ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 2, Summer 2003

PARA-ORTHOGONAL POLYNOMIALS IN FREQUENCY ANALYSIS

LEYLA DARUIS, OLAV NJÅSTAD AND WALTER VAN ASSCHE

1. Introduction. By a trigonometric signal we mean an expression of the form

$$x(m) = \sum_{j=1}^{I} \left(\alpha_j e^{im\omega_j} + \alpha_{-j} e^{im\omega_{-j}} \right), \tag{1.1}$$

and we assume $\alpha_{-j} = \overline{\alpha_j}$, and $\omega_{-j} = -\omega_j \in (0, \pi)$ for $j = 1, 2, \dots, I$. The constants α_j represent *amplitudes*, the quantities ω_j are *frequencies*, and m is discrete time. The frequency analysis problem is to determine the numbers $\{\alpha_j, \omega_j : j = 1, 2, \dots, I\}$, and $n_0 = 2I$ when values $\{x(m) : m = 0, 1, \dots, N-1\}$ (observations) are known.

The Wiener-Levinson method, formulated in terms of Szegő polynomials, can briefly be described as follows (the original ideas of the method can be found in [**12, 20**]). An absolutely continuous measure ψ_N is defined on $[-\pi, \pi]$ (or on the unit circle $\mathbf{T}$ through the transformation $\theta \mapsto z = e^{i\theta}$) by the formula

$$\frac{d\psi_N}{d\theta} = \frac{1}{2\pi} \left| \sum_{m=0}^{N-1} x(m) e^{-im\theta} \right|^2. \tag{1.2}$$

Here N is an arbitrary natural number. The measure gives rise to a positive definite inner product which determines a sequence $\{\Phi_n(\psi_N, z) : n = 0, 1, 2 \dots\}$ of monic orthogonal polynomials (Szegő polynomials). All the zeros of $\Phi_n(\psi_N, z)$ lie in the open unit disk.

Let $\varphi_n(\psi_N, z)$ be the orthonormal polynomials (with positive leading coefficient κ_n^N) with respect to ψ_N. Then we have

$$\varphi_n(\psi_N, z) = \kappa_n^N \Phi_n(\psi_N, z), \tag{1.3}$$

Received by the editors on September 30, 2002.
The first author was partially supported by Laguna University under contract 1802010204 and by Ministerio de Ciencia y Technología del Gobierno Español under contract BF2001-3411.
The third author's research is supported by INTAS 00-272 and research grant G.0184.02 of FWO-Vlaanderen.

where

$$\kappa_n^N = \left(\prod_{j=1}^{n} (1 - |\Phi_j(\psi_N, 0)|^2) \right)^{-1/2}. \tag{1.4}$$

For the basic theory of Szegő polynomials, see e.g., [**1, 2, 7, 18, 19**].

Let $\{\zeta_k : k = 1, 2, \ldots, n_0\}$ be a numbering of the so-called frequency points $\{e^{i\omega_j} : j = \pm 1, \pm 2, \ldots, \pm I\}$, and set $\lambda_k = |\alpha_j|^2$ for $\zeta_k = e^{i\omega_j}$. Let ψ be the discrete measure defined by

$$\psi(\theta) = \sum_{k=1}^{n_0} \lambda_k \delta(e^{i\theta} - \zeta_k). \tag{1.5}$$

Then the measures ψ_N/N converge in the weak* sense to ψ (see [**6, 15**]).

For a fixed degree n, $n \geq n_0$, every subsequence of $\{\Phi_n(\psi_N, z) : N = 1, 2, \ldots\}$ contains a subsequence converging to a polynomial of the form

$$P_n(z) = Q_{n-n_0}(z) \prod_{j=1}^{I} (z - e^{i\omega_j})(z - e^{i\omega_{-j}}), \tag{1.6}$$

where $Q_{n-n_0}(z)$ is a polynomial of degree $n - n_0$. It follows that n_0 of the zeros of $\Phi_n(\psi_N, z)$, closest to the frequency points, converge to these frequency points (see, e.g., [**6, 8, 9, 15**]). Furthermore, for every n there is a constant $K_n < 1$ such that $n - n_0$ of the zeros of $\Phi_n(\psi_N, z)$ are contained in the disk $\{|z| \leq K_n\}$ for all N, (see [**13, 15**] and also [**14**] where more general orthogonal rational functions are used in frequency analysis problems). These properties make it possible to determine the number n_0 of frequency points and to localize these frequency points from the behavior of the zeros of $\Phi_n(\psi_N, z)$ as N increases. For a survey on the use of Szegő polynomials in frequency analysis, see [**11**]). See also [**17**] where a matrix approach is discussed.

In this paper we shall sketch a different approach to the frequency analysis problem, which uses zeros of para-orthogonal polynomials instead of zeros of orthogonal polynomials. A para-orthogonal polynomial is a polynomial of the form

$$B_n(\psi_N, \tau, z) = \Phi_n(\psi_N, z) + \tau \Phi_n^*(\psi_n, z), \qquad \tau \in \mathbf{T}, \tag{1.7}$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ is the reversed polynomial. For convenience we shall suppress the τ in all notation when we are considering a fixed value of τ. The polynomial $B_n(\psi_N, z)$ has n simple zeros $z_1^N, z_2^N, \dots, z_n^N$, all lying on $\mathbf{T}$. The following convergence result will be fundamental in the sequel. A proof can be found in [**10**].

Theorem 1.1. *Let $\{N_k : k = 1, 2, \dots\}$ be an arbitrary subsequence of the sequence of natural numbers, let τ be an arbitrary point on $\mathbf{T}$, and let $n \geq n_0$. Then there exists a subsequence $\{N_{k(\nu)}\}$ and a polynomial $W_{n-n_0}(z)$ of degree $n - n_0$ such that*

$$\lim_{\nu\to\infty} B_n(\psi_{N_{k(\nu)}}, z) = W_{n-n_0}(z)\prod_{k=1}^{n_0}(z - \zeta_k), \tag{1.8}$$

where ζ_k are the frequency points.

It follows that some of the zeros $z_1^N, z_2^N, \dots, z_n^N$ of $B_n(\psi_{N_{k(\nu)}}, z)$ converge to the frequency points, and the rest converge to zeros of $W_{n-n_0}(z)$. A frequency point may also be a zero of $W_{n-n_0}(z)$. We shall occasionally write $B_n(z)$ for the polynomial $W_{n-n_0}(z)\prod_{k=1}^{n_0}(z - \zeta_k)$. We will use the weights in the Szegő quadrature formula for Szegő polynomials to distinguish the frequency points from the zeros of $W_{n-n_0}(z)$. These quadrature weights will also determine the numbers $\lambda_k = |\alpha_j|^2$ for $\zeta_k = e^{i\omega_j}$ and therefore provide useful estimates of the modulus of the amplitudes in the signal.

2. General results The zeros $z_1^N, \dots, z_n^N$ of $B_n(\psi_N, z)$ are nodes in a Szegő quadrature formula for Szegő polynomials with respect to the measure $\psi_N(\theta)/N$. The weights λ_k^N are given by

$$\lambda_k^N = \frac{1}{N}\int_{-\pi}^{\pi} L_k^N(e^{i\theta})\, d\psi_N(\theta), \quad k = 1, 2, \dots, n, \tag{2.1}$$

where

$$L_k^N(z) = \frac{(z - z_1^N)\cdots(z - z_{k-1}^N)(z - z_{k+1}^N)\cdots(z - z_n^N)}{(z_k^N - z_1^N)\cdots(z_k^N - z_{k-1}^N)(z_k^N - z_{k+1}^N)\cdots(z_k^N - z_n^N)} \tag{2.2}$$

are the fundamental polynomials of Lagrange interpolation (see [**1, 2, 7**]). The weights may also be expressed as (see [**3, 5**])

$$\lambda_k^N = \left(\sum_{j=0}^{n-1} |\varphi_j(\psi_N, z_k^N)|^2 \right)^{-1}. \tag{2.3}$$

For more information on Szegő quadrature formulas, see [**1, 2, 3, 5, 7**].

In the following we shall always assume that $n \geq n_0$. The following general result holds.

Theorem 2.1. *Let* λ_k^N*,* $k = 1, 2, \ldots, n$*, and* λ_m*,* $m = 1, 2, \ldots, n_0$*, be defined as in the foregoing. Then*

$$\lim_{N\to\infty} \sum_{k=1}^{n} \lambda_k^N = \sum_{m=1}^{n_0} \lambda_m. \tag{2.4}$$

Proof. On the one hand, since $\psi_N(\theta)/N \xrightarrow{*} \psi(\theta)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \int_{-\pi}^{\pi} d\psi_N(\theta) = \int_{-\pi}^{\pi} d\psi(\theta) = \sum_{m=1}^{n_0} \lambda_m.$$

On the other hand, since the Szegő quadrature is exact for the function $f(\theta) \equiv 1$, we have

$$\frac{1}{N} \int_{-\pi}^{\pi} d\psi_N(\theta) = \sum_{k=1}^{n} \lambda_k^N$$

for all N. From this the result follows. □

We note that the above result is valid irrespective of whether the sequence $\{B_n(\psi_N, z) : N = 1, 2, \ldots\}$ converges or not, or of whether the zeros of the limiting polynomials $B_n(z)$ of convergent sequences are simple or not.

For the sake of completeness, we give a proof of a convergence result, which will be used when convergence of weights belonging to individual zeros is discussed.

Theorem 2.2. *Assume that the sequence* $\{f_p : p = 1, 2, \dots\}$ *of continuous functions on* $\mathbf{T}$ *converges uniformly to* f *on* $\mathbf{T}$*. Then*

$$\lim_{p\to\infty} \frac{1}{p} \int_{-\pi}^{\pi} f_p(e^{i\theta})\, d\psi_p(\theta) = \int_{-\pi}^{\pi} f(e^{i\theta})\, d\psi(\theta). \tag{2.5}$$

Proof. Let $\varepsilon > 0$. Since ψ_p/p converges to ψ in the weak* topology, we have

$$\left| \frac{1}{p} \int_{-\pi}^{\pi} f(e^{i\theta})\, d\psi_p(\theta) - \int_{-\pi}^{\pi} f(e^{i\theta})\, d\psi(\theta) \right| < \frac{\varepsilon}{2}$$

for p sufficiently large. Since f_p converges uniformly to f we have

$$\left| \frac{1}{p} \int_{-\pi}^{\pi} [f_p(e^{i\theta}) - f(e^{i\theta})]\, d\psi_p(\theta) \right| < \frac{\varepsilon}{2}$$

for p sufficiently large, since all the measures ψ_p/p have finite total mass

$$\frac{1}{p} \int_{-\pi}^{\pi} d\psi_p(\theta) = \sum_{k=1}^{n_0} \lambda_k.$$

It follows that

$$\left| \frac{1}{p} \int_{-\pi}^{\pi} f_p(e^{i\theta})\, d\psi_p(\theta) - \int_{-\pi}^{\pi} f(e^{i\theta})\, d\psi(\theta) \right| < \varepsilon$$

for p sufficiently large, which completes the proof. □

Simple zeros in the limit. If the zeros $\zeta_1, \dots, \zeta_{n_0}, \zeta_{n_0+1}, \dots, \zeta_n$ of $B_n(z) = W_{n-n_0}(z) \prod_{k=1}^{n_0} (z - \zeta_k)$ in Theorem 1.1 are distinct, then we set

$$\Lambda_m(z) = \frac{(z-\zeta_1)\cdots(z-\zeta_{m-1})(z-\zeta_{m+1})\cdots(z-\zeta_n)}{(z_m-\zeta_1)\cdots(z_m-\zeta_{m-1})(z_m-\zeta_{m+1})\cdots(z_m-\zeta_n)}. \tag{3.1}$$

We note that $\Lambda_m(\zeta_j) = \delta_{m,j}$, and hence

$$\int_{-\pi}^{\pi} \Lambda_m(e^{i\theta})\, d\psi(\theta) = \lambda_m, \quad \text{for } m = 1, 2, \dots, n_0, \tag{3.2}$$

$$\int_{-\pi}^{\pi} \Lambda_m(e^{i\theta})\, d\psi(\theta) = 0, \quad \text{for } m = n_0 + 1, \dots, n. \tag{3.3}$$

We can now prove the following result.

Theorem 3.1. *Let the situation be as in Theorem* 1.1*, and assume that the zeros of the limiting polynomial* $B_n(z) = B_n(\psi_{N_{k(\nu)}}, z)$ *are distinct and that* $\lim_{\nu\to\infty} z_m^{N_{k(\nu)}} = \zeta_m$*. Then*

$$\lim_{\nu\to\infty} \lambda_m^{N_{k(\nu)}} = \lambda_m, \quad \textit{for } m = 1, 2, \dots, n_0, \tag{3.4}$$

$$\lim_{\nu\to\infty} \lambda_m^{N_{k(\nu)}} = 0, \quad \textit{for } m = n_0 + 1, \dots, n. \tag{3.5}$$

Proof. Since $z_m^{N_{k(\nu)}}$ converges to ζ_m as $\nu \to \infty$, we conclude that $L_m^{N_{k(\nu)}}(e^{i\theta})$ converges to $\Lambda_m(e^{i\theta})$ uniformly for $\theta \in [-\pi, \pi]$. Thus by letting $L_m^{N_{k(\nu)}}$ play the role of f_p, Λ_m the role of f, and $\psi_{N_{k(\nu)}}$ the role of ψ_p in Theorem 2.2, we find that

$$\lim_{\nu\to\infty} \frac{1}{N_{k(\nu)}} \int_{-\pi}^{\pi} L_m^{N_{k(\nu)}}(e^{i\theta})\, d\psi_{N_{k(\nu)}}(\theta) = \int_{-\pi}^{\pi} \Lambda(e^{i\theta})\, d\psi(\theta).$$

Taking into account (2.1)–(2.2) and (3.2)–(3.3), we conclude that (3.4)–(3.5) hold. □

Corollary 3.2. *Assume that every subsequence* $\{N_k : k = 1, 2, \dots\}$ *has a subsequence* $\{N_{k(\nu)} : \nu = 1, 2, \dots\}$ *such that the limiting polynomial* $B_n(\{N_{k(\nu)}\}, z)$ *has* n *distinct zeros. Then*

$$\lim_{N\to\infty} \lambda_m^N = \lambda_m \quad \textit{for } m = 1, 2, \dots, n_0, \tag{3.6}$$

$$\lim_{N\to\infty} \lambda_m^N = 0 \quad \textit{for } m = n_0 + 1, \dots, n. \tag{3.7}$$

Thus the n_0 *zeros of* $B_n(\psi_N, z)$ *corresponding with the eventually largest weights in the quadrature formula approach the frequency points* $\zeta_1, \dots, \zeta_{n_0}$*.*

Proof. It follows from Theorem 3.1 and the assumption in Corollary 3.2 that every subsequence $\{\lambda_m^{N_k} : k = 1, 2, \dots\}$ has a subsequence

$\{\lambda_m^{N_{k(\nu)}} : \nu = 1, 2, \dots\}$ which converges to λ_m for $m = 1, 2, \dots, n_0$, and to 0 for $m = n_0 + 1, \dots, n$. A general convergence property for sequences then ensures that the whole sequence $\{\lambda_m^N : N = 1, 2, \dots\}$ converges to λ_m or 0, respectively. □

It follows from Corollary 3.2 that if, in particular, the sequence $\{B_n(\psi_N, z) : N = 1, 2, \dots\}$ itself converges to a polynomial $B_n(z)$ with distinct zeros, then (3.6)–(3.7) holds.

Example 3.1. We consider the signal

$$x(m) = \alpha \left(e^{\pi m i/2} + e^{-\pi m i/2}\right), \qquad \alpha > 0. \tag{3.8}$$

The frequency points are $\zeta_1 = i$ and $\zeta_2 = -i$. By using the Szegő recursion formulas (see, e.g., [**7, 18**]) it can be shown that, for $n = 3, 4, 5$, the para-orthogonal polynomials $B_n(\psi_N, z)$ converge to the polynomial

$$B_n(\tau, z) = (z^{n-2} + \tau)(z - i)(z + i). \tag{3.9}$$

Let $n = 3$ and $\tau = -1$. Then in addition to the frequency points ζ_1 and ζ_2, the polynomial $B_n(-1, z)$ has the zero $\zeta_3 = 1$. By a suitable ordering of the zeros z_k^N, $k = 1, 2, 3$, of $B_3(\psi_N, z)$ we have $z_k^N \to \zeta_k$ as $N \to \infty$. Then $\lambda_1^N \to \alpha^2$, $\lambda_2^N \to \alpha^2$, and $\lambda_3 \to 0$ as $N \to \infty$. Observations of the limiting behavior of zeros and weights will here indicate that $n_0 = 2$, with two frequency points $\pm i$. For the amplitude we can conclude that the modulus is α.

Example 3.2. With the signal as in Example 3.1, with $\alpha = 1$, $n = 4$ and $\tau = i$ we get the following numerical results. First, we have computed the modulus of the zeros of the Szegő polynomials $\varphi_4(\psi_N, z)$ for some values of N (Table 1). Observe that two of the zeros have modulus close to 1 and the remaining two zeros have small modulus. The zeros with modulus close to 1 approach the frequency points $\pm i$.

TABLE 1. Modulus of the zeros of Szegő polynomials.

$N = 21$	$N = 41$	$N = 81$	$N = 201$	$N = 301$
0.948387	0.974646	0.987417	0.994987	0.996661
0.948387	0.974646	0.987417	0.994987	0.996661
0.230094	0.160236	0.112527	7.08899E−02	5.78321E−02
0.230094	0.160236	0.112527	7.08899E−02	5.78321E−02

Next, we have computed the quadrature weights λ_m^N in the Szegő quadrature formula for some values of N (Table 2). Observe that two of the weights are close to 1 (which is $|\alpha|^2$) and the remaining weights are small. The weights near 1 correspond to the zeros near the frequency points $\pm i$.

TABLE 2. Quadrature coefficients of the Szegő quadrature.

$N = 21$	$N = 41$	$N = 81$	$N = 201$	$N = 301$
0.9567102	0.9767705	0.987955	0.9950733	0.9967007
0.9567102	0.9767705	0.987955	0.9950733	0.9967007
9.090903E-02	4.761901E-02	2.439022E-02	9.900983E-03	6.622512E-03
9.090903E-02	4.761901E-02	2.439022E-02	9.900983E-03	6.622512E-03

Example 3.3. As a numerical example we take $N = 65536$ observations from the signal

$$x(m) = \sum_{k=1}^{4} (A_k \cos(m\omega_k) + B_k \sin(m\omega_k)) + Z_m \tag{3.10}$$

which contains some noise Z_m, which we have taken to be white noise with variance 0.000036 (i.e., all Z_m are uncorrelated random variables). We have taken the following values for the parameters:

k	ω_k	A_k	B_k
1	0.44821001146034	0.13694483364390	0.2190355252614
2	1.34558877237344	−0.17193901065310	−0.00965099052822
3	0.22410500573017	0.15	−0.175
4	0.67279438618672	0.125	0.09

This was obtained by taking the two main frequencies ω_1, ω_2 of the sound of some flute, to which harmonics were added ($\omega_3 = \omega_1/2$ and $\omega_4 = \omega_2/2$). The resulting sound can be heard as `bill.wav`[1]. The zeros of the para-orthogonal polynomial of degree 100 (with $\tau = -1$) were computed as the eigenvalues of a unitary Hessenberg matrix following [**4, 5**] and the weights of the Szegő quadrature were obtained from the first components of the corresponding eigenvectors [**5, 16**]. Figure 1 gives the weights λ_k as a function of θ_k, where $e^{i\theta_k} = z_k^N$. The values of the largest weights and the corresponding θ_k are given in Table 3.

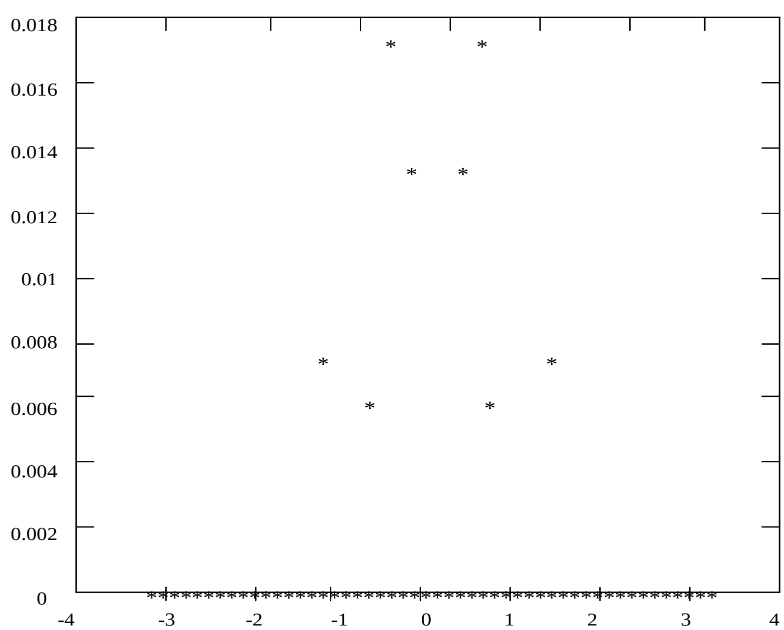

FIGURE 1. Weights for the Szegő quadrature for 100 nodes.

TABLE 3. The largest weights in the Szegő quadrature.

θ	λ	$(A_k^2 + B_k^2)/4$
± 0.4482264345	0.0166596342	0.0166826152
± 1.3455856216	0.0074091825	0.0074140413
± 0.2241051893	0.0132797435	0.01328125
± 0.6727401886	0.0058919407	0.00593125

Observe that the signal (3.10) can be written as

$$x(m) = \sum_{k=1}^{4} \left(\frac{A_k - iB_k}{2} e^{im\omega_k} + \frac{A_k + iB_k}{2} e^{-im\omega_k} \right) + Z_m, \tag{3.11}$$

so that $\alpha_k = \overline{\alpha_{-k}} = (A_k - iB_k)/2$. According to Corollary 3.2, the weights corresponding to the zeros near the frequency points converge to $|\alpha_k|^2 = (A_k^2 + B_k^2)/4$.

4. Zeros that are not frequency points. In this section we shall show that for zeros of $W_{n-n_0}(\{N_{k(\nu)}\}, z)$ which are not frequency points, the weights associated with the corresponding zeros of the para-orthogonal polynomials tend to zero. We recall the connection between the monic orthogonal polynomials $\Phi_n(\psi_N, z)$ and the orthonormal polynomials $\varphi_n(\psi_N, z)$ given in (1.3)–(1.4), as well as the expression (2.3) for the weights λ_k^N. We shall also make use of the fact that for $n < n_0$ we have

$$\lim_{N\to\infty} \Phi_n(\psi_N, z) = \Phi_n(\psi, z), \tag{4.1}$$

with ψ given by (1.5), (see [**8, 15**]) and that

$$\lim_{N\to\infty} |\Phi_{n_0}(\psi_N, 0)| = 1 \tag{4.2}$$

(see, e.g., [**7**]).

In the following theorem, we assume the same situation as in Theorem 1.1.

Theorem 4.1. *Let $n > n_0$, and suppose that the sequence $\{z_m^{N_{k(\nu)}} : \nu = 1, 2, \dots\}$ of zeros of $B_n(\psi_{N_{k(\nu)}}, z)$ converge to a zero ζ_m of $W_{n-n_0}(\{N_{k(\nu)}\}, z)$ which is not a frequency point. Then*

$$\lim_{\nu \to \infty} \lambda_m^{N_{k(\nu)}} = 0. \tag{4.3}$$

Proof. We observe that $|\varphi_{n_0}(\psi_{N_{k(\nu)}}, z_m^{N_{k(\nu)}}|^2$ is a term in the denominator of (2.3). From (1.3)–(1.4) we get

$$|\varphi_{n_0}(\psi_{N_{k(\nu)}}, z_m^{N_{k(\nu)}})|^2 = \frac{|\Phi_{n_0}(\psi_{N_{k(\nu)}}, z_m^{N_{k(\nu)}})|^2}{\prod_{j=1}^{n_0}(1 - |\Phi_j(\psi_{N_{k(\nu)}}, 0)|^2)}. \tag{4.4}$$

The numerator $|\Phi_{n_0}(\psi_{N_{k(\nu)}}, z_m^{N_{k(\nu)}}|$ tends to $|\Phi_{n_0}(\psi, \zeta_m)|$, which is different from zero (since ζ_m is not a frequency point), while $|\Phi_{n_0}(\psi_{N_{k(\nu)}}, 0)|$ tends to 1 by (4.2). It follows from (4.4) that

$$\lim_{\nu \to \infty} |\varphi_{n_0}(\psi_{N_{k(\nu)}}, z_m^{N_{k(\nu)}})| = \infty,$$

and hence by (2.3) that (4.3) is satisfied. □

The result of Theorem 4.1 is valid for the weights corresponding to zeros $z_m^{N_{k(\nu)}}$ tending to ζ_m for any convergent subsequence $\{B_n(\psi_{N_{k(\nu)}}, z) : \nu = 1, 2, \dots\}$. It is thus not required that ζ_m is a simple zero of $W_{n-n_0}(\{N_{k(\nu)}\}, z)$ for any convergent subsequence, only that ζ_m is not a frequency point.

5. Multiple zeros in the limit. We shall now consider a simple case of the situation where the frequency points (and in principle also other zeros ζ_m) occur as multiple zeros of $B_n(\{N_{k(\nu)}\}, z)$. Without loss of generality we formulate the results in terms of convergence of the whole sequence $\{B_n(\psi_N, z) : N = 1, 2, \dots\}$ and hence of the whole sequences $\{z_m^N : N = 1, 2, \dots\}$, $m = 1, 2, \dots, n$, to avoid troublesome formulations and notation in terms of subsequences.

As before, let z_m^N, $m = 1, 2, \dots, n$, be the zeros of $B_n(\psi_N, z)$, and recall that they are all distinct. To simplify the notation we shall suppress the index N in the following calculations. We introduce the polynomial T_n as

$$T_n(z) = (z - z_3) \cdots (z - z_n). \tag{5.1}$$

Then the polynomials L_1 and L_2 in (2.2) may be written as

$$L_1(z) = \frac{(z - z_2)T_n(z)}{(z_1 - z_2)T_n(z_1)}, \quad L_2(z) = \frac{(z - z_1)T_n(z)}{(z_2 - z_1)T_n(z_2)}. \tag{5.2}$$

We easily find that

$$L_1(z) + L_2(z) = \frac{T_n(z)\,[(z - z_2)T_n(z_2) - (z - z_1)T_n(z_1)]}{(z_1 - z_2)T_n(z_1)T_n(z_2)}. \tag{5.3}$$

In the following proposition we do not need to require that the values of the z_k are distinct.

Proposition 5.1. *We may write*

$$L_1(z) + L_2(z) = \frac{T_n(z)P(z)}{T_n(z_1)T_n(z_2)}, \tag{5.4}$$

where P *is a polynomial with the property*

$$P(z_1) = T_n(z_1), \quad \textit{when } z_2 = z_1. \tag{5.5}$$

Proof. From (5.3) we find

$$\begin{aligned} P(z) &= \frac{(z - z_2)T_n(z_2) - (z - z_1)T_n(z_1)}{z_1 - z_2} \\ &= \frac{z(T_n(z_2) - T_n(z_1))}{z_1 - z_2} - \frac{z_2T_n(z_2) - z_1T_n(z_1)}{z_1 - z_2}. \end{aligned}$$

Here $T_n(z_2)$ is a polynomial in z_2. The numerators are both divisible by $z_2 - z_1$, hence $(T_n(z_2) - T_n(z_1))/(z_1 - z_2)$ and $(z_2T_n(z_2) - z_1T_n(z_1))/(z_1 - z_2)$ both extend to polynomials in z_2, which are also defined for $z_2 = z_1$. From this expression it follows immediately that $P(z_1) = T_n(z_2)$, and hence by continuity we have $P(z_1) = T_n(z_1)$ when $z_2 = z_1$. □

We again consider a fixed degree n. We now assume that $z_m^N \to \zeta_m$ as $N \to \infty$ for $m = 1, 2, \ldots, n_0$ and $m = n_0 + p + 1, \ldots, n$, and that $z_{n_0+j}^N \to \zeta_j$ for $j = 1, \ldots, p$, where $1 \leq p \leq n_0$. Here it is assumed that $\zeta_k \neq \zeta_m$ for $k = n_0 + p + 1, \ldots, n$ and $m = 1, \ldots, n_0$, and that the points $\zeta_{n_0+p+1}, \ldots, \zeta_n$ are distinct. Thus frequency points may have multiplicity at most 2 as zeros of $B_n(z)$, while zeros of $B_n(z)$ which are not frequency points are simple. In other words, it is assumed that W_{n-n_0} has only simple zeros. We introduce the notation

$$T_n^j(z) = (z - z_1^N) \cdots (z - z_{j-1}^N)(z - z_{j+1}^N) \cdots (z - z_{n_0+j-1}^N)(z - z_{n_0+j+1}^N) \cdots (z - z_n^N), \tag{5.6}$$

for $j = 1, \ldots, p$. It follows from Proposition 5.1 that for $j = 1, \ldots, p$ we may write

$$\lim_{N\to\infty} \left[L_j^N(z) + L_{n_0+j}^N(z)\right] = \frac{T_n^j(z)P_j(z)}{T_n^j(\zeta_j)^2}, \tag{5.7}$$

with z_m^N replaced by ζ_m for $m = 1, \ldots, n_0$ and $m = n_0 + p + 1, \ldots, n$, and $z_{n_0+j}^N$ replaced by ζ_j for $j = 1, \ldots, p$. Here $P_j(z)$ is a polynomial with the property $P_j(\zeta_j) = T_n^j(\zeta_j)$. Furthermore

$$\lim_{N\to\infty} L_m^N(z) = \Lambda_m(z), \tag{5.8}$$

for $m = p + 1, \ldots, n_0$ and $m = n_0 + p + 1, \ldots, n$, where

$$\Lambda_m(z) = \frac{(z-\zeta_1)^2 \cdots (z-\zeta_p)^2 (z-\zeta_{p+1}) \cdots (z-\zeta_{m-1})(z-\zeta_{m+1}) \cdots (z-\zeta_n)}{(\zeta_m-\zeta_1)^2 \cdots (\zeta_m-\zeta_p)^2 (\zeta_m-\zeta_{p+1}) \cdots (\zeta_m-\zeta_{m-1})(\zeta_m-\zeta_{m+1}) \cdots (\zeta_m-\zeta_n)}. \tag{5.9}$$

The $\zeta_{n_0+1}, \ldots, \zeta_{n_0+p}$ do not exist and do not occur in (5.9). We note that

$$\frac{T_n^j(\zeta_j)P_j(\zeta_j)}{T_n(\zeta_j)^2} = 1, \quad \text{for } j = 1, \ldots, p,$$

while

$$\frac{T_n(\zeta_k)P_j(\zeta_k)}{T_n(\zeta_j)^2} = 0, \quad \text{for } k \neq j.$$

Similarly $\Lambda_m(\zeta_m) = 1$ for $m = p+1, \ldots, n_0$ and $m = n_0+p+1, \ldots, n$, while $\Lambda_m(\zeta_k) = 0$ for $k \neq m$.

Theorem 5.2. *Let* $z_m^N \to \zeta_m$ *as* $N \to \infty$ *for* $m = 1, \dots, n_0$ *and* $m = n_0 + p + 1, \dots, n$, *and* $z_{n_0+j}^N \to \zeta_j$ *for* $j = 1, \dots, p$. *Assume that* $\zeta_1, \dots, \zeta_{n_0}, \zeta_{n_0+p+1}, \dots, \zeta_n$ *are distinct points. Then*

$$\lim_{N\to\infty} (\lambda_j^N + \lambda_{n_0+j}^N) = \int_{-\pi}^{\pi} \frac{T_n^j(e^{i\theta}) P_j(e^{i\theta})}{T_n^j(\zeta_j)^2} \, d\psi(\theta), \tag{5.10}$$

for $j = 1, \dots, p$, *and*

$$\lim_{N\to\infty} \lambda_m^N = \int_{-\pi}^{\pi} \Lambda(e^{i\theta}) \, d\psi(\theta), \tag{5.11}$$

for $m = p + 1, \dots, n_0$ *and* $m = n_0 + p + 1, \dots, n$.

Proof. From the assumptions it follows that $L_j^N(z) + L_{n_0+j}^N(z)$ converges uniformly to $T_n^j(z) P_j(z)/T_n(\zeta_j)^2$ on $\mathbf{T}$ for $j = 1, \dots, p$ and that $L_m^N(z)$ converges uniformly to $\Lambda_m(z)$ on $\mathbf{T}$ for $m = p+1, \dots, n_0$ and $m = n_0 + p + 1, \dots, n$. The result then follows from Theorem 2.2 in the same way as Theorem 3.1 does. □

Corollary 5.3. *Let the assumptions be as in Theorem* 5.2. *Then*

$$\lim_{N\to\infty} \left(\lambda_j^N + \lambda_{n_0+j}^N\right) = \lambda_j, \quad \text{for } j = 1, \dots, p, \tag{5.12}$$

$$\lim_{N\to\infty} \lambda_k^N = \lambda_k, \qquad \text{for } k = p + 1, \dots, n_0, \tag{5.13}$$

$$\lim_{N\to\infty} \lambda_m^N = 0, \quad \text{for } m = n_0 + p + 1, \dots, n. \tag{5.14}$$

Proof. The result follows from Theorem 5.2 and the remarks preceding it, together with the definition of the measure ψ. □

Remark. The argument can be extended to allow double zeros among the zeros $\zeta_{n_0+p+1}, \dots, \zeta_n$. Furthermore in order to obtain the results (5.12)–(5.13) it is not necessary to make any assumptions on $\zeta_{n_0+p+1}, \dots, \zeta_n$, except that they are all different from $\zeta_1, \dots, \zeta_{n_0}$.

Example 5.1. Let the signal be as in Example 3.1, with $n = 3$ and $\tau = -i$. Then $B_3(-i, z) = (z - i)^2 (z + i)$. Here $\zeta_1 = i$, $\zeta_2 = -i$,

$\zeta_3 = \zeta_1 = i$, with $\lambda_1 = \lambda_2 = \alpha^2$. By numbering the zeros of $B_3(\psi_N, -i, z)$ such that $z_1^N \to \zeta_1$, $z_2^N \to \zeta_2$, $\zeta_3^N \to \zeta_1$, we conclude from Corollary 5.3 that

$$\lambda_1^N + \lambda_3^N \to \alpha^2, \quad \lambda_2^N \to \alpha^2.$$

Observations of the limiting behavior of zeros and weights will here indicate that $n_0 = 2$, that the frequency points are $\pm i$, and that the amplitudes have modulus α.

Example 5.2. Let the signal be as in Example 3.1, with $n = 5$ and $\tau = i$. According to (3.9) we have

$$\begin{aligned} B_5(i, z) &= (z^3 + i)(z - i)(z + i) \\ &= (z - i)^2(z + i)\Big(z - \frac{1}{2}[\sqrt{3} - i]\Big)\Big(z + \frac{1}{2}[\sqrt{3} + i]\Big). \end{aligned}$$

Here we set $\zeta_1 = i$, $\zeta_2 = -i$, $\zeta_3 = [\sqrt{3} - i]/2$, $\zeta_4 = -[\sqrt{3} + i]/2$, $\zeta_5 = i$. We number the zeros of $B_5(\psi_N, i, z)$ such that $z_k^N \to \zeta_k$ for $k = 1, 2, 3, 4$, and $z_5^N \to \zeta_1$. It follows from Corollary 5.3 that

$$\lim_{N\to\infty} \left(\lambda_1^N + \lambda_5^N\right) = \alpha^2, \quad \lim_{N\to\infty} \lambda_2^N = \alpha^2, \quad \lim_{N\to\infty} \lambda_3^N = \lim_{N\to\infty} \lambda_4^N = 0.$$

Observations of the limiting behavior of z_k^N and λ_k^N will indicate that $n_0 = 2$, that the frequencies are $\pm\pi/2$, and that the amplitudes have modulus α.

ENDNOTES

1. Available at `http://www.wis.kuleuven.ac.be/bill.wav`

REFERENCES

1. L. Daruis and P. González-Vera, *Szegő polynomials and quadrature formulas on the unit circle*, Appl. Numer. Math. **36** (1999), 79–112.

2. L. Daruis, P. González-Vera and O. Njåstad, *Szegő quadrature formulas for certain Jacobi-type weight functions*, Math. Comp. **71** (2002), 683–701.

3. P. González-Vera, J.C. Santos-Léon and O. Njåstad, *Some results about numerical quadrature on the unit circle*, Adv. Comput. Math. **5** (1996), 297–328.

4. W.B. Gragg, *The QR algorithm for unitary Hessenberg matrices*, J. Comput. Appl. Math. **16** (1986), 1–8.

5. ———, *Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle*, J. Comput. Appl. Math. **46** (1993), 183–198.

6. W.B. Jones, O. Njåstad, E.B. Saff, *Szegő polynomials associated with Wiener-Levinson filters*, J. Comput. Appl. Math. **32** (1990), 387–407.

7. W.B. Jones, O. Njåstad and W.J. Thron, *Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle*, Bull. London Math. Soc. **21** (1989), 113–152.

8. W.B. Jones, O. Njåstad, W.J. Thron, H. Waadeland, *Szegő polynomials applied to frequency analysis*, J. Comput. Appl. Math. **46** (1993), 217–228.

9. W.B. Jones, O. Njåstad and H. Waadeland, *An alternative way of using Szegő polynomials in frequency analysis*, in *Continued fractions and orthogonal functions* (S.C. Cooper and W.J. Thron, eds.), Marcel Dekker, New York, 1994, pp. 141–152.

10. ———, *Asymptotics of zeros of orthogonal and para-orthogonal Szegő polynomials in frequency analysis*, in *Continued fractions and orthogonal functions* (S.C. Cooper and W.J. Thron, eds.), Marcel Dekker, New York, 1994, pp. 153–190.

11. W.B. Jones and V. Petersen, *Continued fractions and Szegő polynomials in frequency analysis and related topics*, Acta Appl. Math. **61** (2000), 149–174.

12. N. Levinson, *The Wiener RMS* (*root mean square*) *error criterion in filter design and prediction*, J. Math. Phys. Mass. Inst. Techn. **25** (1947), 261–278.

13. X. Li, *Asymptotics of columns in the table of orthogonal polynomials with varying measures*, Methods Appl. Anal. **2** (1995), 222–236.

14. O. Njåstad and H. Waadeland, *Generalized Szegő theory in frequency analysis*, J. Math. Anal. Appl. **206** (1997), 289–307.

15. K. Pan and E.B. Saff, *Asymptotics for zeros of Szegő polynomials associated with trigonometric polynomial signals*, J. Approx. Theory **71** (1992), 239–251.

16. A. Sinap, *Gaussian quadrature for matrix valued functions on the unit circle*, ETNA, Electron. Trans. Numer. Anal. **3** (1995), 96–115.

17. A. Sinap and W. Van Assche, *Orthogonal matrix polynomials and applications*, J. Comput. Appl. Math. **66** (1996), 27–52.

18. G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ. **23**, Providence RI, 1975 (4th ed.).

19. W. Van Assche, *Orthogonal polynomials in the complex plane and on the real line*, in *Special functions, q-Series and related topics* (M.E.H. Ismail, D.R. Masson and M. Rahman, eds.), Fields Institute Communications **14**, Amer. Math. Soc., Providence RI, (1997), 211–245.

20. N. Wiener, *Extrapolation, interpolation and smoothing of stationary time series*, The Technology Press of the Massachusetts Institute of Technology and John Wiley and Sons, New York, 1949.

Universidad de La Laguna, Tenerife, Department of Mathematical Analysis, Tenerife, Canary Islands, Spain
E-mail address: ldaruis@ull.es

Norwegian University of Science and Technology, Department of Mathematical Sciences, Trondheim, Norway
E-mail address: njastad@math.ntnu.no

Katholieke Universiteit Leuven, Department of Mathematics, Leuven, Belgium
E-mail address: walter@wis.kuleuven.ac.be