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ON SUMS OF TWO SQUARES AND SUMS OF TWO TRIANGULAR NUMBERS

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ABSTRACT. For each integer $n \geq 0$, $r_2(n)[t_2(n)]$ denotes the number of representations of n by sums of two squares (two triangular numbers). Similarities and differences of the two functions r_2 and t_2 are described, with the major contribution being an apparently new recursive determination of t_2.

1. Introduction. We begin with a definition.

Definition 1.1. As usual, $\mathbf{P} := \{1, 2, 3, \dots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$ and $\mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}$. Then for each $n \in \mathbf{N}$,

$$\begin{aligned} r_2(n) &:= |\{(x,y) \in \mathbf{Z}^2 \mid n = x^2 + y^2\}|, \\ t_2(n) &:= |\{(x,y) \in \mathbf{N}^2 \mid n = x(x+1)/2 + y(y+1)/2\}|. \end{aligned}$$

Also for each $n \in \mathbf{P}$ and each $i \in \{1, 3\}$,

$$d_i(n) := \sum_{\substack{d \mid n \\ d \equiv i \pmod 4}} 1.$$

That the functions r_2 and t_2 are closely related is revealed by the next two theorems and their obvious corollary.

Theorem 1.2 (Jacobi). *For each $n \in \mathbf{P}$,*

$$r_2(n) = 4\{d_1(n) - d_3(n)\}.$$

(*Of course,* $r_2(0) = 1$.)

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Theorem 1.3. *For each* $n \in \mathbf{N}$,

$$t_2(n) = d_1(4n+1) - d_3(4n+1).$$

Corollary 1.4. *For each* $n \in \mathbf{N}$,

$$r_2(4n+1) = 4t_2(n).$$

These results belong to *multiplicative* number theory in the sense that evaluation of $d_1(n) - d_3(n)$, $n \in \mathbf{P}$, entails factorization of n and subsequent appeal to the fundamental theorem of arithmetic. In [**1**, pp. 213–214], the author derived the following *additive* recursive determination of the function r_2.

Theorem 1.5. *For each* $n \in \mathbf{N}$,

$$\sum_{k\geq 0}(-1)^{k(k+1)/2}\, r_2(n-k(k+1)/2) = \begin{cases} (-1)^{m(m+3)/2}(2m+1) & \text{if } n = m(m+1)/2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Put $r_2(x) := 0$, *whenever* $x < 0$.

The major objective of this note is to show that the function t_2 also has an *additive* recursive determination. This is accomplished by the following theorem.

Theorem 1.6. *For each* $n \in \mathbf{N}$,

$$t_2(n) + 2\sum_{k\geq 1}(-1)^k t_2(n-k^2) = \begin{cases} (-1)^m(2m+1) & \text{if } n = m(m+1), \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Put $t_2(x) := 0$ *whenever* $x < 0$.

Proof of this result is supplied in Section 2. For a proof of Jacobi's Theorem 1.2, see [**3**, pp. 241–243], and for proof of Theorem 1.3, see [**2**, pp. 175–176].

2. Proof of Theorem 1.6. Our proof is based on the following three identities, each of which is valid for all complex numbers x such that $|x| < 1$.

$$\prod_1^\infty (1 - x^{2n})(1 - x^{2n-1})^2 = 1 + 2\sum_1^\infty (-1)^k x^{k^2}, \tag{2.1}$$

$$\prod_1^\infty (1 - x^{2n})(1 - x^{2n-1})^{-1} = \sum_0^\infty x^{n(n+1)/2}, \tag{2.2}$$

$$\prod_1^\infty (1 - x^n)^3 = \sum_0^\infty (-1)^k (2k+1) x^{k(k+1)/2}. \tag{2.3}$$

Identities (2.1) and (2.2) are due to Gauss, while (2.3) is due to Jacobi. For proofs of all of them, see [**3**, pp. 282–285]. In passing we observe that the square of the right-hand side of (2.2) generates the sequence $t_2(n)$, $n \in \mathbf{N}$.

We square (2.2) and multiply the resulting identity by (2.1) to get

$$\begin{aligned}\sum_{m=0}^\infty (-1)^m (2m+1) x^{m(m+1)} &= \prod_{n=1}^\infty (1 - x^{2n})^3 \\ &= \sum_{j=0}^\infty t_2(j) x^j \Big\{ 1 + 2\sum_{k=1}^\infty (-1)^k x^{k^2} \Big\} \\ &= \sum_{n=0}^\infty t_2(n) x^n + 2\sum_{n=1}^\infty x^n \sum_{k\ge 1} (-1)^k t_2(n-k^2)\end{aligned}$$

(In the first step we effected the substitution $x \to x^2$ in (2.3).) Now, equating coefficients of x^n, $n \in \mathbf{N}$, we prove our theorem.

Recall that a *rectangular number* is one of the form $m(m+1)$, $m \in \mathbf{N}$. Our next result is then an immediate consequence of Theorem 1.6.

Corollary 2.1. *For each $n \in \mathbf{N}$, $t_2(n)$ is odd if and only if n is a rectangular number.*

Of course, this result can be established directly. If (i) $n \in \mathbf{N}$ is a rectangular number, so that $n = m(m+1)$, for some $m \in \mathbf{N}$, then

$n = m(m+1)/2 + m(m+1)/2$. And all other pairs $(x,y) \in \mathbf{N}^2$, if any, satisfy the condition $(x,y) \neq (y,x)$. Accordingly, these pairs are paired as $(x,y), (y,x)$ to yield two distinct representations of n:

$$n = x(x+1)/2 + y(y+1)/2, \quad n = y(y+1)/2 + x(x+1)/2.$$

Clearly the count of $(m,m), (x_1,y_1), (y_1,x_1), \dots, (x_r,y_r), (y_r,x_r)$ is odd. If (ii) $n \in \mathbf{N}$ is *not* a rectangular number, then *all* pairs $(x,y) \in \mathbf{N}^2$, possibly 0 in number, satisfy the condition $(x,y) \neq (y,x)$. In any case, the count of these $(x_1,y_1), (y_1,x_1), \dots, (x_r,y_r), (y_r,x_r)$ is even.

The following brief table is compiled solely on the strength of Theorem 1.6.

TABLE 2.1.

n	$t_2(n)$	n	$t_2(n)$	n	$t_2(n)$
0	1	13	2	26	0
1	2	14	0	27	2
2	1	15	2	28	2
3	2	16	4	29	2
4	2	17	0	30	1
5	0	18	2	31	4
6	3	19	0	32	0
7	2	20	1	33	0
8	0	21	4	34	2
9	2	22	2	35	0
10	2	23	0	36	4
11	2	24	2	37	2
12	1	25	2	38	2

Concluding remarks. We began this discussion by observing the close relation between the functions $r_2 : \mathbf{N} \to \mathbf{N}$ and $t_2 : \mathbf{N} \to \mathbf{N}$. This is vividly demonstrated by Corollary 1.4. However, we should point out that the two functions differ markedly with respect to parity of their values. To be sure, the function r_2 has exactly one odd value (i.e., $r_2(0) = 1$ and $r_2(n) \equiv 0 \pmod 4$, for each $n \in \mathbf{P}$), while t_2 has infinitely many odd values and infinitely many even values. (Corollary 2.1 gives a precise statement.)

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REFERENCES

1. J.A. Ewell, *On the counting function for sums of two squares*, Acta Arith. **40** (1982), 213–215.

2. ———, *On representations of numbers by sums of two triangular numbers*, Fibonacci Quart. **30** (1992), 175–178.

3. G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, 4th ed., Clarendon Press, Oxford, 1960.

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