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INTEGRAL TRANSFORMS OF FUNCTIONALS IN $L_2(C_0[0,T])$

BYOUNG SOO KIM AND DAVID SKOUG

ABSTRACT. In this paper we give a necessary and sufficient condition that a functional $F(x)$ in $L_2(C_0[0,T])$ has an integral transform $\mathcal{F}_{\alpha,\beta}F(x)$ which also belongs to $L_2(C_0[0,T])$.

1. Introduction. Let $C_0[0,T]$ denote one-parameter Wiener space; that is, the space of all **R**-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote Wiener measure. $(C_0[0,T],\mathcal{M},m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)m(dx).$$

Let $L_2(C_0[0,T])$ be the space of all real or complex-valued functionals F satisfying

$$\int_{C_0[0,T]} |F(x)|^2 m(dx) < \infty. \tag{1.1}$$

Let $K = K[0,T]$ be the space of **C**-valued continuous functions defined on $[0,T]$ which vanish at $t=0$. Next we state the definition of the integral transform $\mathcal{F}_{\alpha,\beta}$ introduced in [**6**] and used in [**5**], for functionals F defined on K.

Definition 1. Let F be a functional defined on K. For each pair of nonzero complex numbers α and β, the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$\mathcal{F}_{\alpha,\beta}F(y) = \int_{C_0[0,T]} F(\alpha x+\beta y)m(dx), \quad y \in K, \tag{1.2}$$

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if it exists.

Remark 1. (i) When $\alpha = 1$ and $\beta = i$, $\mathcal{F}_{\alpha,\beta}F$ is the Fourier-Wiener transform introduced by Cameron in [**1**] and used by Cameron and Martin in [**2**].

(ii) When $\alpha = \sqrt{2}$ and $\beta = i$, $\mathcal{F}_{\alpha,\beta}F$ is the modified Fourier-Wiener transform introduced by Cameron and Martin in [**3**].

In this paper we give a necessary and sufficient condition that a functional $F(x)$ in $L_2(C_0[0,T])$ has an integral transform $\mathcal{F}_{\alpha,\beta}F$ also belonging to $L_2(C_0[0,T])$.

2. Integral transforms of the Fourier-Hermite functionals. For $n = 0, 1, 2, \dots$, let $H_n(u)$ denote the Hermite polynomial

$$H_n(u) = (-1)^n (n!)^{-1/2} e^{u^2/2} \frac{d^n}{du^n}(e^{-u^2/2}). \tag{2.1}$$

Then, as is well known, the set

$$\{(2\pi)^{-1/4} H_n(u) e^{-u^2/4} : n = 0, 1, 2, \dots\} \tag{2.2}$$

is a CON set on $\mathbf{R}$.

Let $\{a_p(t) : p = 1, 2, \dots\}$ be a CON set of functions of bounded variation on $[0,T]$. Define

$$\Phi_{n,p}(x) = H_n\bigg(\int_0^T a_p(t)\,dx(t)\bigg), \quad n = 0, 1, 2, \dots, \quad p = 1, 2, \dots, \tag{2.3}$$

and

$$\Psi_{n_1,\dots,n_p}(x) = \Psi_{n_1,\dots,n_p,0,\dots,0}(x) = \Phi_{n_1,1}(x) \cdots \Phi_{n_p,p}(x). \tag{2.4}$$

The functionals in (2.4) are called the Fourier-Hermite functionals.

In [**4**], Cameron and Martin showed that the Fourier-Hermite functionals form a CON set in $L_2(C_0[0,T])$. That is to say that every functional $F(x)$ in $L_2(C_0[0,T])$ has a Fourier-Hermite development which converges in the $L_2(C_0[0,T])$ sense to $F(x)$; namely, that

$$F(x) = \operatorname*{l.i.m.}_{N\to\infty} \sum_{n_1,\dots,n_N=0}^{N} A^F_{n_1,\dots,n_N} \Psi_{n_1,\dots,n_N}(x), \tag{2.5}$$

where $A^F_{n_1,\dots,n_N}$ is the Fourier-Hermite coefficient,

$$A^F_{n_1,\dots,n_N} = \int_{C_0[0,T]} F(x)\Psi_{n_1,\dots,n_N}(x)m(dx). \tag{2.6}$$

Throughout this paper, in order to ensure that various integrals exist, we will assume that $\beta = a + bi$ is a nonzero complex number satisfying the inequality

$$\mathrm{Re}\,(1-\beta^2) = 1 + b^2 - a^2 > 0. \tag{2.7}$$

Note that $\mathrm{Re}\,(1-\beta^2) = 1+b^2-a^2 > 0$ if and only if the point $(a, b) \in \mathbf{R}^2$ lies in the open region, determined by the hyperbola $a^2 - b^2 = 1$, containing the b-axis. Hence, for all $|\beta| \le 1$, $\beta \ne \pm 1$, $\mathrm{Re}\,(1-\beta^2) > 0$. Next we define

$$\alpha \equiv \sqrt{1-\beta^2}, \quad -\pi/4 < \arg(\alpha) < \pi/4 \tag{2.8}$$

and note that $\alpha^2 + \beta^2 = 1$ and $\mathrm{Re}\,(\alpha^2) = \mathrm{Re}\,(1-\beta^2) > 0$.

Our first lemma plays a key role in finding the integral transform of the Fourier-Hermite functionals.

Lemma 1. *Let β be a nonzero complex number satisfying inequality* (2.7) *and let α be defined by equation* (2.8). *Let $r \in \mathbf{R}$. Then, for $n = 0, 1, 2, \dots$,*

$$\int_{\mathbf{R}} H_n(u)\exp\left\{-\frac{1}{2\alpha^2}(u-r\beta)^2\right\}du = \alpha\sqrt{2\pi}\,\beta^n H_n(r). \tag{2.9}$$

Proof. Since $H_n(u)$ is a polynomial of degree n and, since $\mathrm{Re}\,(\alpha^2) > 0$, the integral exists and

$$\begin{aligned} I_n &\equiv \int_{\mathbf{R}} H_n(u)\exp\left\{-\frac{1}{2\alpha^2}(u-r\beta)^2\right\}du \\ &= (-1)^n(n!)^{-1/2}\int_{\mathbf{R}} e^{u^2/2}\frac{d^n}{du^n}(e^{-u^2/2})\exp\left\{-\frac{1}{2\alpha^2}(u-r\beta)^2\right\}du \\ &= (-1)^n(n!)^{-1/2}e^{r^2/2}\int_{\mathbf{R}}\exp\left\{-\frac{\beta^2}{2\alpha^2}\left(u-\frac{r}{\beta}\right)^2\right\}\frac{d^n}{du^n}(e^{-u^2/2})\,du. \end{aligned}$$

Then, integrating by parts n times, we obtain

$$\begin{aligned} I_n &= (n!)^{-1/2} e^{r^2/2} \int_{\mathbf{R}} e^{-u^2/2} \frac{d^n}{du^n} \bigg(\exp\bigg\{ -\frac{\beta^2}{2\alpha^2} \Big(u - \frac{r}{\beta} \Big)^2 \bigg\} \bigg)\, du \\ &= (n!)^{-1/2} (-\beta)^n e^{r^2/2} \int_{\mathbf{R}} e^{-u^2/2} \frac{d^n}{dr^n} \bigg(\exp\bigg\{ -\frac{\beta^2}{2\alpha^2} \Big(u - \frac{r}{\beta} \Big)^2 \bigg\} \bigg)\, du \\ &= (n!)^{-1/2} (-\beta)^n e^{r^2/2} \frac{d^n}{dr^n} \bigg(\int_{\mathbf{R}} \exp\bigg\{ -\frac{1}{2\alpha^2} + \frac{r\beta}{\alpha^2} u - \frac{r^2}{2\alpha^2} \bigg\}\, du \bigg) \\ &= (n!)^{-1/2} (-\beta)^n \alpha \sqrt{2\pi}\, e^{r^2/2} \frac{d^n}{dr^n} (e^{-r^2/2}) \\ &= \alpha \sqrt{2\pi}\, \beta^n H_n(r), \end{aligned}$$

which completes the proof of Lemma 1. □

Remark 2. Equation (2.9) holds for all $r \in \mathbf{C}$ since $H_n(r)$ is a polynomial of degree n and so both sides of equation (2.9) are analytic functions of r throughout $\mathbf{C}$.

Next, using Lemma 1, we obtain a formula for the integral transform of the Fourier-Hermite functionals given by equation (2.4).

Theorem 2. *Let α and β be as in Lemma* 1. *Then, for each $y \in K$,*

$$\mathcal{F}_{\alpha,\beta} \Psi_{n_1,\dots,n_p}(y) = \beta^{n_1+\cdots+n_p} \Psi_{n_1,\dots,n_p}(y). \tag{2.10}$$

Proof. For $j = 1, 2, \dots$, let $r_j \equiv \int_0^T a_j(t)\, dy(t) = \langle a_j, y \rangle$, which we know exists for all $y \in K$ since a_j is of bounded variation on $[0, T]$. Then for every $y \in K$,

$$\begin{aligned} \mathcal{F}_{\alpha,\beta} \Psi_{n_1,\dots,n_p}(y) &= \int_{C_0[0,T]} \Psi_{n_1,\dots,n_p}(\alpha x + \beta y) m(dx) \\ &= \int_{C_0[0,T]} H_{n_1}(\alpha \langle a_1, x \rangle + \beta \langle a_1, y \rangle) \cdots \\ &\qquad \cdots H_{n_p}(\alpha \langle a_p, x \rangle + \beta \langle a_p, y \rangle) m(dx) \\ &= \prod_{j=1}^p \Big[(2\pi)^{-1/2} \int_{\mathbf{R}} H_{n_j}(\alpha u_j + \beta r_j) e^{-u_j^2/2}\, du_j \Big]. \end{aligned}$$

Note that for all positive α and all $\beta \in \mathbf{C}$,

$$\int_{\mathbf{R}} H_n(\alpha u + \beta r) e^{-u^2/2}\, du = \frac{1}{\alpha} \int_{\mathbf{R}} H_n(u) e^{-(u-r\beta)^2/2\alpha^2}\, du.$$

But each side of the above expression is an analytic function of α throughout the region $\{\alpha \in \mathbf{C} : \mathrm{Re}\,(\alpha^2) > 0\}$. Hence, by the uniqueness theorem for analytic functions, the above equality holds for all α with $\mathrm{Re}\,(\alpha^2) > 0$ and all $\beta \in \mathbf{C}$ and so

$$\mathcal{F}_{\alpha,\beta} \Psi_{n_1,\dots,n_p}(y) = \prod_{j=1}^{p} \Big[(2\pi\alpha^2)^{-1/2} \int_{\mathbf{R}} H_{n_j}(u_j) e^{-(u_j - r_j\beta)^2/2\alpha^2}\, du_j\Big].$$

Then, using Lemma 1, we obtain equation (2.10), the desired result. □

Our first corollary follows immediately from equation (2.10) and the fact that $\|\Psi_{n_1,\dots,n_p}\|_2 = 1$.

Corollary 3. *Let* α *and* β *be as in Lemma* 1. *Then*

$$\|\mathcal{F}_{\alpha,\beta} \Psi_{n_1,\dots,n_p}\|_2 = |\beta|^{n_1+\dots+n_p}. \tag{2.11}$$

Corollary 4. *Choosing* $\alpha = \sqrt{2}$ *and* $\beta = i$ *in equation* (2.10) *we obtain Lemma* 5.1 [**3**, *p.* 104]; *namely, that*

$$\mathcal{F}_{\sqrt{2},i}\, \Psi_{n_1,\dots,n_p}(y) = i^{n_1+\dots+n_p} \Psi_{n_1,\dots,n_p}(y) \tag{2.12}$$

for all $y \in K$.

3. Integral transforms of functionals belonging to $L_2(C_0[0,T])$.

For $F \in L_2(C_0[0,T])$ let (2.5) denote the Fourier-Hermite expression of $F(x)$ with the Fourier-Hermite coefficients $A^F_{n_1,\dots,n_N}$ given by equation (2.6). For $N = 1, 2, \dots,$ let

$$F_N(x) = \sum_{n_1,\dots,n_N=0}^{N} A^F_{n_1,\dots,n_N} \Psi_{n_1,\dots,n_N}(x). \tag{3.1}$$

Then, by Theorem 2, we know that for each $N = 1, 2, \ldots$, $\mathcal{F}_{\alpha,\beta}F_N$ exists for all α and β as in Lemma 1, $\mathcal{F}_{\alpha,\beta}F_N$ is an element of $L_2(C_0[0,T])$ such that, for each $y \in K$,

$$\mathcal{F}_{\alpha,\beta}F_N(y) = \sum_{n_1,\ldots,n_N=0}^{N} A^F_{n_1,\ldots,n_N}\beta^{n_1+\cdots+n_N}\Psi_{n_1,\ldots,n_N}(y). \tag{3.2}$$

Furthermore,

$$\|\mathcal{F}_{\alpha,\beta}F_N\|_2^2 = \sum_{n_1,\ldots,n_N=0}^{N} |A^F_{n_1,\ldots,n_N}\beta^{n_1+\cdots n_N}|^2. \tag{3.3}$$

Definition 2. Let $F \in L_2(C_0[0,T])$ be given by (2.5). Then, for each nonzero complex numbers α and β, we define the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F to be

$$\mathcal{F}_{\alpha,\beta}F(x) = \operatorname*{l.i.m.}_{N\to\infty} \mathcal{F}_{\alpha,\beta}F_N(x), \quad x \in C_0[0,T] \tag{3.4}$$

if it exists; that is to say, if

$$\lim_{N\to\infty}\int_{C_0[0,T]} |\mathcal{F}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_N(x)|^2 m(dx) = 0. \tag{3.5}$$

Lemma 5. *Let $F \in L_2(C_0[0,T])$ be given by equation* (2.5) *with Fourier-Hermite coefficients given by* (2.6)*. Let α and β be as in Lemma* 1 *and assume that $\mathcal{F}_{\alpha,\beta}F$ exists and is in $L_2(C_0[0,T])$. Then*

$$A^{\mathcal{F}_{\alpha,\beta}F}_{n_1,\ldots,n_N} = A^F_{n_1,\ldots,n_N}\beta^{n_1+\cdots+n_N} \tag{3.6}$$

for each $N = 1, 2, \ldots$.

Proof. Fix $N = 1, 2, \ldots$. For any given $\varepsilon > 0$, take a natural number M satisfying $\|\mathcal{F}_{\alpha,\beta}F - \mathcal{F}_{\alpha,\beta}F_M\|_2 < \varepsilon$ and $M \geq N$. Then we have

$$\begin{aligned}
&|A^{\mathcal{F}_{\alpha,\beta}F}_{n_1,\ldots,n_N} - A^F_{n_1,\ldots,n_N}\beta^{n_1+\cdots+n_N}| \\
&= \left|\int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}F(x)\Psi_{n_1,\ldots,n_N}(x)m(dx) - A^F_{n_1,\ldots,n_N}\beta^{n_1+\cdots+n_N}\right| \\
&\leq \left|\int_{C_0[0,T]} [\mathcal{F}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_M(x)]\Psi_{n_1,\ldots,n_N}(x)m(dx)\right| \\
&\quad + \left|\int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}F_M(x)\Psi_{n_1,\ldots,n_N}(x)m(dx) - A^F_{n_1,\ldots,n_N}\beta^{n_1+\cdots+n_N}\right|.
\end{aligned}$$

But, by the Hölder inequality,

$$\left|\int_{C_0[0,T]}[\mathcal{F}_{\alpha,\beta}F(x)-\mathcal{F}_{\alpha,\beta}F_M(x)]\Psi_{n_1,\dots,n_N}(x)m(dx)\right| \\ \le \|\mathcal{F}_{\alpha,\beta}F-\mathcal{F}_{\alpha,\beta}F_M\|_2<\varepsilon$$

and from (3.2) we know that

$$\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}F_M(x)\Psi_{n_1,\dots,n_N}(x)m(dx)=A^F_{n_1,\dots,n_N}\beta^{n_1+\cdots+n_N}.$$

Hence

$$|\mathcal{A}^{\mathcal{F}_{\alpha,\beta}F}_{n_1,\dots,n_N}-A^F_{n_1,\dots,n_N}\beta^{n_1+\cdots+n_N}|<\varepsilon$$

which establishes equation (3.6). □

The following theorem is our main result. It gives a necessary and sufficient condition that a functional F in $L_2(C_0[0,T])$ has an integral transform $\mathcal{F}_{\alpha,\beta}F$ belonging to $L_2(C_0[0,T])$.

Theorem 6. *Let $F\in L_2(C_0[0,T])$ be given by equation* (2.5) *with Fourier-Hermite coefficients given by* (2.6). *Let α and β be as in Lemma* 1. *Then $\mathcal{F}_{\alpha,\beta}F$ exists and is an element of $L_2(C_0[0,T])$ if and only if*

$$\lim_{N\to\infty}\sum_{n_1,\dots,n_N=0}^{N}|A^F_{n_1,\dots,n_N}\beta^{n_1+\cdots+n_N}|^2<\infty. \tag{3.7}$$

Furthermore, if (3.7) *holds, then the Fourier-Hermite expression of $\mathcal{F}_{\alpha,\beta}F$ is given by*

$$\mathcal{F}_{\alpha,\beta}F(y)=\underset{N\to\infty}{\text{l.i.m.}}\sum_{n_1,\dots,n_N=0}^{N}A^F_{n_1,\dots,n_N}\beta^{n_1+\cdots+n_N}\Psi_{n_1,\dots,n_N}(y). \tag{3.8}$$

Proof. Assume that $\mathcal{F}_{\alpha,\beta}F$ exists and is an element of $L_2(C_0[0,T])$. By (3.5) we have that, for any given $\varepsilon>0$,

$$\int_{C_0[0,T]}|\mathcal{F}_{\alpha,\beta}F(x)-\mathcal{F}_{\alpha,\beta}F_N(x)|^2m(dx)<\varepsilon$$

for sufficiently large N, and so

$$\begin{aligned}\bigg(\sum_{n_1,\dots,n_N=0}^{N}|A^F_{n_1,\dots,n_N}\beta^{n_1+\dots+n_N}|^2\bigg)^{1/2}&\\ &= \|\mathcal{F}_{\alpha,\beta}F_N\|_2\\ &\le \|\mathcal{F}_{\alpha,\beta}F\|_2+\|\mathcal{F}_{\alpha,\beta}-\mathcal{F}_{\alpha,\beta}F_N\|_2\\ &\le \|\mathcal{F}_{\alpha,\beta}F\|_2+\varepsilon.\end{aligned}$$

Hence we have

$$\lim_{N\to\infty}\sum_{n_1,\dots,n_N=0}^{N}|A^F_{n_1,\dots,n_N}\beta^{n_1+\dots+n_N}|^2\le\|\mathcal{F}_{\alpha,\beta}F\|_2^2<\infty.$$

To prove the converse, suppose that (3.7) holds. Let $M>N$, let

$$I_M=\{(n_1,\dots,n_M):n_1,\dots,n_M=0,1,\dots,M\},$$

and let

$$\begin{aligned}I_N=\{(n_1,\dots,n_M):n_1,\dots,n_N=0,1,\dots,N&\\ \text{and } n_{N+1}=\dots=n_M=0\}.&\end{aligned}$$

Then

$$\begin{aligned}&\int_{C_0[0,T]}|\mathcal{F}_{\alpha,\beta}F_M(x)-\mathcal{F}_{\alpha,\beta}F_N(x)|^2m(dx)\\ &=\int_{C_0[0,T]}\Big|\sum_{I_M-I_N}A^F_{n_1,\dots,n_M}\beta^{n_1+\dots+n_M}\Psi_{n_1,\dots,n_M}(x)\Big|^2m(dx)\\ &=\sum_{I_M-I_N}|A^F_{n_1,\dots,n_M}\beta^{n_1+\dots+n_M}|^2\\ &=\sum_{n_1,\dots,n_M=0}^{M}|A^F_{n_1,\dots,n_M}\beta^{n_1+\dots+n_M}|^2-\sum_{n_1,\dots,n_N=0}^{N}|A^F_{n_1,\dots,n_N}\beta^{n_1+\dots+n_N}|^2\end{aligned}$$

which goes to 0 as $M,N\to\infty$. Hence $\{\mathcal{F}_{\alpha,\beta}F_N\}$ is a Cauchy sequence in $L_2(C_0[0,T])$ and, since $L_2(C_0[0,T])$ is complete,

$$\mathcal{F}_{\alpha,\beta}F(x)=\underset{N\to\infty}{\text{l.i.m.}}\,\mathcal{F}_{\alpha,\beta}F_N(x),\quad x\in C_0[0,T]$$

exists and is an element of $L_2(C_0[0,T])$. □

Corollary 7. *Let* F, α *and* β *be as in Theorem* 6. *Furthermore, assume that* $|\beta| \le 1$. *Then* $\mathcal{F}_{\alpha,\beta}F$ *exists, belongs to* $L_2(C_0[0,T])$, *and*

$$\begin{aligned} \|\mathcal{F}_{\alpha,\beta}F\|_2^2 &= \lim_{N\to\infty} \sum_{n_1,\dots,n_N=0}^{N} |A^F_{n_1,\dots,n_N} \beta^{n_1+\dots+n_N}|^2 \\ &\le \lim_{N\to\infty} \sum_{n_1,\dots,n_N=0}^{N} |A^F_{n_1,\dots,n_N}|^2 = \|F\|_2^2. \end{aligned} \tag{3.9}$$

In addition,

$$\|\mathcal{F}_{\alpha,\beta}F\|_2 = \|F\|_2 \tag{3.10}$$

if and only if $|\beta| = 1$.

Corollary 8. *Let* p *be a fixed positive integer, and let* $F(x) = f(\langle a_1, x\rangle, \dots, \langle a_p, x\rangle)$ *where* f *is such that*

$$f(u_1,\dots,u_p) \exp\Big\{-\frac{1}{4}\sum_{j=1}^{p} u_j^2\Big\} \in L_2(\mathbf{R}^p). \tag{3.11}$$

Let α *and* β *be as in Lemma* 1. *Then the integral transform* $\mathcal{F}_{\alpha,\beta}F$ *exists and is an element of* $L_2(C_0[0,T])$. (*Note that in this case we don't need the restriction* $|\beta| \le 1$).

Proof. Since zero is an admissible value for each n_j in the Fourier-Hermite coefficient $A^F_{n_1,\dots,n_N}$ of F, we need only consider the two cases $N = p$ and $N > p$. Then, using (2.2), (2.3), (2.4) and (2.6), a direct calculation shows that, for all nonnegative indices $n_1, \dots, n_N$,

$$A^F_{n_1,\dots,n_N} = \begin{cases} 0 \quad N > p \quad \text{and} \quad n_N > 0, \\ \Big(\frac{1}{2\pi}\Big)^{p/2} \int_{\mathbf{R}^p} f(u_1,\dots,u_p) H_{n_1}(u_1)\dots \\ H_{n_p}(u_p) \exp\Big\{-\frac{1}{2}\sum_{j=1}^{p} u_j^2\Big\} d\vec{u} \quad N = p. \end{cases}$$

Moreover, by (3.11), we know that $|A^F_{n_1,\dots,n_p}| < \infty$ for all nonnegative indices $n_1,\dots,n_p$. Hence,

$$\lim_{N\to\infty}\sum_{n_1,\dots,n_N=0}^{N} |A^F_{n_1,\dots,n_N}\beta^{n_1+\cdots+n_N}|^2 \le |\beta|^{2p^2}\sum_{n_1,\dots,n_p=0}^{p}|A^F_{n_1,\dots,n_p}|^2 < \infty,$$

and so, by Theorem 6, $\mathcal{F}_{\alpha,\beta}F$ exists and is an element of $L_2(C_0[0,T])$. □

Next, choosing $\alpha = \sqrt{2}$ and $\beta = i$, we obtain the main theorem of [**3**].

Corollary 9. *Every functional $F(x) \in L_2(C_0[0,T])$ has a Fourier-Wiener transform $G(y) \in L_2(C_0[0,T])$. The functional $G(y)$ has $F(-x)$ as its transform and F and G satisfy Plancherel's relation*

$$\int_{C_0[0,T]} |F(x)|^2 m(dx) = \int_{C_0[0,T]} |G(y)|^2 m(dy). \tag{3.12}$$

Proof. Using Corollary 7 and Theorem 6, we obtain that $G(y) \in L_2(C_0[0,T])$ is given by

$$G(y) = \operatorname*{l.i.m.}_{N\to\infty}\sum_{n_1,\dots,n_N=0}^{N} A^F_{n_1,\dots,n_N} i^{n_1+\cdots+n_N}\Psi_{n_1,\dots,n_N}(y),$$

and that

$$\mathcal{F}_{\sqrt{2},i}G(y) = \operatorname*{l.i.m.}_{N\to\infty}\sum_{n_1,\dots,n_N=0}^{N} A^F_{n_1,\dots,n_N}(-1)^{n_1+\cdots+n_N}\Psi_{n_1,\dots,n_N}(y).$$

But it is easy to see that

$$(-1)^{n_1+\cdots+n_N}\Psi_{n_1,\dots,n_N}(y) = \Psi_{n_1,\dots,n_N}(-y)$$

and so $\mathcal{F}_{\sqrt{2},i}G(y) = F(-y)$. Equation (3.12) then follows immediately from the Fourier-Hermite expressions for $F(x)$ and $G(y)$ and the fact that $\{\Psi_{n_1,\dots,n_N}\}$ is an orthonormal set. □

Remark 3. In [**7**], Lee, for the abstract Wiener space (H, B, p_1) (also see [**5**]) and the class $\mathcal{E}_a(B)$ of the restrictions to B of exponential type analytic functionals on $[B]$, the complexification of B, established the following theorem [**7**, Theorem 2.6].

Theorem 2.6 [**7**]. *For* $F \in \mathcal{E}_a(B)$,

$$\int_B |\mathcal{F}_{\alpha,\beta}F(y)|^2 p_1(dy) = \int_B |F(y)|^2 p_1(dy)$$

if and only if $\alpha^2 + \beta^2 = 1$ *and* $|\beta| = 1$.

He then pointed out that Theorem 2.6 ensures that $\mathcal{F}_{\alpha,\beta}$ can be extended from $\mathcal{E}_a(B)$ to $L_2(p_1)$ as a unitary operator.

4. Further results. Recall that, throughout this paper, we have assumed that $\beta = a + bi$ was a nonzero complex number satisfying inequality (2.7); namely, that $\operatorname{Re}(1 - \beta^2) > 0$. Furthermore, in Corollary 7, we showed that if β also satisfies the inequality $|\beta| \le 1$, then $\mathcal{F}_{\alpha,\beta}$ exists as an element of $L_2(C_0[0,T])$ for all $F \in L_2(C_0[0,T])$ with α given by (2.8). In the example below we show that for any complex number β with $|\beta| > 1$ and $\operatorname{Re}(1 - \beta^2) > 0$, there exists a functional $F \in L_2(C_0[0,T])$ (of course F depends on β) such that $\mathcal{F}_{\alpha,\beta}F$ doesn't exist as an element of $L_2(C_0[0,T])$.

Example 10. Let $\beta = a + bi$ be such that $|\beta| = k > 1$ and $\operatorname{Re}(1 - \beta^2) > 0$. Let α be given by (2.8). Let $\Psi_{(0)}(x) \equiv \Psi_0(x)$, $\Psi_{(1)}(x) \equiv \Psi_1(x)$, $\Psi_{(2)}(x) \equiv \Psi_{1,1}(x)$, $\Psi_{(3)}(x) \equiv \Psi_{1,1,1}(x)$, etc. For $N = 1, 2, \dots$, let

$$F_N(x) = \sum_{n=0}^{N} k^{-n} \Psi_{(n)}(x). \tag{4.1}$$

Then, since $\{\Psi_{(n)} : n = 0, 1, 2, \dots\}$ is an orthonormal set of functionals in $L_2(C_0[0,T])$,

$$\int_{C_0[0,T]} |F_N(x)|^2 m(dx) = \sum_{n=0}^{N} k^{-2n} = \frac{k^2}{k^2-1}\left[1 - \left(\frac{1}{k^2}\right)^{N+1}\right],$$

and so

$$\lim_{N\to\infty} \|F_N\|_2^2 = \frac{k^2}{k^2-1}.$$

But, for $N > M$,

$$\|F_N - F_M\|_2^2 = \sum_{n=M+1}^{N} \frac{1}{k^{2n}} \longrightarrow 0$$

as $M, N \to \infty$. Hence $\{F_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $L_2(C_0[0,T])$ and, since $L_2(C_0[0,T])$ is complete,

$$F(x) \equiv \operatorname*{l.i.m.}_{N\to\infty} F_N(x)$$

is an element of $L_2(C_0[0,T])$. In fact,

$$F(x) = \sum_{n=0}^{\infty} k^{-n} \Psi_{(n)}(x)$$

is the Fourier-Hermite series for F; i.e., the Fourier-Hermite coefficients for F are $A_0^F = 1$, and for $n_p \neq 0$,

$$A^F_{n_1,\dots,n_p} = \begin{cases} 0 & n_1 n_2 \dots n_p \neq 1 \\ k^{-p} & n_1 n_2 \dots n_p = 1. \end{cases} \tag{4.2}$$

Next, using (4.2) and the fact that $|\beta| = k$, we observe that

$$\begin{aligned}\sum_{n_1,\dots,n_N=0}^{N} |A^F_{n_1,\dots,n_N} \beta^{n_1+\dots+n_N}|^2 &= |A_0^F \beta^0|^2 + |A_1^F \beta^1|^2 + |A^F_{1,1}\beta^2|^2 \\ &\quad + \dots + |A^F_{1,\dots,1}\beta^N|^2 \\ &= 1 + 1 + 1 + \dots + 1 = N + 1.\end{aligned}$$

Hence, by Theorem 6, $\mathcal{F}_{\alpha,\beta}F$ doesn't exist as an element of $L_2(C_0[0,T])$. However, note that $\mathcal{F}_{\alpha,\beta}F$ does exist as an element of $L_2(C_0[0,T])$ if $|\beta| < k$ with $\mathrm{Re}\,(1-\beta^2) > 0$ and $\alpha = \sqrt{1-\beta^2}$. $\square$

Our final results in this paper involve the inverse transform of $\mathcal{F}_{\alpha,\beta}$. In order to ensure the existence of the inverse transform of $\mathcal{F}_{\alpha,\beta}$, we need to put an additional assumption on $\beta = a+bi$; namely, that

$$\mathrm{Re}\left(1-\frac{1}{\beta^2}\right) > 0. \tag{4.3}$$

But $\mathrm{Re}\,[1-(1/\beta^2)] > 0 \Leftrightarrow (a^2+b^2)^2 - (a^2-b^2) > 0$. But the graph of $(a^2+b^2)^2-(a^2-b^2) = 0$ is the lemniscate $r^2 = \cos(2\theta)$. Hence $\mathrm{Re}\,[1-(1/\beta^2)] > 0$ if and only if the point $(a,b) \in \mathbf{R}^2$ lies outside the lemniscate $(a^2+b^2)^2-(a^2-b^2) = 0$.

Theorem 11. *Let F, β and α be as in Theorem* 6*, and assume that* (3.7) *holds. Furthermore, assume that β satisfies inequality* (4.3)*. Then, for $\alpha' \equiv \sqrt{1-1/\beta^2}$ and $\beta' = \pm 1/\beta$, we have that*

$$\mathcal{F}_{\alpha',\beta'}\mathcal{F}_{\alpha,\beta}F(y) = F(\beta\beta' y), \quad y \in C_0[0,T]. \tag{4.4}$$

That is to say,

$$\mathcal{F}_{\alpha',1/\beta}\mathcal{F}_{\alpha,\beta}F(y) = F(y), \quad y \in C_0[0,T],$$

and

$$\mathcal{F}_{\alpha',-1/\beta}\mathcal{F}_{\alpha,\beta}F(y) = F(-y), \quad y \in C_0[0,T].$$

Proof. Since $\mathcal{F}_{\alpha,\beta}F$ exists, the Fourier-Hermite expression of it is given by

$$\mathcal{F}_{\alpha,\beta}F(y) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{n_1,\dots,n_N=0}^{N} A^F_{n_1,\dots,n_N}\beta^{n_1+\cdots+n_N}\Psi_{n_1,\dots,n_N}(y).$$

Now

$$\lim_{N\to\infty} \sum_{n_1,\dots,n_N=0}^{N} |A^F_{n_1,\dots,n_N}\beta^{n_1+\cdots+n_N}(\beta')^{n_1+\cdots+n_N}|^2 = \lim_{N\to\infty} \sum_{n_1,\dots,n_N=0}^{N} |A^F_{n_1,\dots,n_N}|^2 = \|F\|_2^2 < \infty.$$

Hence, by Theorem 6, $\mathcal{F}_{\alpha',\beta'}\mathcal{F}_{\alpha,\beta}F$ exists and is given by

$$\begin{aligned}\mathcal{F}_{\alpha',\beta'}\mathcal{F}_{\alpha,\beta}F(y) &= \underset{N\to\infty}{\text{l.i.m.}} \sum_{n_1,\dots,n_N=0}^{N} A^F_{n_1,\dots,n_N}(\beta\beta')^{n_1+\cdots+n_N}\Psi_{n_1,\dots,n_N}(y)\\ &= F(\beta\beta' y)\end{aligned}$$

which completes the proof of Theorem 11. □

Remark 4. Some special cases of Theorem 11:

$$\mathcal{F}_{\sqrt{2},-i}\,\mathcal{F}_{\sqrt{2},i}\,F(y) = F(y),$$

$$\mathcal{F}_{\sqrt{2},i}\,\mathcal{F}_{\sqrt{2},i}\,F(y) = F(-y),$$

$$\mathcal{F}_{(7-4\sqrt{2}i)^{1/2}/3,(2+\sqrt{2}i)/3}\,\mathcal{F}_{1+i/\sqrt{2},1-i/\sqrt{2}}\,F(y) = F(y).$$

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Department of Mathematics, Yonsei University, Seoul, 120-749, Korea
E-mail address: mathkbs@dreamwiz.com

Department of Mathematics and Statistics, University of Nebraska-Lincoln, Lincoln, NE 68588-0323
E-mail address: dskoug@math.unl.edu