y(x) = h(x)y$ for every $x \in X$. Then the set $S = \{g_y : y \in B\}$ is bounded in $CV_0(X,E)$. Thus, according to Theorem 2.1, the set $M_\pi(S)$ is equicontinuous on X. This implies that there exists a neighborhood G of x_0 such that

$$\|\pi_x(g_y(x)) - \pi_{x_0}(g_y(x_0))\| < \varepsilon, \quad \text{for every } x \in G \text{ and } y \in B.$$

Further, it implies that $\|\pi_x - \pi_{x_0}\| < \varepsilon$, for every $x \in G \cap K_0$. This proves that $\pi : X \to B(E)$ is continuous in the uniform operator topology. To establish condition (ii), we fix $x_0 \in X$ and consider a bounded set $B \subseteq E$. Choose $f \in CV_0(X)$ such that $f(x_0) = 1$. For each $y \in B$, we define the function $g_y : X \to E$ as $g_y(x) = f(x)y$, for every $x \in X$. Clearly the set $S = \{g_y : y \in B\}$ is bounded in $CV_0(X, E)$ and hence the set $M_\pi(S)$ is relatively compact in $CV_0(X, E)$. Again in view of Theorem 2.1, the set $M_\pi(S)(x_0) = \{\pi_{x_0}(y) : y \in B\}$ is relatively compact in E. This proves that π_{x_0} is a compact operator on E. Now we shall establish condition (iii). Let $\varepsilon > 0$ and let $B = \{y \in E : \|y\| \leq 1\}$. For $x_0 \in X$, we choose $f \in CV_0(X)$ such that $f(x_0) \neq 0$. Again, we may select $v \in V$ such that $v(x_0) \geq 1$. Now there exists $\lambda_v > 0$ such that $v(x)|f(x)| \leq \lambda_v$, for every $x \in X$. For each $y \in B$, if we define the function $g_y : X \to E$ as $g_y(x) = f(x)y$, for every $x \in X$, then the set $F = \{g_y : y \in B\}$ is bounded in $CV_0(X, E)$. Since the set $M_\pi(F)$ is relatively compact in $CV_0(X, E)$, it follows from Theorem 2.1 that the set $v.M_\pi(F)$ vanishes at infinity. Thus there exists a compact set K of X such that $\|v(x)\pi_x(g_y(x))\| < \varepsilon\lambda_v$, for every $x \in X \setminus K$ and $y \in B$. That is, for each $x \in X \setminus K$, we have $\|\pi_x(v(x)f(x)y)\| < \varepsilon\lambda_v$, for every $y \in B$. Now it readily follows that $\|\pi_x\| < \varepsilon$, for every $x \in X \setminus K$. Finally, to establish condition (iv), let $F \subseteq CV_0(X, E)$ be any nonzero bounded set. Fix $x_0 \in X$ and $\varepsilon > 0$. Let K_0 be a neighborhood of x_0 such that $\overline{K_0}$ is compact. Then there exists $v \in V$ such that $\beta = \inf\{v(x) : x \in \overline{K_0}\} > 0$. Let $B = \{v(x)f(x) : x \in X, f \in F\}$. Then B is bounded and there exists $m > 0$ such that $\|v(x)f(x)\| \leq m$ for every $x \in X$ and $f \in F$. Since $\pi : X \to B(E)$ is continuous, there exists a neighborhood G_1 of x_0 such that $\|(\pi_x - \pi_{x_0})(y)\| < (\varepsilon\beta/2m)$, for every $x \in G_1$ and for all $y \in E$ such that $\|y\| \leq 1$. Further, it implies that

$$(*) \qquad \|\pi_x(f(x)) - \pi_{x_0}(f(x))\| < \frac{\varepsilon}{2},$$

for every $x \in G_1 \cap K_0$ and $f \in F$. Again, since the set $M_\pi(F)$ is relatively compact in $CV_0(X, E)$ and by Theorem 2.1, it is equicontinuous on X. Thus there exists a neighborhood G_2 of x_0 such that

$$(**) \qquad \|\pi_x(f(x)) - \pi_{x_0}(f(x_0))\| < \frac{\varepsilon}{2}$$

for every $x \in G_2$ and $f \in F$. Let $G = G_1 \cap G_2 \cap K_0$. Then from $(*)$ and $(**)$, it follows that $\|\pi_{x_0}(f(x) - f(x_0))\| < \varepsilon$, for every $x \in G$ and $f \in F$. With this the proof of the necessary part is complete.

Now we shall show that these conditions are sufficient for the compactness of M_π on $CV_0(X,E)$. Condition (iii) implies that there exists $\lambda > 0$ such that $\|\pi_x\| \le \lambda$ for every $x \in X$. Let $v \in V$. Then there exists $u \in V$ such that $\lambda v \le u$. Now it is easy to see that $v(x)\|\pi_x(y)\| \le u(x)\|y\|$, for every $x \in X$ and $y \in E$. Therefore, according to [**10**, Theorem 2.1], it follows that M_π is a multiplication operator on $CV_0(X,E)$. Now we shall show that M_π is a compact operator on $CV_0(X,E)$. Let $F \subseteq CV_0(X,E)$ be a bounded set. We shall show that the set $M_\pi(F)$ satisfies all the conditions of Theorem 2.1. Fix $x_0 \in X$ and $\varepsilon > 0$. Let K_0 be a neighborhood of x_0 such that $\overline{K}_0$ is compact. Then there exists $v \in V$ such that $\propto = \inf\{v(x) : x \in \overline{K}_0\} > 0$. Let $B = \{v(x)f(x) : x \in X, f \in F\}$. Then there exists $k > 0$ such that $\|v(x)f(x)\| \le k$ for every $x \in X$ and $f \in F$. Since $\pi : X \to B(E)$ is continuous, there exists a neighborhood G_1 of x_0 such that $\|(\pi_x - \pi_{x_0})\| < (\varepsilon\alpha/2k)$ for every $x \in G_1$. Further, it implies that

$$\|\pi_x(f(x)) - \pi_{x_0}(f(x))\| < \frac{\varepsilon}{2}, \tag{1}$$

for all $x \in G_1 \cap K_0$ and $f \in F$. Again, by condition (iv), there exists a neighborhood G_2 of x_0 such that

$$\|\pi_{x_0}(f(x) - f(x_0))\| < \frac{\varepsilon}{2} \tag{2}$$

for every $x \in G_2$ and $f \in F$. Let $G = G_1 \cap G_2 \cap K_0$. Then, using (1) and (2), we have $\|\pi_x(f(x)) - \pi_{x_0}(f(x_0))\| < \varepsilon$, for every $x \in G$ and $f \in F$. This proves that the set $M_\pi(F)$ is equicontinuous on X. Also, for each $x \in X$, the set $M_\pi(F)(x)$ is relatively compact in E since π_x is a compact operator on E, and the set $\{f(x) : f \in F\}$ is bounded in E. Finally we show that, for each $v \in V$, the set $v.M_\pi(F)$ vanishes at infinity. Fix $v \in V$ and $\varepsilon > 0$. Let $B = \{v(x)f(x) : x \in X, f \in F\}$. Then there exists $\lambda > 0$ such that $\|v(x)f(x)\| \le \lambda$, for every $x \in X$ and $f \in F$. According to condition (iii), there exists a compact set $K \subseteq X$ such that $\|\pi_x\| < \varepsilon/\lambda$ for every $x \in X \setminus K$. That is, $\|\pi_x(y)\| < \varepsilon/\lambda$, for every $x \in X \setminus K$ and $y \in E$ such that $\|y\| \le 1$. Further, it readily follows that $v(x)\|\pi_x(f(x))\| < \varepsilon$ for every $x \in X \setminus K$ and $f \in F$. This proves that the set $v.M_\pi(F)$ vanishes at infinity on X. Thus all the conditions of Theorem 2.1 are satisfied by the set $M_\pi(F)$. Hence M_π is a compact multiplication operator on $CV_0(X, E)$. With this the proof of the theorem is complete. $\square$

In case V is a system of constant weights, we can extend the above result to the locally convex Hausdorff space E. This we shall prove in the following theorem.

Theorem 2.4. *Let X be a locally compact Hausdorff space, and let E be a quasi-complete locally convex Hausdorff space. Let V be a system of constant weights on X. Then $M_\pi : CV_0(X, E) \to CV_0(X, E)$ is a compact multiplication operator if and only if*

(i) *$\pi : X \to B(E)$ is continuous in the topology of uniform convergence on bounded subsets of E;*

(ii) *for every $p \in \operatorname{cs}(E)$, there exists $q \in \operatorname{cs}(E)$ such that $p(\pi_x(y)) \leq q(y)$, for every $x \in X$ and for every $y \in E$;*

(iii) *for every $x \in X$, π_x is a compact operator on E;*

(iv) *$\pi : X \to B(E)$ vanishes at infinity uniformly, i.e., for every $p \in \operatorname{cs}(E)$, bounded set $B \subseteq E$ and $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that $\|\pi_x\|_{p,B} < \varepsilon$, for every $x \in X \setminus K$ (where $\|\pi_x\|_{p,B} = \sup\{p(\pi_x(y)) : y \in B\}$,*

(v) *for every bounded set $F \subseteq CV_0(X, E)$, $x_0 \in X$, $p \in \operatorname{cs}(E)$ and $\varepsilon > 0$, there exists a neighborhood G of x_0 such that*

$$p(\pi_{x_0}(f(x) - f(x_0))) < \varepsilon \quad \textit{for every } x \in G \textit{ and } f \in F.$$

Proof. Assume that conditions (i) through (v) hold. According to [**10**, Theorem 2.1], conditions (i) and (ii) imply that M_π is a multiplication operator on $CV_0(X, E)$. Let $F \subseteq CV_0(X, E)$ be a bounded set. In order to show that M_π is a compact operator, it is enough to prove that the set $M_\pi(F)$ satisfies all the conditions of Corollary 2.2. Fix $x_0 \in X$, $p \in \operatorname{cs}(E)$ and $\varepsilon > 0$. Consider the set $B = \{f(x) : x \in X, f \in F\}$. Obviously the set B is bounded in E. By condition (i), there exists a neighborhood G_1 of x_0 such that $\|\pi_x - \pi_{x_0}\|_{p,B} < \varepsilon/2$ for every $x \in G_1$. Further it implies that

$$\text{(a)} \qquad p(\pi_x(f(x)) - \pi_{x_0}(f(x))) < \frac{\varepsilon}{2}$$

for every $x \in G_1$ and $f \in F$. Again, by condition (v), there exists a neighborhood G_2 of x_0 such that

$$\text{(b)} \qquad p(\pi_{x_0}(f(x)) - \pi_{x_0}(f(x_0))) < \frac{\varepsilon}{2}$$

for every $x \in G_2$ and $f \in F$. Let $G = G_1 \cap G_2$. Then, from (a) and (b), it follows that $p(\pi_x(f(x)) - \pi_{x_0}(f(x_0))) < \varepsilon$ for every $x \in G$ and $f \in F$. This proves that the set $M_\pi(F)$ is equicontinuous on X. Let $x_0 \in X$. Then clearly the set $M_\pi(F)(x_0) = \{\pi_{x_0}(f(x_0)) : f \in F\}$ is relatively compact in E since π_{x_0} is a compact operator on E and the set $\{f(x_0) : f \in F\}$ is bounded in E. Finally we show that the set $M_\pi(F)$ vanishes at infinity on X. For this, we fix $p \in \operatorname{cs}(E)$ and $\varepsilon > 0$. Since the set $B = \{f(x) : x \in X, f \in F\}$ is bounded in E, according to condition (iv) there exists a compact set $K \subseteq X$ such that $\|\pi_x\|_{p,B} < \varepsilon$ for every $x \in X \setminus K$. Further, it implies that $p(\pi_x(f(x))) < \varepsilon$ for every $x \in X \setminus K$ and $f \in F$. This proves that the set $M_\pi(F)$ vanishes at infinity uniformly on X. With this the proof of the sufficient part is complete.

Now we shall show that conditions (i) through (v) are necessary for M_π to be a compact multiplication operator on $CV_0(X, E)$. Let $x_0 \in X$. Fix $p \in \operatorname{cs}(E)$, bounded set $B \subseteq E$ and $\varepsilon > 0$. Let G_1 be a neighborhood of x_0 in X such that $\overline{G}_1$ is compact. Choose $f \in CV_0(X)$ such that $f(\overline{G}_1) = 1$. For each $y \in B$, we define the function $g_y : X \to E$ as $g_y(x) = f(x)y$, for every $x \in X$. Clearly the set $S = \{g_y : y \in B\}$ is bounded in $CV_0(X, E)$. According to Corollary 2.2, the set $M_\pi(S)$ is equicontinuous on X. Thus there exists a neighborhood G_2 of x_0 such that

$$p(\pi_x(g_y((x)) - \pi_{x_0}(g_y(x_0))) < \varepsilon,$$

for every $x \in G_2$ and $y \in B$. Further, it implies that

$$\|\pi_x - \pi_{x_0}\|_{p,B} < \varepsilon,$$

for every $x \in G_1 \cap G_2$. This establishes condition (i). The proof of condition (ii) follows from [**10**, Theorem 2.1]. Again, let $x_0 \in X$ and $B \subseteq E$ be a bounded set. Select $f \in CV_0(X)$ such that $f(x_0) = 1$. For each $y \in B$, define the function $g_y(x) = f(x)y$ for every $x \in X$. Let $S = \{g_y : y \in B\}$. Then the set $M_\pi(S)$ is relatively compact in

$CV_0(X,E)$. By Corollary 2.2, the set $M_\pi(S)(x_0)=\pi_{x_0}(B)$ is relatively compact in E. This proves that π_{x_0} is a compact operator on E. To prove condition (iv), we suppose that $\pi : X \to B(E)$ does not vanish at infinity on X. This implies that there exists $q \in \mathrm{cs}\,(E)$, bounded set $B \subseteq E$ and $\varepsilon > 0$ such that for every compact set $K \subseteq X$, there exists $x_k \in X \setminus K$ for which $\|\pi_{x_k}\|_{q,B} \geq \varepsilon$. Further, it implies that there exists $y_k \in B$ such that $q(\pi_{x_k}(y_k)) \geq \varepsilon$. In this fashion, for each compact set $K \subseteq X$, we select $f_k \in CV_0(X)$ such that $0 \leq f_k \leq 1$ and $f_k(x_k)=1$. For each compact set $K \subseteq X$, we define the function $g_k : X \to E$ as $g_k(x) = f_k(x)y_k$ for every $x \in X$. It is easy to see that the set $S = \{g_k : K \subseteq X,\text{ a compact set}\}$ is bounded in $CV_0(X,E)$. Again, since the set $M_\pi(S)$ is relatively compact in $CV_0(X,E)$, by Corollary 2.2, the set $M_\pi(S)$ vanishes at infinity on X. Thus there exists a compact set $K_0 \subseteq X$ such that $q(\pi_x(g_k(x))) < \varepsilon/2$, for every $x \in X \setminus K_0$ and for every compact set $K \subseteq X$. From this, it follows that $q(\pi_x(f_{k_0}(x)y_{k_0})) < \varepsilon/2$, for every $x \in X \setminus K_0$. In particular, for $x = x_{k_0}$, we have

$$\varepsilon \leq q(\pi_{x_{k_0}}(y_{k_0})) < \frac{\varepsilon}{2},$$

which is a contradiction. This proves that $\pi : X \to B(E)$ vanishes at infinity on X. Finally we shall establish condition (v). Let $F \subseteq CV_0(X,E)$ be a bounded set. Fix $x_0 \in X$, $p \in \mathrm{cs}\,(E)$ and $\varepsilon > 0$. If we consider the set $B = \{f(x) : x \in X, f \in F\} \subseteq E$, then there exists a neighborhood G_1 of x_0 such that $\|\pi_x - \pi_{x_0}\|_{p,B} < \varepsilon/2$ for every $x \in G_1$. Thus we get

$$(*) \qquad p(\pi_x(f(x)) - \pi_{x_0}(f(x))) < \frac{\varepsilon}{2},$$

for every $x \in G_1$ and $f \in F$. Again, since the set $M_\pi(F)$ is relatively compact in $CV_0(X,E)$, according to Corollary 2.2, the set $M_\pi(F)$ is equicontinuous on X. Thus there exists a neighborhood G_2 of x_0 such that

$$(**) \qquad p(\pi_x(f(x)) - \pi_{x_0}(f(x_0))) < \frac{\varepsilon}{2},$$

for every $x \in G_2$ and $f \in F$. Let $G = G_1 \cap G_2$. Then, from $(*)$ and $(**)$ we have

$$p(\pi_{x_0}(f(x)) - \pi_{x_0}(f(x_0))) < \varepsilon,$$

for every $x \in G$ and $f \in F$. This proves condition (v). With this the proof of the theorem is complete.

Example 2.5. Let $X = \mathbf{N}$, the set of natural numbers with discrete topology, and let $V = K^+(\mathbf{N})$, the set of positive constant functions on $\mathbf{N}$. Let $E = C_b(\mathbf{R})$ be a Banach space of bounded continuous complex-valued functions on $\mathbf{R}$, the set of reals with the usual topology. For each $n \in \mathbf{N}$, let $\phi_n : \mathbf{R} \to \mathbf{R}$ be the map defined by $\phi_n(t) = n$, for every $t \in \mathbf{R}$. Then each ϕ_n induces the compact composition operator C_{ϕ_n} on $C_b(\mathbf{R})$, where C_{ϕ_n} is defined as $C_{\phi_n}(f) = f \circ \phi_n$, for every $f \in C_b(\mathbf{R})$. Now, if we define $\pi : \mathbf{N} \to B(E)$ as $\pi(n) = (1/n)C_{\phi_n}$, for every $n \in \mathbf{N}$, then, in view of Theorem 2.3, it follows that M_π is a compact multiplication operator on $C_0(\mathbf{N}, E)$. In case we take $E = C(\mathbf{R})$, with compact-open topology, then in view of Theorem 2.4, the mapping $\pi : \mathbf{N} \to B(E)$, defined as above does not induce the compact multiplication operator M_π on $C_0(\mathbf{N}, E)$. But if $\pi : \mathbf{N} \to B(E)$ is defined as $\pi(n) = (1/n)A$, for every $n \in \mathbf{N}$, where A is any nonzero compact homomorphism on $C(\mathbf{R})$, then it turns out that M_π is a compact multiplication operator on $C_0(\mathbf{N}, E)$.

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