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ZETA FUNCTIONS AND REGULARIZED DETERMINANTS ON PROJECTIVE SPACES

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ABSTRACT. A Hermite type formula is introduced and used to study the zeta function over the real and complex n-projective space. This approach allows to compute the residua at the poles and the value at the origin as well as the value of the derivative at the origin that gives the regularized determinant of the associated Laplacian operator.

1. Introduction. Zeta functions on the sphere (and in general on a closed Riemannian manifold) were first introduced by Minakshisundaram and Peijel as extensions of the classical Riemannian zeta function [**12**]. An analytic definition, by a Mellin transform of the trace of the heat operator associated to the Laplacian in the standard metric, shows how the residua at the poles are given by the coefficients in the asymptotic expansion of the trace of the heat operator [**1**]. Such an expansion can be obtained in a large number of cases using global analysis [**16, 5, 3, 4**], and in particular the first coefficients in the case of the Laplacian on a Riemannian manifold can be computed using local invariants associated to the curvature tensor [**10**]. The constant term that corresponds to the value of the zeta function at the origin is important in physics [**15**] and in conformal theory [**2**], being associated to the conformal anomaly. On the other hand, the derivative of the zeta function at the origin gives the Atiyah regularized determinant of the Laplacian [**13**]. Early approaches to calculate these quantities give explicit results for the two-sphere [**15, 9**], while more recently an explicit formula for the residua has been obtained in [**7**] for the n-sphere, using a result that allows to write a Dirichlet series as a sum of classical Hurwitz zeta functions [**6**]. This method fails to compute the derivative, but an alternative one is provided by [**8**], where a factorization theorem for zeta regularized products is introduced and explicit results for the low-dimensional spheres are given.

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The aim of this paper is to show how these zeta functions can be treated very easily by classical methods using a Hermite type formula in exactly the same way as for the one-dimensional case of the Riemann (Hurwitz) zeta function. This approach allows us to deal not only with the zeta function on the spheres, but also with the zeta function on the real and complex projective spaces. For all these cases, we localize the poles and give explicit formulae for the residua. We show that the origin $s = 0$ is a regular point and compute the value of the zeta function together with the value of its first derivative at $s = 0$.

The main feature of this approach is that it can be used more generally to deal with the zeta function of any operator whose eigenvalues are explicitly known with their multiplicity. There is work in progress where further cases are under consideration.

2. A Hermite type formula. Consider the function

$$z(s,a,b,c) = \sum_{n=1}^{\infty} \frac{P_d(cn)}{(cn+a)^s(cn+b)^s},$$

of the complex variable s for $\mathrm{Re}\,(s) > \frac{d+1}{2}$, where P_d is a polynomial of degree d and a, b, c real constants with $a, b > -1$, $c > 0$. Introduce the function of the complex variable z:

$$\phi(z,s,a,b,c) = \frac{P_d(z)}{(cz+a)^s(cz+b)^s},$$

then we have the following

Proposition 1. *For* $\mathrm{Re}\,(s) > \frac{d+1}{2}$,

$$z(s,a,b,c) = \frac{1}{2}\phi(1,s,a,b,c) + \int_1^{\infty} \phi(z,s,a,b,c)\,dz + I(s),$$

where $I(s)$ *is an integral function of* s.

Proof. The infinite sum is actually a finite sum of infinite sums of the following type that can be treated by using the Plana theorem when

$\mathrm{Re}\,(s) > \frac{d+1}{2}$,

$$\begin{aligned}\sum_{n=1}^{\infty}&\psi_l(n,s,a,b,c)\\ &= \sum_{n=1}^{\infty} \frac{c_l(cn)^l}{(cn+a)^s(cn+b)^s}\\ &= \frac{1}{2}\psi_l(1,s,a,b,c) + \int_1^{\infty} \psi_l(z,s,a,b,c)\,dz\\ &\quad - 2c_l \int_0^{\infty} (c^2+y^2)^{l/2}[(a+c)^2+y^2]^{-s/2}[(b+c)^2+y^2]^{-s/2}\\ &\quad \times \sin\left[l \arctan\frac{y}{c} - s\left(\arctan\frac{y}{a+c} + \arctan\frac{y}{b+c}\right)\right]\frac{dy}{e^{2\pi y}-1}.\end{aligned}$$

Indeed, the last integral converges for all values of s. Furthermore, the last integral converges uniformly by standard estimates on arctan, so the given formula defines an analytic function of s. □

From Proposition 1, it is clear that all the poles and relative residua come from the first two terms. This expression can also be used to compute the value of z and its derivative at $s = 0$.

3. Formulae for the residua and the value at the origin. Spheres and real projective spaces can be treated together as follows; complex and projective spaces will be considered afterwards.

The polynomial P_d with $c = 1, 2$ for the sphere S^k and the projective space $\mathbf{R}P^k$, respectively, $k > 1$, is:

$$P_{k-1}(x) = Q_k(x) = \frac{2x+k-1}{(k-1)!}\prod_{i=1}^{k-2}(x+i).$$

It is now convenient to distinguish odd and even cases. We introduce the numbers $b_{k,l}$, by

Definition 1. For $k = 1, 2, 3, \ldots$, the numbers $b_{k,l}$ are defined by the equations

$$\prod_{i=1}^{2h-2}(x+i)=\sum_{l=0}^{h-1}b_{2h,l}[x^2+(2h-1)x]^l,$$
$$(x+h)\prod_{i=1}^{2h-1}(x+i)=\sum_{l=0}^{h}b_{2h+1,l}[x^2+2hx]^l.$$

In particular, $b_{2h,h-1}=b_{2h+1,h}=1$, $b_{2h,0}=(2h-2)!$, $b_{2h+1,0}=h(2h-1)!$.

This allows us to write the zeta function as a sum of some standard ones. Let

$$z_k(s,c)=\sum_{n=1}^{\infty}\frac{Q_k(cn)}{[cn(cn+k-1)]^s},$$

be the zeta function in dimension k, then

$$\zeta(s,S^k)=z_k(s,1),$$
$$\zeta(s,\mathbf{R}P^k)=z_k(s,2).$$

Now consider the two functions

$$z_{\text{even}}(s,a,b,c)=\sum_{n=1}^{\infty}\frac{2(cn+a)+b-a}{[(cn+a)(cn+b)]^s},$$
$$z_{\text{odd}}(s,a,b,c)=\sum_{n=1}^{\infty}\frac{1}{[(cn+a)(cn+b)]^s},$$

then

Lemma 1. *For $h=1,2,\dots,$*

$$z_{2h}(s,c)=\frac{1}{(2h-1)!}\sum_{l=0}^{h-1}b_{2h,l}z_{\text{even}}(s-l,0,2h-1,c),$$
$$z_{2h+1}(s,c)=\frac{2}{(2h)!}\sum_{l=0}^{h}b_{2h+1,l}z_{\text{odd}}(s-l,0,2h,c).$$

To write simpler formulae, let us also introduce the function

$$f(s,l,a,b,c) = \int_0^\infty [(a+c)^2+y^2]^{\frac{l-s}{2}}[(b+c)^2+y^2]^{-s/2} \times \sin\left[(l-s)\arctan\frac{y}{a+c} - s\arctan\frac{y}{b+c}\right]\frac{dy}{e^{2\pi y}-1}.$$

By expanding the trigonometric functions and recalling the standard integral representation of the Bernoulli numbers B_n, we get[1]

$$f(-n,l,a,b,c) = \frac{1}{4}\sum_{i=1}^{E(\frac{n+l+1}{2})}\sum_{j=0}^{E(\frac{n}{2})}\binom{n+l}{2i-1}\binom{n}{2j}(a+c)^{n+l+1-2i}(b+c)^{n-2j}\frac{B_{2(i+j)}}{i+j} + \frac{1}{4}\sum_{i=1}^{E(\frac{n+1}{2})}\sum_{j=0}^{E(\frac{n+l}{2})}\binom{n}{2i-1}\binom{n+l}{2j}(a+c)^{n+l-2j}(b+c)^{n+1-2i}\frac{B_{2(i+j)}}{i+j},$$

where $E(q)$ denotes the integer part of the rational number q.

We can now state the main properties of the functions $z_{\text{even/odd}}$.

Lemma 2. *The function $z_{\text{even}}(s,a,b,c)$ has a simple pole at $s=1$ with residuum $1/c$, while for $m=0,1,2,\ldots,$*

$$z_{\text{even}}(-m,a,b,c) = \frac{1}{2}[2(c+a)+b-a](c+a)^m(c+b)^m - \frac{(c+a)^{m+1}(c+b)^{m+1}}{(m+1)c} - 4f(-m,1,a,b,c) - 2(b-a)f(-m,0,a,b,c).$$

Proof. Proceeding as in Proposition 1, we get

$$z_{\text{even}}(s,a,b,c) = \frac{1}{2}\frac{2(c+a)+b-a}{(c+a)^s(c+b)^s} + \frac{1}{c}\frac{1}{(c+a)^{s-1}(c+b)^{s-1}}\frac{1}{s-1} - 4\int_0^\infty [(a+c)^2+y^2]^{1/2-s/2}[(b+c)^2+y^2]^{-s/2}$$

$$\times \sin\left[(1-s)\arctan\frac{y}{a+c} - s\arctan\frac{y}{b+c}\right]\frac{dy}{e^{2\pi y}-1}$$
$$+ 2(b-a)\int_0^\infty [(a+c)^2+y^2]^{-s/2}[(b+c)^2+y^2]^{-s/2}$$
$$\times \sin\left[s\left(\arctan\frac{y}{a+c} + \arctan\frac{y}{b+c}\right)\right]\frac{dy}{e^{2\pi y}-1};$$

then the poles are given by the second term, while the values at the non positive integers can be computed by using the formulae introduced above. □

Lemma 3. *The function* $z_{\text{odd}}(s,0,b,c)$ *has simple poles at* $s = 1/2 - m$, $m = 0,1,2,\dots$, *with residua*

$$\mathrm{Res}_1\left(z_{\text{odd}}(s,0,b,c), s = \frac{1}{2} - m\right) = \frac{(-1)^m}{2^{m+1}}\frac{(2m-1)!!}{m!}\frac{1}{c}\left(\frac{b}{2}\right)^{2m}.$$

For $m = 0,1,2,\dots$,

$$\begin{aligned} &z_{\text{odd}}(-m,0,b,c)\\ &\quad = \frac{1}{2}c^m(c+b)^m + \frac{(-1)^{m+1}}{m+1}\frac{2^m}{c}\left(\frac{b}{c}\right)^{2m-1}\frac{\Gamma(m+1)}{(2m+1)!!}\\ &\qquad - \frac{(bc)^m}{m+1}\frac{\Gamma(2m+1)}{\Gamma(-m)\Gamma(m+1)}\\ &\qquad \times \sum_{i=0}^{m}\frac{\Gamma(i-m)\Gamma(i+m+1)}{i!\Gamma(2m+i+1)}\left(\frac{c}{b}\right)^i - 2f(-m,0,0,b,c). \end{aligned}$$

Proof. For the odd case, difficulties arise when dealing with the first integral. If we restrict ourselves to the interesting case of $a = 0$, we get

$$z_{\text{odd}}(s,0,b,c) = \frac{1}{2}\frac{1}{c^s(c+b)^s} + c^{-s}\int_1^\infty x^{-s}(cx+b)^{-s}\,dx$$
$$+ 2\int_0^\infty [c^2+y^2]^{-s/2}[(b+c)^2+y^2]^{-s/2}$$
$$\times \sin\left[s\left(\arctan\frac{y}{c} + \arctan\frac{y}{b+c}\right)\right]\frac{dy}{e^{2\pi y}-1},$$

and we have the following two possible ways of treating it

$$c^{-s}\int_1^\infty x^{-s}(cx+b)^{-s}\,dx = c^{-s}(c+b)^{-s}\frac{1}{2s-1}F\left(s,1;2s;\frac{b}{c+b}\right) =$$

when $\mathrm{Re}\,(s) > 1/2$ and where F is the hypergeometric function, and

$$= \frac{1}{\sqrt{\pi}}\frac{1}{c}\left(\frac{b}{2}\right)^{1-2s}\frac{\Gamma(1-s)\Gamma(s+1/2)}{2s-1} - c^{-s}\int_0^1 x^{-s}(cx+b)^{-s}\,dx,$$

when $1/2 < \mathrm{Re}\,(s) < 1$, where the last integral is actually convergent if $\mathrm{Re}\,(s) < 1$, and can be expressed in terms of a hypergeometric function

$$c^{-s}\int_0^1 x^{-s}(cx+b)^{-s}\,dx = \frac{(bc)^{-s}}{1-s}F\left(s,1-s;2-s;-\frac{c}{b}\right).$$

In particular, we use the first representation to get the analytical continuation in the negative half plane and to compute the residua at the poles, and the second one to compute the value at $s=-m$, $m=0,1,2,\ldots$, and the derivative at $s=0$, see Section 4. (Both expressions are good to compute the residua.) $\square$

From the previous lemma, we immediately get the following

Proposition 2. *The function z_{2h} has simple poles at $s=n$, for $n=1,2,3,\ldots,h$, with residua*

$$\mathrm{Res}_1(z_{2h}(s,c), s=n) = \frac{1}{c}\frac{b_{2h,n-1}}{(2h-1)!}.$$

The function z_{2h+1} has simple poles at $s=1/2+h-m$, $m=0,1,2,\ldots$, with residua

$$\mathrm{Res}_1\left(z_{2h+1}(s,c), s=\frac{1}{2}+h-m\right)$$
$$= \frac{1}{c}\frac{2}{(2h)!}\sum_{l=0}^{\min(h,m)}\frac{(-1)^{m-l}b_{2h+1,l}}{2^{m-l+1}}\frac{(2(m-l)-1)!!}{(m-l)!}\left(\frac{2b}{2}\right)^{2(m-l)}.$$

These should be compared with Theorem 2 of [**7**], where we note that a factor 1/2 is missing.

For what concerns the value of $z_k(0,c)$, this can be easily computed (using any mathematical software) from the formulae in Lemmas 1, 2 and 3. Explicit results for $n = 2, 3$ and 4 are given in Section 4.

Next consider the complex projective spaces $\mathbf{C}P^k$, $k > 1$. The polynomial P_d is

$$P_{k-1}(x) = Q_k(x) = \frac{k(2x+k)}{[(k-1)!]^2} \prod_{i=1}^{k-1} (x+i)^2,$$

and

$$\zeta(s, \mathbf{C}P^k) = \frac{1}{4^s} \sum_{n=1}^{\infty} \frac{Q_k(n)}{n^s (n+k)^s}.$$

If we introduce the numbers $a_{k,l}$ by

Definition 2. For $k = 1, 2, 3, \ldots$, the numbers $a_{k,l}$ are defined by the equations

$$\prod_{i=1}^{k-1} (x+i)^2 = \sum_{l=0}^{k-1} a_{k,l} (x^2 + kx)^l,$$

(in particular, $a_{k,k-1} = 1$, $a_{k,0} = [(k-1)!]^2$); then we can write

Lemma 4. *For* $k = 1, 2, 3, \ldots$,

$$\zeta(s, \mathbf{C}P^k) = \frac{1}{4^s} \frac{k}{[(k-1)!]^2} \sum_{l=0}^{k-1} a_{k,l} z_{\mathrm{even}}(s-l, 0, k, 1).$$

From this and Lemma 2, we immediately get

Proposition 3. *The function* $\zeta(s, \mathbf{C}P^k)$ *has simple poles at* $s = n$ *for* $n = 1, 2, \ldots, k$*, with residua*

$$\mathrm{Res}_1(\zeta(s, \mathbf{C}P^k), s = n) = \frac{1}{4^n} \frac{k}{[(k-1)!]^2} a_{k,n-1}.$$

4. Determinant and low dimensional cases. In this last section we show how to use Proposition 1 to compute the regularized determinant by considering the case of $k=3$. Next we give explicit results for dimensions 2, 3 and 4. Even if this is beyond the purposes of these notes, we observe that such results show that a unique formula for the determinant for a general k is likely to exist, and could be determined using the approach outlined here.

We have $k=3$, $h=1$, $a=0$ and $b=2$. Writing $z_3(s,c)$ as in Proposition 1, we decompose the second term as in Lemma 1, but the third one by developing the numerator. We get

$$\begin{aligned} z_3(s,c) &= \sum_{n=1}^{\infty} \frac{(cn+1)^2}{[cn(cn+2)]^s} \\ &= \frac{(c+1)^2}{2[c(c+2)]^s} + \int_1^{\infty} [cx(cx+2)]^{-s}\, dx + \int_1^{\infty} [cx(cx+2)]^{1-s}\, dx \\ &\quad - 2\int_0^{\infty} (c^2+y^2)^{-s/2}[(c+2)^2+y^2]^{-s/2} \\ &\quad \times \sin[-s(\theta+\phi)]\frac{dy}{e^{2\pi y}-1} - 4\int_0^{\infty} (c^2+y^2)^{1/2-s/2} \\ &\quad \times [(c+2)^2+y^2]^{-s/2}\sin[(1-s)\theta - s\phi]\frac{dy}{e^{2\pi y}-1} \\ &\quad - 2\int_0^{\infty} (c^2+y^2)^{1-s/2}[(c+2)^2+y^2]^{-s/2}\sin[(2-s)\theta - s\phi]\frac{dy}{e^{2\pi y}-1}, \end{aligned}$$

where $\theta = \arctan y/c$, $\phi = \arctan y/(c+1)$. The derivative at $s=0$ of the first term is immediate,

$$\frac{d}{ds}\frac{(c+1)^2}{2[c(c+2)]^s}\bigg|_{s=0} = -(c+1)\log\sqrt{c(c+1)},$$

while that of the second one can be computed with little effort by using the representation introduced in the proof of Lemma 3. We get

$$\begin{aligned} -\frac{8}{9c} - \frac{4}{3} - \frac{2c}{3} - \frac{2c^2}{9} + \left(\frac{1}{3c} + 1 + c + \frac{c^2}{3}\right)\log c \\ + \left(\frac{2}{3c} + 1 + c + \frac{c^2}{3}\right)\log\left(1+\frac{2}{c}\right). \end{aligned}$$

For the last three integrals we get

$$
\begin{gathered}
2\int_0^\infty (\theta+\phi)\frac{dy}{e^{2\pi y}-1}\\
+2\int_0^\infty \sqrt{c^2+y^2}\big\{\log(c^2+y^2)+\log[(c+2)^2+y^2]\big\}\sin\theta\frac{dy}{e^{2\pi y}-1}\\
+4\int_0^\infty \sqrt{c^2+y^2}(\theta+\phi)\cos\theta\frac{dy}{e^{2\pi y}-1}\\
+\int_0^\infty (c^2+y^2)\big\{\log(c^2+y^2)+\log[(c+2)^2+y^2]\big\}\sin 2\theta\frac{dy}{e^{2\pi y}-1}\\
+2\int_0^\infty \sqrt{c^2+y^2}\,(\theta+\phi)\cos 2\theta\frac{dy}{e^{2\pi y}-1}=
\end{gathered}
$$

that can be expressed in terms of derivatives of the Hurwitz zeta function,

$$
\begin{aligned}
&=\zeta_H'(-2,c)+\zeta_H'(-2,c+2)+2\zeta_H'(-1,c)-2\zeta_H'(-1,c+2)\\
&\quad+(3-2c)(\zeta_H'(0,c)+\zeta_H'(0,c+2))+\frac{44}{9}+\frac{4}{3}c-\frac{10}{3}c^2+\frac{2}{9}c^3\\
&\quad+\left(\frac{3}{2}-3c+\frac{3}{2}c^2-\frac{1}{3}c^3\right)\log c\\
&\quad+\left(-\frac{19}{6}+\frac{3}{2}c^2+c-\frac{1}{3}c^3\right)\log(c+2).
\end{aligned}
$$

Collecting and simplifying, we get the results shown below, together with the other low dimensional cases.

$$
\begin{gathered}
\zeta(0,S^2)=-\frac{2}{3}=-0.\bar{6},\\
\zeta'(0,S^2)=4\zeta_R'(-1)-\frac{1}{2}=-1.162\ldots;\\
\zeta(0,\mathbf{R}P^2)=-\frac{11}{3}=-3.\bar{6}\ldots,\\
\zeta'(0,\mathbf{R}P^2)=4\zeta_R'(-1)-3\log 3+\frac{5}{2}=-1.459\ldots;\\
\zeta(0,S^3)=-1,\\
\zeta'(0,S^3)=2\zeta_R'(-2)+2\zeta_R'(0)+\log 2=-1.206\ldots;
\end{gathered}
$$

$$\zeta(0,\mathbf{R}P^3) = -\frac{1}{2} = -0.5,$$
$$\zeta'(0,\mathbf{R}P^3) = 2\zeta_R'(-2) - 2\zeta_R'(0) - \frac{13}{6}\log 2 + 2\log 3 - 8 = -5.527\dots;$$
$$\zeta(0,S^4) = -\frac{61}{90} = -0.6\bar{7},$$
$$\zeta'(0,S^4) = \frac{2}{3}\zeta_R'(-3) + \frac{13}{3}\zeta_R'(-1) + \log 3 - \frac{15}{16} = -0.5516\dots,$$
$$\zeta(0,\mathbf{R}P^4) = \frac{7}{45} = 0.1\bar{5},$$
$$\zeta'(0,\mathbf{R}P^4) = \frac{2}{3}\zeta_R'(-3) + \frac{13}{3}\zeta_R'(-1) + \frac{13}{6}\log 2 + \log 3 - \frac{35}{16}\log 5 + \frac{45}{16} = -4.684\dots,$$
$$\zeta(0,\mathbf{C}P^2) = -\frac{89}{30} = -2.9\bar{6},$$
$$\zeta'(0,\mathbf{C}P^2) = 8\zeta_R'(-3) + 24\zeta_R'(-1) + \frac{149}{15}\log 2 - 4\log 3 - \frac{203}{12} = -18.353\dots.$$

The values for $\zeta'(0,S^k)$ agree with the ones provided by [**8**, Section 4]; the value for $\zeta(0,S^2)$ confirms the one originally given by [**15**], against that of [**9**], see also Section 5 for further remarks; the values for $\zeta(0,S^3)$ agree with the one computed from the formula in the corollary of Section 3 of [**7**], see also Section 3 of [**6**]. Numerical computations were done by using Maple.

5. Remarks. To conclude, some remarks are in order. The first one concerns the one-dimensional case, where we get the following relations

$$\zeta(s,S^1) = 2\zeta_R(2s),$$
$$\zeta(s,\mathbf{R}P^1) = 2^{-2s}\zeta(s,S^1),$$
$$\zeta(s,\mathbf{C}P^1) = 2^{-2s}\zeta(s,S^2).$$

The first two relations are well known, while the third one can be easily read out from the results in the sections above. We can then complete the table in Section 4:

$$\zeta(0,S^1) = -1, \quad \zeta'(0,S^1) = 4\zeta_R'(0) = -3.676\dots;$$
$$\zeta(0,\mathbf{R}P^1) = -1, \quad \zeta'(0,\mathbf{R}P^1) = 4\zeta_R'(0) + 2\log 2 = -2.29\dots;$$
$$\zeta(0,\mathbf{C}P^1) = -\tfrac{2}{3} = -0.\bar{6}, \quad \zeta'(0,\mathbf{C}P^1) = 4\zeta_R'(0) + \tfrac{4}{3}\log 2 - \tfrac{1}{2} = -3.252\dots.$$

Also notice that, while the value of the zeta function at $s = 0$ seems to depend only on the topology, the value of its derivative, and hence the regularized determinant, does not. For what concerns the first statement, just recall that, for any closed Riemannian manifold M of dimension m,

$$\zeta(0, M) = a_m(M) - \dim\ker\Delta_M,$$

where Δ_M is the Laplacian operator in the standard metric and $a_m(M)$ is the coefficient of the constant term in the asymptotic expansion of the heat operator $e^{-t\Delta_M}$, see, for example, [**14**]. In particular, compare with the value computed above for $\zeta(0, S^2)$, where $a_2(S^2) = 1/(24\pi)\int_{S^2} R_{S^2}(x)\, d\,\mathrm{vol}\,(x) = 1/3$.

Our final remark concerns the possibility of using the approach introduced to treat the case of the Laplacian coupled with a constant potential. This is an important tool to face the problem of a generic potential, see [**11**] for the one-dimensional case. For the sake of simplicity, we restrict ourselves to the case of the two-sphere. The eigenvalues are then $\lambda_n = n(n+1) + q^2$, and the zeta function is

$$\zeta(s, \Delta_{S^2} + q^2) = \sum_{n=1}^{\infty} \frac{2n+1}{[n(n+1) + q^2]^s}.$$

By expanding the power of the binomial (for finite q), this becomes

$$\zeta(s, S^2) - s\zeta(s+1, S^2)q^2 + \sum_{k=2}^{\infty} \binom{-s}{k} \zeta(s+k, S^2) q^{2k}.$$

We can compute

$$\zeta(0, \Delta_{S^2} + q^2) = \zeta(0, S^2) - q^2.$$
$$\zeta'(0, \Delta_{S^2} + q^2) = \zeta'(0, S^2) - 2\gamma q^2 - \log G(q),$$

where $\gamma = -\psi(1)$ and $G(q)$ is the integral function of order 2 and zeros $\pm i\sqrt{n(n+1)}$, of multiplicity $2n+1$, defined by the canonical product of genus 2:

$$G(q) = \prod_{n=1}^{\infty} \left[1 + \frac{q^2}{n(n+1)}\right]^{2n+1} e^{-\frac{2n+1}{n(n+1)}q^2}.$$

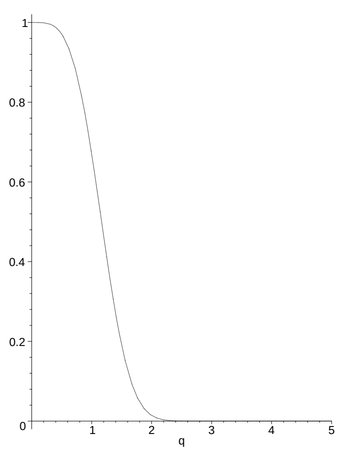

FIGURE 1. $G(q)$.

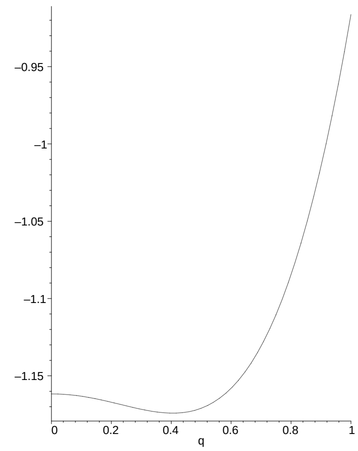

FIGURE 2. $\zeta'(0, \Delta_{S^2} + q^2)$.

The main difference with respect to the one-dimensional case is due to the slower rate of increasing of the eigenvalues; this reflects in the appearance of a further contribution coming from the singularity of $\zeta(s, S^2)$ at $s = 1$, through the term $s\zeta(s+1, S^2)$. In Figures 1 and 2 the functions $G(q)$ and $\zeta'(0, \Delta_{S^2} + q^2)$ are plotted, where the infinite product is approximated by the product on the first 1000 terms.

ENDNOTES

1. Observe that $f(0, 0, a, b, c) = 0$.

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