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SPECTRAL ESTIMATES FOR THE COMMUTATOR OF TWO-DIMENSIONAL HILBERT TRANSFORMATION AND THE OPERATOR OF MULTIPLICATION WITH A C^1 FUNCTION

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1. Introduction and notation. Let Γ be a set of simple non-intersecting closed contours of Lyapunov type and S_Γ be the singular integral along Γ:

$$(S_\Gamma \varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(s)}{s-t}\, ds, \quad t \in \Gamma;$$

the contour Γ is considered oriented in some manner. It is well known that the operator S_Γ is bounded in each of the spaces $L^p(\Gamma)$, $1 < p < \infty$.

Also, it is known that if a function $t \mapsto a(t)$ satisfies a Hölder condition on Γ or if $a \in C(\Gamma)$, then the operator $aS_\Gamma - S_\Gamma a$ is compact on $L^p(\Gamma)$, see [**6**]. In the special case, when Γ is an interval on the real axis, S_Γ is the Hilbert transformation.

Instead of S_Γ it is possible to consider some other singular integral operator and study its spectral properties.

Let Ω be a domain in $\mathbf{C}$. Denote by $L^2(\Omega)$ the space of complex-valued functions on Ω such that the norm

$$\|f\| = \left(\int_\Omega |f(\xi)|^2 \, dA(\xi) \right)^{1/2}$$

is finite. Here dA denotes Lebesgue measure on Ω.

It is known (see [**8**]) that the formula

$$H_\Omega f(z) = -\frac{1}{\pi} \text{ p.v.} \int_\Omega \frac{f(\xi)}{(\xi - z)^2}\, dA(\xi)$$

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defines a bounded operator on $L^2(\Omega)$. (Moreover, it is bounded on $L^p(\Omega)$ for all $1 < p < \infty$.) In the case $\Omega = \mathbf{C}$, H_Ω is unitary and is called the two-dimensional Hilbert transformation.

In this paper we consider spectral characteristics of the operator $aH_\Omega - H_\Omega a$.

For an arbitrary compact operator A on a Hilbert space, denote by $s_n(A)$ the sequence of eigenvalues of the operator $(A^*A)^{1/2}$, in decreasing order (taking into account multiplicity). We shall call the numbers $s_n(A)$ singular values of the operator A. Denote by c_p Schatten-von Neumann class of operators.

For a compact operator A we shall use $\mathbf{N}_t(A)$ to denote the singular values distribution function, that is,

$$\mathbf{N}_t(A) = \sum_{s_n(A) \geq t} 1, \quad t > 0.$$

By $a_n \sim b_n$, $n \to \infty$, we denote the fact that $\lim_{n\to\infty} a_n/b_n = 1$. By $\mathrm{Lip}_\alpha(\Omega)$, $0 < \alpha \leq 1$, we denote the class of all complex-valued functions defined on Ω for which there exists a constant M such that

$$|f(z) - f(\xi)| \leq M|z - \xi|^\alpha$$

for all $z, \xi \in \Omega$.

By $|\Omega|$ we denote the area of Ω. In the rest of the paper, by $\int_S R(z,\xi) \cdot dA(\xi)$ we shall denote the integral operator on the space $L^2(S)$ (where $S \subset \mathbf{C}$) whose kernel is the function $R(\cdot,\cdot)$.

2. Main result.

Theorem 1. a) *If* $a \in \mathrm{Lip}_\alpha(\Omega)$ (Ω *is a bounded domain* $\mathbf{C}$) *then* $aH_\Omega - H_\Omega a \in c_p$ *for every* $p > 2/\alpha$.

b) *If* $a \in C^1(\overline{\Omega})$, *where* Ω *is a bounded domain with a rectifiable boundary, then*

$$s_n(cH_\Omega - H_\Omega a) \sim \left(\frac{1}{\pi n}\int_\Omega \left(\left|\frac{\partial a}{\partial \xi}\right|^2 + \left|\frac{\partial a}{\partial \bar{\xi}}\right|^2\right) dA(\xi)\right)^{1/2}, \quad n \to \infty.$$

Corollaries. 1′ *If* $a \in C^1(\overline{\Omega})$*, then* $aH_\Omega - H_\Omega a \in c_2$ *if and only if* $a \equiv \text{const}$.

2′ *If* a *is univalent analytic function on* $\overline{\Omega}$*, then*

$$s_n(aH_\Omega - H_\Omega a) \sim \sqrt{\frac{|a(\Omega)|}{\pi n}}$$

where $a(\Omega)$ *is the image of* Ω *under mapping* a*.*

For the proof of Theorem 1 we need several lemmas.

Lemma 1 [**3**]. *Let* Ω *be a bounded domain in* $\mathbf{R}^N$, $L \in L^\infty(\Omega \times \Omega)$, $k > 0$, $k \in L^1(0 + \infty)$ *and* $k \in L^2(d, +\infty)$ *for all* $d > 0$*. Then for the operator* $T : L^2(\Omega) \to L^2(\Omega)$ *defined by*

$$Tf(x) = \int_\Omega L(x,y)k(\|x-y\|^N)f(y)\,dy$$

there holds

$$s_n(T) \le c\|L\|_\infty \left(\int_0^d k(t)\,dt + n^{-1/2}\left(\int_d^\infty k^2(t)\,dt \right)^{1/2} \right)$$

for every $d > 0$*, where* c *depends only on* Ω*.*

Lemma 2 [**4**]. a) *If* T' *and* T'' *are compact operators and* $T = T' + T''$, $\lim_{t\to 0^+} t^\gamma \mathbf{N}_t(T') = c(T')$ *and* $s_n(T'') = o(n^{-1/\gamma})$, $\gamma > 0$*, then there exists the limit* $\lim_{t\to 0^+} t^\gamma \mathbf{N}_t(T)$ *and it is equal to* $c(T')$*.*

b) *Let* T *be a compact operator, and suppose that, for every* $\varepsilon > 0$*, there exists a decomposition* $T = T'_\varepsilon + T''_\varepsilon$ *where* T'_ε, T''_ε *are compact operators such that*

(i) *There exists* $\lim_{t\to 0^+} t^\gamma \mathbf{N}_t(T'_\varepsilon) = c(T'_\varepsilon)$, $c(T'_\varepsilon)$ *being a bounded function in the neighborhood of* $\varepsilon = 0$*.*

(ii) $\overline{\lim}_{n\to\infty} s_n(T''_\varepsilon) \cdot n^{1/\gamma} \le \varepsilon$*.*

Then there exists $\lim_{\varepsilon\to 0^+} c(T'_\varepsilon) = c(T)$ *and*

$$\lim_{t\to 0^+} t^\gamma \mathbf{N}_t(T) = c(T).$$

Lemma 3. *If $\Omega_1, \Omega_2 \subset \mathbf{C}$ are bounded measurable sets, $\Omega_1 \subset \Omega_2$ and $B_i : L^2(\Omega_i) \to L^2(\Omega_i)$, $i = 1, 2$, are the linear operators defined by*

$$B_i f(z) = -\frac{1}{\pi} \int_{\Omega_i} \frac{p(\xi)(z-\xi) + q(\xi)(\bar{z}-\bar{\xi})}{(z-\xi)^2} f(\xi)\, dA(\xi),$$
$$p, q \in C(\overline{\Omega_2})$$

then

$$\mathbf{N}_t(B_1) \le \mathbf{N}_t(B_2).$$

Proof. Let $P : L^2(\Omega_2) \to L^2(\Omega_1)$ be the orthoprojector. From $B_1 = PB_2P$ we have $s_n(B_1) \le s_n(B_2)$. The statement of the lemma follows. □

Lemma 4. *If Ω is a bounded Jordan measurable set in $\mathbf{C}$ and a, b some complex constants, then when $n \to \infty$*

$$(1)\quad s_n\biggl(-\frac{1}{\pi} \int_{\Omega} \frac{a \cdot (z-\xi) + b \cdot (\bar{z}-\bar{\xi})}{(z-\xi)^2} \cdot dA\,(\xi)\biggr) \sim \sqrt{\frac{|\Omega|(|a|^2 + |b|^2)}{\pi n}}$$

Proof. Let $T = -\frac{1}{\pi} \int_{\Omega} \frac{a(z-\xi)+b\cdot(\bar{z}-\bar{\xi})}{(z-\xi)^2} \cdot dA\,(\xi)$. The function $k(z) = -(1/\pi)(a \cdot z + b \cdot \bar{z})/z^2$, $z = x_1 + ix_2$ is homogeneous of order -1 and

$$T = \int_{\Omega} k(z-\xi) \cdot dA\,(\xi).$$

Let $\tilde{k}$ be the Fourier transform of k, which is understood, for example, in the sense of Riesz summation:

$$\tilde{k}(x) = \lim_{r\to\infty} \int_{|t|\le r} \left(1 - \frac{|t|^2}{r^2}\right)^s k(t) e^{-i\langle x,t\rangle}\, dt.$$

(Here $x = (x_1, x_2)$, $t = (t_1, t_2)$, $\langle x, t\rangle = x_1 t_1 + x_2 t_2$, $dt = dt_1 dt_2$.)

After calculation we obtain

$$(2)\qquad \tilde{k}(x) = -\frac{2i}{(x_1 + ix_2)^2}((b-a)x_1 - i(a+b)x_2).$$

According to the result of Birman-Solomjak [**2**, pp. 75–76], we get

$$\lim_{t\to 0^+} t^2 \mathbf{N}_t(T) = \gamma(k)\cdot|\Omega|$$

where $\gamma(k) = (2\pi)^{-2}\mathrm{mes}\,\{x\in\mathbf{R}^2 : |\tilde{k}(x)|>1\}$. Having in mind (2) we can write

$$\gamma(k) = \frac{1}{4\pi^2}\,\mathrm{mes}\left\{z : \frac{2|b\bar{z}-az|}{|z|^2} > 1\right\},\quad z = x_1 + ix_2.$$

Putting $z = re^{i\theta}$, we get

$$\begin{aligned}\mathrm{mes}\,\{z : 2|b\bar{z}-az| > |z|^2\}\\ &= \mathrm{mes}\,\{re^{i\theta} : 0\le\theta\le 2\pi, 0<r<2|be^{-i\theta}-ae^{i\theta}|\}\\ &= \frac{1}{2}\int_0^{2\pi}(2|be^{-i\theta}-ae^{i\theta}|)^2\,d\theta\\ &= 4\pi(|a^2|+|b|^2).\end{aligned}$$

So $\gamma(k) = \frac{1}{\pi}(|a^2|+|b|^2)$ and we have

$$\lim_{t\to 0^+} t^2\mathbf{N}_t(T) = \frac{|\Omega|}{\pi}(|a^2|+|b|^2)$$

whence, putting $t = s_n(T)$, we get (1). □

Lemma 5. *If Ω is a bounded domain in $\mathbf{C}$ with the rectifiable boundary, p, $q\in C(\overline{\Omega})$, then when $n\to\infty$,*

$$s_n\left(-\frac{1}{\pi}\int_\Omega \frac{p(\xi)(z-\xi)+q(\xi)(\bar{z}-\bar{\xi})}{(z-\xi)^2}\cdot dA\,(\xi)\right) \sim \left(\frac{1}{n\pi}\int_\Omega(|p(\xi)|^2+|q(\xi)|^2)\,dA(\xi)\right)^{1/2}.$$

Proof. Consider first the case when Ω is a square the sides of which are parallel to the coordinate axes. Divide Ω into N squares Δ_i of equal sides and denote their centers by ξ_i. Let

$$T = -\frac{1}{\pi}\int_\Omega \frac{p(\xi)(z-\xi)+q(\xi)(\bar{z}-\bar{\xi})}{(z-\xi)^2}\cdot dA(\xi)$$

$$T_i = -\frac{1}{\pi}\int_\Omega \frac{p(\xi_i)(z-\xi)+q(\xi_i)(\bar{z}-\bar{\xi})}{(z-\xi)^2}\cdot dA(\xi)$$

and

$$P_i f(z) = \chi_{\Delta_i}(z) f(z), \quad i = 1, 2, \dots, N.$$

(Here $\mathcal{X}_S(\cdot)$ denotes the characteristic function of S.)

Then

$$T = \sum_{i=1}^{N} P_i T_i P_i + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} P_i T P_j + \sum_{i=1}^{N} P_i (T - T_i) P_i.$$

It can be shown directly that $P_i T P_j \in c_2$ for $i \neq j$ and therefore

$$\lim_{n\to\infty} n^{1/2} s_n \bigg(\sum_{\substack{i,j=1 \\ i \neq j}}^{N} P_i T P_j \bigg) = 0. \tag{3}$$

Let $\varepsilon > 0$. Choose N large enough so that $|p(\xi) - p(\xi_i)| < \varepsilon$ and $|q(\xi) - q(\xi_i)| < \varepsilon$ for all $\xi \in \Delta_i$ and $i = 1, 2, \dots, N$. (This is possible because $p, q \in C(\overline{\Omega})$.)

Then, according to Lemma 4, we have

$$\sqrt{n} s_n(P_i(T - T_i)P_i) \leq C \cdot \varepsilon \sqrt{|\Delta_i|}, \quad i = 1, 2, \dots, N \tag{4}$$

where C is independent of ε, i.

From (4) it follows that

$$\mathbf{N}_t(P_i(T - T_i)P_i) \leq \frac{(C\varepsilon)^2}{t^2} |\Delta_i|, \quad i = 1, 2, \dots, N. \tag{5}$$

Since the operators $P_i(T - T_i)P_i$, $i = 1, 2, \dots, N$, are mutually orthogonal, we have

$$\mathbf{N}_t \bigg(\sum_{i=1}^{N} P_i (T - T_i) P_i \bigg) = \sum_{i=1}^{N} \mathbf{N}_t (P_i (T - T_i) P_i)$$

and thus, from (5) we get

$$t^2 \mathbf{N}_t \bigg(\sum_{i=1}^{N} P_i (T - T_i) P_i \bigg) \leq \sum_{i=1}^{N} (C\varepsilon)^2 |\Delta_i| = (C\varepsilon)^2 |\Omega|.$$

From this it follows that

$$s_n\bigg(\sum_{i=1}^{N} P_i(T-T_i)P_i\bigg) \le \frac{C\cdot\varepsilon\sqrt{|\Omega|}}{\sqrt{n}}$$

which together with (3) shows that for the operator

$$E_N = \sum_{\substack{i,j=1\\ i\neq j}} P_iTP_j + \sum_{i=1}^{N} P_i(T-T_i)P_i$$

there holds

$$\varlimsup_{n\to\infty} \sqrt{n}s_n(E_N) \le C\varepsilon\sqrt{|\Omega|}. \tag{6}$$

Since, by Lemma 4,

$$s_n(P_iT_iP_i) \sim \sqrt{\frac{|\Delta_i|\cdot(|p(\xi_i)|^2+|q(\xi_i)|^2)}{n\pi}}, \quad n\to\infty,$$

we get

$$\mathbf{N}_t(P_iT_iP_i) \sim \frac{|\Delta_i|}{\pi t^2}\,(|p(\xi_i)|^2+|q(\xi_i)|^2), \quad t\to 0^+$$

whence, because of the orthogonality of the summands in the sum $\sum_{i=1}^{N} P_iT_iP_i$ it follows that

$$\lim_{t\to 0^+} t^2\mathbf{N}_t\bigg(\sum_{i=1}^{N} P_iT_iP_i\bigg) = \frac{1}{\pi}\sum_{i=1}^{N} |\Delta_i|\cdot(|p(\xi_i)|^2+|q(\xi_i)|^2).$$

Having in mind that $T=\sum_{i=1}^{N} P_iT_iP_i + E_N$, the previous equality (6) and Lemma 2 b) yield

$$\begin{aligned}\lim_{t\to 0^+} t^2\mathbf{N}_t(T) &= \lim_{N\to\infty}\frac{1}{\pi}\sum_{i=1}^{N} |\Delta_i|\cdot(|p(\xi_i)|^2+|q(\xi_i)|^2)\\ &= \frac{1}{\pi}\int_\Omega (|p(\xi)|^2+|q(\xi)|^2)\,dA.\end{aligned} \tag{7}$$

Putting $t = s_n(T)$ in (7) we get the statement of the lemma in the case when Ω is square.

Now let $\Omega = \cup_{i=1}^{S} K_i$, where K_i are (equal) squares with sides parallel to the axes and with disjoint interiors. Let

$$P_i f(z) = \mathcal{X}_{K_i}(z) f(z), \quad i = 1, 2, \dots, S$$
$$T = -\frac{1}{\pi} \int_{\Omega} \frac{p(\xi)(z-\xi) + q(\xi)(\bar{z}-\bar{\xi})}{(z-\xi)^2} \cdot dA\,(\xi).$$

Then

$$T = \sum_{i=1}^{S} P_i T P_i + \sum_{\substack{i,j=1 \\ i \neq j}}^{S} P_i T P_j .$$

For $i \neq j$, we have $P_i T P_j \in c_2$ and therefore

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{S} P_i T P_j \in c_2 .$$

Hence

$$s_n\biggl(\sum_{\substack{i,j=1 \\ i \neq j}}^{S} P_i T P_j \biggr) = o(n^{-1/2}). \tag{8}$$

Since

$$\lim_{t \to 0^+} t^2 \mathbf{N}_t(P_i T P_i) = \frac{1}{\pi} \int_{K_i} (|p(\xi)|^2 + |q(\xi)|^2)\, dA$$

because of (7), and the summands in $\sum_{i=1}^{S} P_i T P_i$ are orthogonal, we get

$$\mathbf{N}_t\biggl(\sum_{i=1}^{S} P_i T P_i \biggr) = \sum_{i=1}^{S} \mathbf{N}_t(P_i T P_i)$$

and hence

$$\begin{aligned} \lim_{t \to 0^+} t^2 \mathbf{N}_t\biggl(\sum_{i=1}^{S} P_i T P_i \biggr) &= \frac{1}{\pi} \sum_{i=1}^{S} \int_{K_i} (|p(\xi)|^2 + |q(\xi)|^2)\, dA \\ &= \frac{1}{\pi} \int_{\Omega} (|p(\xi)|^2 + |q(\xi)|^2)\, dA \end{aligned} \tag{9}$$

From (8), (9) and Lemma 2a) we get

$$\lim_{t\to 0^+} t^2\mathbf{N}_t(T) = \frac{1}{\pi}\int_\Omega (|p(\xi)|^2 + |q(\xi)|^2)\, dA \tag{10}$$

and hence, putting $t = s_n(T)$, we get the statement in the case when Ω is a disjoint union of squares.

Now consider the general case, i.e., let Ω be a bounded domain with rectifiable boundary. Since $p, q \in C(\overline{\Omega})$ we can extend p, q to continuous functions $\tilde{p}, \tilde{q}$, respectively, on some neighborhood of $\overline{\Omega}$. The set Ω is Jordan measurable. Denote by Ω_N and $\overline{\Omega}_N$ finite disjoint unions of equal squares such that

$$\underline{\Omega}_N \subset \Omega \subset \overline{\Omega}_N$$

and

$$\begin{aligned} |\underline{\Omega}_N| &\longrightarrow |\Omega| \\ |\overline{\Omega}_N| &\longrightarrow |\Omega|, \quad N \to \infty. \end{aligned}$$

Let $\underline{T}_N$ and $\overline{T}_N$ be linear operators on $L^2(\underline{\Omega}_N)$ and $L^2(\overline{\Omega}_N)$, respectively, defined by

$$\begin{aligned} \underline{T}_N &= -\frac{1}{\pi}\int_{\underline{\Omega}_N} \frac{p(\xi)(z-\xi) + q(\xi)(\bar{z}-\bar{\xi})}{(z-\xi)^2}\cdot dA\,(\xi) \\ \overline{T}_N &= -\frac{1}{\pi}\int_{\overline{\Omega}_N} \frac{\tilde{p}(\xi)(z-\xi) + \tilde{q}(\xi)(\bar{z}-\bar{\xi})}{(z-\xi)^2}\cdot dA\,(\xi). \end{aligned}$$

According to Lemma 3, we have

$$\mathbf{N}_t(\underline{T}_N) \le \mathbf{N}_t(T) \le \mathbf{N}_t(\overline{T}_N)$$

whence

$$t^2\mathbf{N}_t(\underline{T}_N) \le t^2\mathbf{N}_t(T) \le t^2\mathbf{N}_t(\overline{T}_N),$$

i.e.,

(11)

$$\varliminf_{t\to 0^+} t^2\mathbf{N}_t(\underline{T}_N) \le \varliminf_{t\to 0^+} t^2\mathbf{N}_t(T) \le \varlimsup_{t\to 0^+} t^2\mathbf{N}_t(T) \le \varlimsup_{t\to 0^+} t^2\mathbf{N}_t(\overline{T}_N).$$

Since, by (10),

$$\underline{\lim}_{t\to 0^+} t^2\mathbf{N}_t(\underline{T}_N) = \lim_{t\to 0^+} t^2\mathbf{N}_t(\underline{T}_N) = \frac{1}{\pi}\int_{\underline{\Omega}_N} (|p(\xi)|^2 + |q(\xi)|^2)\, dA$$

and

$$\overline{\lim}_{t\to 0^+} t^2\mathbf{N}_t(\overline{T}_N) = \lim_{t\to 0^+} t^2\mathbf{N}_t(\overline{T}_N) = \frac{1}{\pi}\int_{\overline{\Omega}_N} (|\tilde{p}(\xi)|^2 + |\tilde{q}(\xi)|^2)\, dA.$$

Equation (11) implies

$$\begin{aligned}\frac{1}{\pi}\int_{\underline{\Omega}_N} (|p(\xi)|^2 + |q(\xi)|^2)\, dA &\leq \underline{\lim}_{t\to 0^+} t^2\mathbf{N}_t(T)\\ &\leq \overline{\lim}_{t\to 0^+} t^2\mathbf{N}_t(T)\\ &\leq \frac{1}{\pi}\int_{\overline{\Omega}_N} (|\tilde{p}(\xi)|^2 + |\tilde{q}(\xi)|^2)\, dA.\end{aligned}$$

From the last relation for $N \to \infty$, we obtain

$$\lim_{t\to 0^+} t^2\mathbf{N}_t(T) = \frac{1}{\pi}\int_{\Omega} (|p(\xi)|^2 + |q(\xi)|^2)\, dA. \tag{12}$$

Putting $t = s_n(T)$ in (12), we get

$$ns_n^2(T) \sim \frac{1}{\pi}\int_{\Omega} (|p(\xi)|^2 + |q(\xi)|^2)\, dA(\xi),$$

i.e.,

$$s_n(T) \sim \left(\frac{1}{\pi n}\int_{\Omega} (|p(\xi)|^2 + |q(\xi)|^2)\, dA(\xi)\right)^{1/2}, \quad n\to\infty.$$

Lemma 5 is proved. □

Proof of Theorem 1. a) Let $0 < \alpha < 1$ and $|a(z) - a(\xi)| \leq M|z-\xi|^\alpha$, $z, \xi \in \Omega$. The kernel of the operator $aH_\Omega - H_\Omega a$ is equal to the function

$$\frac{a(z) - a(\xi)}{(z-\xi)^2}.$$

Let δ be the diameter of Ω. Applying Lemma 1 with

$$L(z,\xi) = \frac{a(z)-a(\xi)}{|z-\xi|^\alpha}\left(\frac{|z-\xi|}{z-\xi}\right)^2; \quad k(t) = \begin{cases} t^{\alpha/2-1} & 0<t<\delta, \\ 1/1+t^2 & t>\delta, \end{cases}$$

and $d = 1/n$ we get

$$\begin{aligned} &s_n(aH_\Omega - H_\Omega a) \\ &\quad \le c\|L\|_\infty\left(\int_0^{1/n} t^{a/2-1}\,dt + n^{-1/2}\left(\int_{1/n}^{\delta} t^{\alpha-2}\,dt + \int_\delta^\infty \frac{dt}{1+t^2}\right)^{1/2}\right) \\ &\quad = O(n^{-\alpha/2}). \end{aligned}$$

Thus $s_n(aH_\Omega - H_\Omega a) = O(n^{-\alpha/2})$ and hence $aH_\Omega - H_\Omega a \in c_p$ for all $p > 2/\alpha$.

If $\alpha = 1$, then $aH_\Omega - H_\Omega a \in c_p$ for all $p > 2$ which can be obtained by direct application of the Russo theorem [**7**].

b) Let $a \in C^1(\overline{\Omega})$. Then

$$(13) \qquad \begin{cases} a(z)-a(\xi) = (\partial a/\partial\xi)(z-\xi) + (\partial z/\partial\bar\xi)(\bar z - \bar\xi) + R(z,\xi) \\ \lim_{z\to\xi}(R(z,\xi)/|z-\xi|) = 0, (\partial a/\partial\xi), (\partial a/\partial\bar\xi) \in C(\overline{\Omega}). \end{cases}$$

Thus

$$(14) \qquad aH_\Omega - H_\Omega a = V + S$$

where the operators V and S act on $L^2(\Omega)$ and are defined in the following way

$$\begin{aligned} Vf(z) &= -\frac{1}{\pi}\int_\Omega \frac{(\partial a/\partial\xi)(z-\xi) + (\partial a/\partial\bar\xi)(\bar z-\bar\xi)}{(z-\xi)^2} f(\xi)\,dA(\xi) \\ Sf(z) &= -\frac{1}{\pi}\int_\Omega \frac{R(z,\xi)}{(z-\xi)^2} f(\xi)\,dA(\xi). \end{aligned}$$

According to Lemma 5 we get

$$(15) \qquad s_n(V) \sim \left(\frac{1}{\pi n}\int_\Omega \left(\left|\frac{\partial a}{\partial\xi}\right|^2 + \left|\frac{\partial a}{\partial\bar\xi}\right|^2\right) dA(\xi)\right)^{1/2}, \quad n\to\infty.$$

Let us prove that

$$\lim_{n\to\infty} n^{1/2} s_n(S) = 0. \tag{16}$$

From (13) it follows that S can be written in the form

$$Sf(z) = \int_\Omega \frac{\omega(z,\xi)}{|z-\xi|} f(\xi)\, dA(\xi),$$
$$\omega \in C(\overline{\Omega} \times \overline{\Omega}),$$
$$\omega(\xi,\xi) = 0 \quad \text{for every } \xi \in \Omega.$$

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|\omega(z,\xi)| < \varepsilon$ when $|z-\xi| < \delta (z, \xi \in \Omega)$. Let

$$K'_\delta(z,\xi) = \begin{cases} \frac{\omega(z,\xi)}{|z-\xi|}; & |z-\xi| < \delta, z, \xi \in \Omega, \\ 0; & |z-\xi| \geq \delta, z, \xi \in \Omega, \end{cases}$$

$$K''_\delta(z,\xi) = \begin{cases} 0; & |z-\xi| < \delta, z, \xi \in \Omega, \\ \frac{\omega(z,\xi)}{|z-\xi|}; & |z-\xi| \geq \delta, z, \xi \in \Omega, \end{cases}$$

and

$$T'_\delta f(z) = \int_\Omega K'_\delta(z,\xi) f(\xi)\, dA(\xi)$$
$$T''_\delta f(z) = \int_\Omega K''_\delta(z,\xi) f(\xi)\, dA(\xi).$$

Then $S = T'_\delta + T''_\delta$. By [**1**, Lemma 4], there holds

$$s_n(T'_\delta) \leq C\varepsilon n^{-1/2} \tag{17}$$

where C is independent of ε, δ.

Since $\int_\Omega \int_\Omega |K''_\delta(z,\xi)|^2\, dA(z)\, dA(\xi) < \infty$, we have $T''_\delta \in c_2$, so

$$\lim_{n\to\infty} n^{1/2} s_n(T''_\delta) = 0. \tag{18}$$

Since

$$s_{2n}(S) \leq s_n(T'_\delta) + s_n(T''_\delta),$$

from (17) and (18) it follows that

$$\overline{\lim_{n\to\infty}} \sqrt{2n} s_{2n}(S) \le C \cdot \varepsilon$$

and since ε is arbitrary, we have

$$\lim_{n\to\infty} \sqrt{2n} s_{2n}(S) = 0$$

which implies (16).

Now the statement of Theorem 1 follows from (14), (15), (16) and the Ky-Fan theorem [**5**].

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