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# WARPED PRODUCTS IN REAL SPACE FORMS

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ABSTRACT. We prove a general inequality in terms of scalar curvature, the warping function, and the squared mean curvature for warped products isometrically immersed in real space forms. We also determine the warped products in real space forms which satisfy the equality case of the inequality.

**1. Introduction.** Let $B$ and $F$ be two Riemannian manifolds with Riemannian metrics $g_B$ and $g_F$, respectively, and $f > 0$ be a differentiable function on $B$. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The *warped product* $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_*(X)\|^2 + f^2(\pi(x))\|\eta * (X)\|^2 \tag{1.1}$$

for any tangent vector $X \in T_xM$. Thus we have $g = g_B + f^2 g_F$. The function $f$ is called the *warping function* of the warped product (cf. [**15**]).

For a submanifold $N$ in a Riemannian manifold $\widetilde{M}$ we denote by $\nabla$ and $\widetilde{\nabla}$ the Levi-Civita connections of $N$ and $\widetilde{M}$, respectively. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.2}$$

$$\widetilde{\nabla}_X \xi = -A\xi X + D_X \xi \tag{1.3}$$

for vector fields $X, Y$ tangent to $N$ and vector field $\xi$ normal to $N$, where $h$ denotes the second fundamental form, $D$ the normal connection and $A$ the shape operator of the submanifold.

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The mean curvature vector $\vec{H}$ is defined by

$$\vec{H} = \frac{1}{n}\,\mathrm{trace}\,h = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{1.4}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame of the tangent bundle $TN$ of $N$. The *squared mean curvature* is given by $H^2 = \langle \vec{H}, \vec{H}\rangle$, where $\langle\ ,\ \rangle$ denotes the inner product. A submanifold $N$ is called *totally geodesic* in $\widetilde{M}$ if the second fundamental form of $N$ in $\widetilde{M}$ vanishes identically.

We denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_pN$, $p \in N$. The *scalar curvature* $\tau$ of $N$ is defined by

$$\tau = \sum_{i<j} K(e_i \wedge e_j). \tag{1.5}$$

One of the most fundamental problems in submanifold theory is the following.

*Problem* 1. *Find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold.*

Many famous results in differential geometry, such as isoperimetric inequality, Chern-Lashof's inequality, and Gauss-Bonnet's theorem among others, can be regarded as results in this respect. In the last few years, several interesting new results concerning Problem 1 have been obtained by various authors (see, for example, [**2–11, 16**]).

According to a well-known theorem of Nash, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension. In particular, Nash's theorem implies that every warped product can be isometrically immersed into some Riemannian manifolds of constant sectional curvature. Thus, it is natural to study Problem 1 for warped products in Riemannian manifolds of constant sectional curvature.

In this paper we provide a solution to this problem. More precisely, we prove the following general result for warped products isometrically immersed in a real space form.

**Theorem 1.** *Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian manifold of constant sectional curvature $c$. Then we have*

(1) *The scalar curvature $\tau$ of the warped product $N_1 \times_f N_2$ satisfies*

$$\tau \leq \frac{\Delta f}{n_1 f} + \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2}(n+1)(n-2)c, \tag{1.6}$$

*where $n_1 = dim\, N_1$, $n = dim\, N_1 \times N_2$, $H^2$ is the squared mean curvature of $\phi$ and $\Delta$ is the Laplacian operator of $N_1$.*

(2) *If $n = 2$, the equality sign of* (1.6) *holds automatically.*

(3) *If $n \geq 3$, then the equality sign of* (1.6) *holds identically if and only if one of the following two cases occurs*:

(3a) *$N_1 \times_f N_2$ is a Riemannian manifold of constant sectional curvature $c$ whose warping function satisfies $\Delta f = cf$ and $N_1 \times_f N_2$ is immersed in $R^m(c)$ as a totally geodesic submanifold, or*

(3b) *In a neighborhood of each point of the open dense subset where $H^2$ is positive, the manifold $N_1 \times_f N_2$ can be locally isometrically immersed as a rotational hypersurface in a totally geodesic submanifold $R^{n+1}(c)$ of $R^m(c)$ with a geodesic of $R^{n+1}(c)$ as its profile curve.*

**2. Preliminaries.** Let $N$ be an $n$-dimensional submanifold of a Riemannian $n$-manifold $R^m(c)$ of constant sectional curvature $c$. We choose a local field of orthonormal frame $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ in $R^m(c)$ such that, restricted to $N$, the vectors $e_1, \dots, e_n$ are tangent to $N$ and $e_{n+1}, \dots, e_m$ are normal to $N$.

Let $\{h^r_{ij}\}$, $i, j = 1, \dots, n$; $r = n+1, \dots, m$, denote the coefficients of the second fundamental form $h$ with respect to $e_1, \dots, e_n, e_{n+1}, \dots, e_m$.

Denote by $R$ the Riemann curvature tensors of $N$. Then the *equation of Gauss* is given by (see, for instance, [**1**])

$$\begin{aligned} R(X, Y; Z, W) = c\{&\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle\} \\ &+ \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \end{aligned} \tag{2.1}$$

for vectors $X, Y, Z, W$ tangent to $N$.

For the second fundamental form $h$ we define the covariant derivative $\overline{\nabla} h$ of $h$, with respect to the connection in $TN \oplus T^\perp N$, by

$$(\overline{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.2}$$

The *equation of Codazzi* is given by

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z). \tag{2.3}$$

Let $\nu$ be a subbundle of the normal bundle $T^\perp N$ of $N$ in $\widetilde{M}$. Then $\nu$ is called a *parallel normal subbundle* if, for each normal vector field $\xi$ in $\nu$ and each vector $X$ tangent to $N$, we have $D_X\xi \in \nu$.

Let $H^{n+1}(c)$, $c<0$, denote the hypersurface of $\mathbf{R}^{n+2}$ given by

$$H^{n+1}(c) = \Big\{ x \in \mathbf{R}^{n+2} \mid x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = \frac{1}{c},\ x_{n+2} > 0 \Big\}.$$

If we endow $H^{n+1}(c)$ with the Riemannian metric induced by the Lorentzian metric

$$ds^2 = dx_1^2 + \cdots + dx_{n+1}^2 - dx_{n+2}^2$$

on $\mathbf{R}^{n+2}$, then $H^{n+1}(c)$ has constant negative curvature $c$.

Let $S^{n+1}(c)$, $c>0$, be the hypersurface of radius $c^{-1}$ of $\mathbf{E}^{n+2}$, centered at the origin, i.e.,

$$S^{n+1}(c) = \Big\{ x \in \mathbf{E}^{n+2} \mid x_1^2 + \cdots + x_{n+2}^2 = \frac{1}{c} \Big\}.$$

Then $S^{n+1}(c)$ has constant positive sectional curvature $c$.

Next we briefly recall what is a rotation hypersurface of a real space form $\widetilde{M}^{n+1}(c)$, $c \neq 0$, following [**13**]. We always consider $\widetilde{M}^{n+1}(c)$ as a hypersurface in $(\mathbf{R}^{n+2}, ds^2)$. Let $P^3$ be a three-dimensional linear subspace of $\mathbf{R}^{n+2}$ that intersects $\widetilde{M}^{n+1}(c)$. We denote the intersection by $\widetilde{M}^2(c)$, if $c<0$ we take only the upper part. Let $P^2$ be any linear subspace in $P^3$. We recall that any isometry of $\widetilde{M}^{n+1}(c)$ is the restriction to $\widetilde{M}^{n+1}(c)$ of an orthogonal transformation of $(\mathbf{R}^{n+2}, ds^2)$, and conversely.

Let $O(P^2)$ be the group of orthogonal transformations (with positive determinant) that leaves $P^2$ pointwise fixed. We take any curve $\alpha$ in $\widetilde{M}^2(c)$ which does not intersect $P^2$. The orbit of $\alpha$ under $O(P^2)$ is called the *rotation hypersurface with profile curve $\alpha$ and axis $P^2$*. The orbit of $\alpha(s)$ for a fixed $s$ is a sphere, and if $c<0$, then this sphere is elliptic, hyperbolic or parabolic according to $P^2$ respectively being Lorentzian, Riemannian or degenerate.

In order to give a parametrization of a rotation hypersurface of the different types, we introduce the vector $u \in P^3$ such that $P^2$ coincides with $u^\perp = \{v \in P^3 \mid \langle v, u \rangle = 0\}$. We can always assume that $u$ has length 1, $-1$ or 0, according to $P^2$ respectively being Lorentzian, Riemannian or degenerate and that $\langle u, \alpha' \rangle > 0$. Let $\delta = \langle u, u \rangle$. We define the map $Q$ as the orthogonal projection of $P^3$ on $u^\perp$ if $\delta \neq 0$ and as the identity map of $P^3$ if $\delta = 0$. Further, let $P^{n-1}$ be the orthogonal complement of $P^3$ in $\mathbf{R}^{n+2}$ and let $P^n$ be the linear space, spanned by $P^{n-1}$ and $u$. If $\delta = 1$, respectively $\delta = -1$, then $P^n$ is Riemannian, respectively Lorentzian, and we can define a mapping $\phi$ of $M^{n-1}(\delta)$ into $P^n$ by considering $M^{n-1}(\delta)$ as a unit hypersphere in $P^n$. If $\delta = 0$, then we can define a mapping $\phi$ of $M^{n-1}(0)$ into $P^n$ by identifying $M^{n-1}(0)$ and $P^{n-1}$ and defining

$$\phi(p) = p - \frac{1}{2}\langle p, p \rangle u. \tag{2.4}$$

Then a parametrization of the rotation hypersurface with profile curve $\alpha$ around the axis $P^2$ is given by

$$f(s, p) = Q(\alpha(s)) + \langle \alpha(s), u \rangle \varphi(p). \tag{2.5}$$

If we assume that $s$ is the arc length of $\alpha$, then it follows immediately that the rotation hypersurface $M^n$ is intrinsically the warped product $I \times_f M^{n-1}(\delta)$, where $I$ is an open interval of $\mathbf{R}$ and $f$ is defined by $f(s) = \langle \alpha(s), u \rangle$.

The second fundamental form of the rotation hypersurface $M^n$ is given by

$$h\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = \frac{f'' + cf}{\sqrt{\delta - cf^2 - f'^2}} \tag{2.6}$$

and

$$h(X, Y) = -\frac{\sqrt{\delta - cf^2 - f'^2}}{f}\langle X, Y \rangle \tag{2.7}$$

for $X$ and $Y$ tangent to $M^{n-1}(\delta)$.

We need the following lemma from [**2**].

**Lemma 2.** *Let $a_1, \dots, a_n, c$ be $n+1$, $n \geq 2$, real numbers such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + c\Big). \tag{2.8}$$

*Then $2a_1a_2 \geq c$ with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.*

**3. Proof of Theorem 1.** Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian manifold of constant sectional curvature $c$. We denote by $n_1, n_2$ and $n$ the dimensions of $N_1, N_2$ and $N_1 \times N_2$, respectively, so that $n = n_1 + n_2$.

From equation (2.1) of Gauss, we have

$$2\tau = n^2 H^2 - \|h\|^2 + n(n-1)c. \tag{3.1}$$

Let

$$\delta = 2\tau - \frac{n^2(n-2)}{n-1}\, H^2 - (n+1)(n-2)c. \tag{3.2}$$

Then (3.1) and (3.2) yield

$$n^2 H^2 = (n-1)\|h\|^2 + (n-1)(\delta - 2c). \tag{3.3}$$

Let $X$ and $Z$ be two unit local vector fields tangent to $N_1$ and $N_2$, respectively. We choose an orthonormal frame $e_1, \dots, e_m$ such that $e_1 = X, e_{n+1} = Z$ and $e_{n+1}$ is parallel to the mean curvature vector. Then (3.3) gives

$$\begin{aligned}\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big\{ &\sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2 \\ &+ \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^{r})^2 + \delta - 2c\Big\}.\end{aligned} \tag{3.4}$$

By applying Lemma 2 to (3.4) we obtain

$$2h_{11}^{n+1}h_{n_1+1n_1+1}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \delta - 2c, \tag{3.5}$$

from which we get

$$\begin{aligned} K(e_1 \wedge e_{n_1+1}) \geq{}& \sum_{r=n+1}^{m}\sum_{j\in\Omega_{1n_1+1}}\{(h_{1j}^r)^2 + (h_{n_1+1j}^r)^2\} \\ &+ \frac{1}{2}\sum_{i,j\in\Omega_1 n_1+1}^{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j\in\Omega_1 n_1+1}(h_{ij}^r)^2 \\ &+ \frac{1}{2}\sum_{r=n+2}^{m}(h_{11}^r + h_{n_1+1n_1+1}^r)^2 + \frac{\delta}{2} \geq \frac{\delta}{2}, \end{aligned} \tag{3.6}$$

where $\Omega_{1n_1+1} = \{1, \dots, n\} \setminus \{1, n_1+1\}$.

Since $N_1 \times_f N_2$ is a warped product, we have $\nabla_X Z = \nabla_Z X = (X \ln f)Z$ for unit vector fields $X, Z$ tangent to $N_1, N_2$, respectively. Hence, we find

$$\begin{aligned} K(X \wedge Z) &= \langle \nabla_Z\nabla_X X - \nabla_X\nabla_Z X, Z\rangle \\ &= \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \end{aligned} \tag{3.7}$$

Combining (3.2), (3.6) and (3.7) yields

$$\tau \leq \frac{1}{f}\{(\nabla_{e_1}e_1)f - e_1^2 f\} + \frac{n^2(n-2)}{2(n-1)}\,H^2 + \frac{1}{2}(n+1)(n-2)c. \tag{3.8}$$

If the equality sign of (3.8) holds, then all inequalities in (3.5) and (3.6) become equalities. Thus, we have

$$\begin{gathered} h_{1j}^{n+1} = 0, \qquad h_{jn_1+1}^{n+1} = 0, \qquad h_{ij}^{n+1} = 0, \quad i \neq j; \\ h_{1j}^r = h_{jn_1+1}^r = h_{ij}^r = 0, \qquad h_{11}^r + h_{n_1+1n_1+1}^r = 0, \\ i, j \in \Omega_{1n_1+1}, \quad r = n+2, \dots, m. \end{gathered} \tag{3.9}$$

In other words, the shape operators take the following forms:

(3.10)
$$A_{n+1}=\begin{pmatrix} a & 0 & \cdots & & & & \\ 0 & & & & & & \\ \vdots & & \mu I_{n_1-1} & & & \mathbf{0} & \\ 0 & & & & & & \\ & & & b & 0 & \cdots & 0 \\ & & & 0 & & & \\ & \mathbf{0} & & \vdots & & \mu I_{n_2-1} & \\ & & & 0 & & & \end{pmatrix}, \quad a+b=\mu;$$

(3.11)
$$A_r=\begin{pmatrix} h^r_{11} & 0 & \cdots & 0 & h^r_{1n_1+1} & 0 & \cdots & 0 \\ 0 & & & & 0 & & & \\ \vdots & & \mathbf{0} & & \vdots & & \mathbf{0} & \\ 0 & & & & 0 & & & \\ h^r_{1n_1+1} & 0 & \cdots & 0 & -h^r_{11} & 0 & \cdots & 0 \\ 0 & & & & 0 & & & \\ \vdots & & \mathbf{0} & & \vdots & & \mathbf{0} & \\ 0 & & & & 0 & & & \end{pmatrix}$$

for $r=n+2,\dots,m$, where $I_k$ denotes the identity matrix of order $k$.

Similar to (3.8), we also have

$$\tau \le \frac{1}{f}\{(\nabla e_\alpha e_\alpha)f - e_\alpha^2 f\} + \frac{n^2(n-2)}{2(n-1)}\,H^2 + \frac{1}{2}(n+1)(n-2)c \tag{3.12}$$

for $\alpha = 1,2,\dots,n_1$. Hence, by summing up $\alpha$ from 1 to $n_1$, we obtain

$$n_1\tau \le \frac{\Delta f}{f} + \frac{n_1 n^2(n-2)}{2(n-1)}\,H^2 + \frac{1}{2}n_1(n+1)(n-2)c, \tag{3.13}$$

which implies (1.6).

If the equality sign of (1.6) holds identically, then the equality sign of (3.12) holds for each $\alpha \in \{1,\dots,n_1\}$. Thus, for each $\alpha \in \{1,\dots,n_1\}$ and each $t \in \{n_1,\dots,n\}$, we have

$$h^{n+1}_{\alpha j}=0, \qquad h^{n+1}_{ij}=0, \qquad h^{n+1}_{ij}=0, \quad i\ne j; \tag{3.14}$$

$$h^r_{\alpha j}=h^r_{ij}=h^r_{ij}=0, \qquad h^r_{\alpha\alpha}+h^r_{tt}=0 \tag{3.15}$$

for $i, j \in \Omega_{\alpha t} = \{1, \dots, n\} \setminus \{\alpha, t\}$; $r = n+2, \dots, m$.

If $n = 2$, then $n_1 = n_2 = 1$. Thus, by (3.7), we have $\tau = \Delta f$. Hence, the equality sign of (1.6) holds automatically.

Next, suppose that $n = n_1 + n_2 \geq 3$. Then (3.15) implies that $A_{n+2} = \cdots = A_m = 0$. Moreover, from $n \geq 3$ and (3.14), we find

(a) $A_{n+1} = 0$ or

(b) $a = 0$, $b = \mu \neq 0$ or $a = \mu \neq 0$, $b = 0$.

*Case* (a). $A_{n+1} = 0$. In this case $N_1 \times_f N_2$ is a totally geodesic submanifold of $R^m(c)$. Hence, $N_1 \times_f N_2$ is a real space form of constant sectional curvature $c$ which implies that the scalar curvature of $N_1 \times_f N_2$ is given by $\tau = n(n-1)c/2$.

Since a totally geodesic submanifold is minimal, the equality sign of (1.6) and $\tau = n(n-1)c/2$ imply $\Delta f = cf$. Therefore, $f$ is either a harmonic function or an eigenfunction of the Laplacian with eigenvalue $c$, according to $c = 0$ or $c \neq 0$, respectively.

Conversely, suppose that $N_1 \times_f N_2$ is a warped product decomposition of a real space form $R^n(c)$ such that the warping function $f$ satisfies $\Delta f = cf$. Clearly, for each integer $m > n$, $N_1 \times_f N_2$ can be locally isometrically immersed in a real space form $R^m(c)$ of the same curvature as a totally geodesic submanifold. It is easy to verify that such a totally geodesic immersion satisfies the equality case of inequality (1.6).

*Case* (b). Either $a = 0$, $b = \mu \neq 0$ or $a = \mu \neq 0$, $b = 0$. In this case $A_{n+1}$ has exactly two distinct eigenvalues $0, \mu$ with multiplicities $1, n-1$, respectively, on the open subset $U = \{p \in N_1 \times_f N_2 : H^2(p) > 0\}$. The first normal subbundle, $\operatorname{Im} h$ is of rank one on $U$. Without loss of generality, we may assume that

$$\begin{gathered} h(e_1, e_1) = 0, \qquad h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu e_{n+1}, \\ h(e_i, e_j) = 0, \quad 1 \leq i \neq j \leq n \end{gathered} \tag{3.16}$$

on $U$. From (3.16) we find

$$\begin{aligned} (\overline{\nabla}_{e_k} h)(e_j, e_j) &= (e_k \mu) e_{n+1} + \mu D_{e_k} e_{n+1}, \\ (\overline{\nabla}_{e_j} h)(e_j, e_k) &= -(\omega_j^k(e_j) + \omega_k^j(e_j)) e_{n+1} \end{aligned} \tag{3.17}$$

for $j \in \{2, \dots, n\}$ and $k \in \{1, \dots, n\}$. Thus, the equation of Codazzi implies that

$$\mu D_{e_k} e_{n+1} + (\omega_j^k(e_j) + \omega_k^j(e_j) + e_k\mu)e_{n+1} = 0, \quad j \neq k,\ j > 1. \tag{3.18}$$

Since $D_X e_{n+1}$ is perpendicular to $e_{n+1}$, (3.18) implies that $D_X e_{n+1} = 0$ for $X$ being one of $e_1, \dots, e_n$. Therefore, the first normal subbundle, $\operatorname{Im} h = \operatorname{Span}\{e_{n+1}\}$, is a parallel normal subbundle on $U$. Hence, by applying a result of Erbacher [**14**], $M$ has essential codimension one. Thus, in a neighborhood of each point of $U$, the warped product $N_1 \times_f N_2$ is isometrically immersed in a totally geodesic submanifold $R^{n+1}(c)$ of $R^m(c)$. Because the shape operator of $U$ has one eigenvalue of multiplicity $n-1$ and the other eigenvalue is zero, it follows from a result of [**13**] (see also Section 2 of [**12**]) that $M$ is a rotation hypersurface whose profile curve is a geodesic of $R^{n+1}(c)$. By applying the continuity of the squared mean curvature $H^2$, we know that $U$ is a dense open subset of $N_1 \times_f N_2$.

Conversely, suppose that $M$ is rotation hypersurface in a real space form $R^{n+1}(c)$ whose profile curve $\alpha$ is a geodesic of $R^{n+1}(c)$. Let us assume that the profile curve $\alpha$ is parametrized by an arc length function $s$. Then $M$ is isometric to a warped product $I \times_f M^{n-1}(\delta)$, where $I$ is an open interval of $\mathbf{R}$ and the shape operator of $M$ has exactly two distinct eigenvalues 0 and $\mu$ of multiplicities 1 and $n-1$, respectively. Thus, the square mean curvature is given by

$$H^2 = \frac{(n-1)^2\mu^2}{n^2}. \tag{3.19}$$

Moreover, by applying the equation of Gauss, we know that the scalar curvature of the rotation hypersurface is given by

$$\tau = \frac{n(n-1)}{2}\, c + \frac{(n-1)(n-2)}{2}\mu^2. \tag{3.20}$$

Because $\alpha = \alpha(s)$ is a geodesic in $R^{n+1}(c)$ parametrized by arc length function, equation (2.6) implies that the warping function $f$ satisfies the differential equation: $f''(s) + cf(s) = 0$. Hence, we get

$$\Delta f = cf. \tag{3.21}$$

From (3.19), (3.20) and (3.21), we can easily verify that the rotation hypersurface $I \times_f M^{n-1}(\delta)$ in $R^{n+1}(c)$ satisfies the equality case of (1.6) identically. $\square$

**4. Remarks.** In this section we provide some remarks related with Theorem 1.

*Remark* 1. In view of Theorem 1, it is interesting to point out that every Riemannian manifold of constant sectional curvature $c$ can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$.

For example, the unit $n$-sphere $S^n(1)$ is locally isometric to $I \times_{\cos x} S^{n-1}(1)$ with warped metric $g = dx^2 + \cos^2 x g_1$; the Euclidean $n$-space $\mathbf{E}^n$ is locally isometric to $I \times_x S^{n-1}(1)$ with warped metric $g = dx^2 + x^2 g_1$; and the unit hyperbolic space $H^n(-1)$ is locally isometric to $\mathbf{R} \times_{e^x} \mathbf{E}^{n-1}$ with warped metric $g = dx^2 + e^{2x} g_0$, where $I = (0, \infty)$, $g_0$ is the standard Euclidean metric of $\mathbf{E}^{n-1}$ and $g_1$ is the standard metric on $S^{n-1}(1)$.

Besides those warped product decompositions of real space form given above, there are some other warped product decompositions of real space form whose warping function also satisfies $\Delta f = cf$. For example, let $\{x_1, \dots, x_{n_1}\}$ be a standard Euclidean coordinate system of $\mathbf{E}^{n_1}$ and $\rho$ be the function defined by $\rho = \sum_{j=1}^{n_1} a_j x_j + b$ where $a_1, \dots, a_{n_1}, b$ are real numbers with $\sum_{j=1}^{n_1} a_j^2 = 1$. Then the warped product $\mathbf{E}^{n_1} \times_\rho S^{n_2}(1)$ is a flat space whose warping function is a harmonic function.

In fact, one may prove that *those warped product decompositions of flat spaces are the only warped product decompositions of flat spaces whose warping functions are harmonic functions.*

*Remark* 2. In view of Theorem 1 it is also natural to look for a complex version of Theorem 1. However, the following proposition shows that the complex version of Theorem 1 does not occur unless the warped products are essentially Riemannian products, in which case the problem has already been investigated in [**1**].

**Proposition 3.** *Let $N_1 \times_f N_2$ be a warped product of two Hermitian manifolds and $\phi : N_1 \times_f N_2 \to \widetilde{M}$ be a holomorphic isometric immersion of $N_1 \times_f N_2$ into a Kaehler manifold $\widetilde{M}$. Then the warping function $f$ must be a constant function.*

*Proof.* Since $N_1 \times_f N_2$ is a warped product with warping function $f$, it is known that, for each $u \in N_1$, $\{u\} \times N_2$ is a totally umbilical submanifold of $N_1 \times_f N_2$ whose mean curvature vector is given by $-\text{grad}\,(\ln f)$ (cf. [**1, 15**]) where $\text{grad}\,(\ln f)$ is the gradient of $\ln f$.

On the other hand, since $\widetilde{M}$ is Kaehlerian and $\phi$ is holomorphic and isometric, we also know that $N_1, N_2$ and $N_1 \times_f N_2$ are also Kaehler manifolds with respect to their induced Kaehler structures. Hence, for each $u \in N_1$, $\{u\} \times N_2$ is a minimal submanifold of $N_1 \times_f N_2$. Therefore, by the minimality and total umbilicity, we conclude that each $\{u\} \times N_2$ is totally geodesic in $N_1 \times_f N_2$ which implies that the warping function $f$ is a constant function. □

*Remark* 3. Let $N_1$ and $N_2$ be two Hermitian manifolds and let $f$ be a positive function on $N_1$. Then the warped product $N_1 \times_f N_2$ is also a Hermitian manifold. On the other hand, from the proof of Proposition 3, we also have the following.

**Proposition 4.** *The warped product $N_1 \times_f N_2$ of two Kaehler manifolds $N_1$ and $N_2$ is Kaehlerian if and only if the warping function is a constant.*

## REFERENCES

**1.** B.Y. Chen, *Geometry of submanifolds and its applications*, Science University of Tokyo, Tokyo, 1981.

**2.** ———, *Some pinching and classification theorems for minimal submanifolds*, Archiv. Math. **60** (1993), 568–578.

**3.** ———, *Mean curvature and shape operator of isometric immersions in real-space-forms*, Glasgow Math. J. **38** (1996), 87–97.

**4.** ———, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension*, Glasgow Math. J. **41** (1999), 33–41.

**5.** ———, *Some new obstructions to minimal and Lagrangian isometric immersions*, Japan J. Math. **26** (2000), 105–127.

**6.** ———, *A series of Kählerian invariants and their applications to Kählerian geometry*, Beiträge Algebra Geom. **42** (2001), 165–178.

**7.** ———, *Riemannian geometry of Lagrangian submanifolds*, Taiwanese J. Math. **5** (2001), 681–723.

**8.** ———, *Geometry of warped products as Riemannian submanifolds and related problem*, Soochow J. Math. **28** (2002), 125–157.

**9.** ———, *On isometric minimal immersions from warped products into real space forms*, Proc. Edinburgh Math. Soc. **45** (2002), 579–587.

**10.** P.J. de Smet, F. Dillen, L. Verstraelen and L. Vrancken, *The normal curvature of totally real submanifolds of $S^6(1)$*, Glasgow Math. J. **40** (1998), 199–204.

**11.** ———, *A pointwise inequality in submanifold theory*, Arch. Math. (Brno) **35** (1999), 115–128.

**12.** F. Dillen, *Semi-parallel hypersurfaces of a real space form*, Israel J. Math. **75** (1991), 193–202.

**13.** M. do Carmo and M. Dajczer, *Rotation hypersurfaces in spaces of constant curvature*, Trans. Amer. Math. Soc. **277** (1983), 685–709.

**14.** J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geometry **5** (1971), 333–340.

**15.** B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

**16.** C. Scharlach, U. Simon, L. Verstraelen and L. Vrancken, *A new intrinsic curvature invariant for centroaffine hypersurfaces*, Beiträge Algebra Geom. **38** (1997), 437–458.

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