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ORTHOGONAL LAURENT POLYNOMIALS AND QUADRATURE FORMULAS FOR UNBOUNDED INTERVALS: I. GAUSS-TYPE FORMULAS

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Dedicated to W.B. Jones on the occasion of his 70th birthday

ABSTRACT. We study the convergence of quadrature formulas for integrals over the positive real line with an arbitrary distribution function. The nodes of the quadrature formulas are the zeros of orthogonal Laurent polynomials with respect to the distribution function and with respect to a certain nesting. This ensures a maximal domain of validity and the quadratures are therefore called Gauss-type formulas. The class of functions for which convergence holds is characterized in terms of the moments of the distribution function. Moreover, error estimates are given when f satisfies certain continuity conditions. Finally, these results are applied to the family of distributions $d\varphi(x) = x^{\alpha}\exp\{-(x^{\gamma_1} + x^{-\gamma_2})\}\,dx$, $\gamma_1, \gamma_2 \geq 1/2$, $\alpha \in \mathbf{R}$.

1. Introduction. The main aim of this work is the approximate calculation of integrals of the form

$$I(f) = \int_0^\infty f(x)\,d\varphi(x),$$

φ being a distribution function on $\mathbf{R}^+$, i.e., a real-valued, bounded, nondecreasing function with infinitely many points of increase on any

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interval $[a,b] \subset \mathbf{R}^+ = [0,\infty)$ and f a Riemann-Stieltjes integrable function with respect to $d\varphi$ whose singularities can only be the origin and/or infinity. It will also be assumed that the following integrals (moments) exist

$$c_n = \int_0^\infty x^n \, d\varphi(x) < \infty, \quad \forall\, n \in \mathbf{Z}. \tag{1.1}$$

The integral $I(f)$ will be approximated by means of a quadrature formula of the form

$$I_n(f) = \sum_{k=1}^{n} \lambda_k f(x_k),$$

which is characterized by the coefficients or weights $\{\lambda_k\}_{k=1}^n$ and the nodes $\{x_k\}_{k=1}^n$ which are supposed to lie on $(0,\infty)$ and satisfy $x_j \neq x_k$ if $j \neq k$. An appropriate election of the $2n$ parameters is required if we want $I_n(f)$ to be a good estimation of $I(f)$. This will be done in the subsequent sections by imposing that $I_n(f)$ integrates exactly as many functions as possible in the space

$$\Lambda = \operatorname{span}\{x^j : j \in \mathbf{Z}\}.$$

The elements in Λ are called Laurent polynomials or L-polynomials. We also consider the subspaces

$$\Lambda_{m,n} = \operatorname{span}\{x^j : m \leq j \leq n\}, \tag{1.2}$$

with $m, n \in \mathbf{Z}$ and $m \leq n$. In order to construct a sequence of nested subspaces like (1.2), whose union is contained in Λ, let us start from two nondecreasing sequences $\{p(n)\}$ and $\{q(n)\}$ of nonnegative integers verifying $p(n) + q(n) = n$, $n = 0, 1, 2, \dots$. Set $\mathcal{L}_n = \Lambda_{-p(n),q(n)}$ and $\Lambda_{-p,q} = \cup_{n=0}^{\infty} \mathcal{L}_n$ where $p = \lim_{n\to\infty} p(n)$ and $q = \lim_{n\to\infty} q(n)$. Note that $\mathrm{d}im(\mathcal{L}_n) = n+1$ and $\mathcal{L}_n \subset \mathcal{L}_{n+1}$. For $p = 0$ and $q = \infty$, $\Lambda_{0,\infty} = \Pi$ (the space of all polynomials), while for $p = q = \infty$, it results in $\Lambda_{-p,q} = \Lambda_{-\infty,\infty} = \Lambda$. In the sequel we denote for a nonnegative integer k, the space of polynomials of degree k at most as Π_k. Since $\{x^j\}_{j=-p(n-1)}^{q(n-1)}$ represents a Markov system in $(0,\infty)$ for n distinct nodes $x_{1n}, \dots, x_{n,n}$ on $(0,\infty)$, there exist uniquely defined weights $\lambda_{1,n}, \dots, \lambda_{n,n}$ such that the quadrature formula $I_n(f) = \sum_{i=1}^{n} \lambda_{i,n} f(x_{i,n})$ is exact in $\mathcal{L}_{n-1}$.

Furthermore, by taking $R_n \in \mathcal{L}_{n-1}$, such that $R_n(x_{i,n}) = f(x_{i,n})$, $i = 1, \dots, n$ and writing

$$R_n(x) = \sum_{i=1}^{n} f(x_{i,n}) L_{i,n}(x)$$

where $L_{i,n} \in \mathcal{L}_{n-1}$ and $L_{i,n}(x_{k,n}) = \delta_{i,k}$, it follows that

$$I_n(f) = I(f), \quad \forall\, f \in \mathcal{L}_{n-1} \quad \text{and} \quad \lambda_{i,n} = I(L_{i,n}), \quad i = 1, \dots, n.$$

For this reason, the quadrature formula defined above will be called of interpolatory type in $\mathcal{L}_{n-1}$.

Quadrature formulas based upon Laurent polynomials were introduced earlier by Jones, Thron and Waadeland [**13**] in connection with the solution of the so-called strong Stieltjes moment problem. Algebraic properties for such quadrature formulas were given by Jones and Thron [**12**], see also [**11**]. These papers motivated the development of a theory on orthogonal Laurent polynomials (see e.g., [**6**]) which is parallel to the theory known for the usual orthogonal polynomials. For an alternative approach, see [**15**]. Connections of the quadrature formulas with two-point Padé approximation have been studied by the present authors in a series of papers [**5, 2, 3**]. On the other hand, McCabe and Ranga [**20**] have considered the selection when $\varphi'(x)$, $d\varphi(x) = \varphi'(x)\,dx$, satisfies certain symmetry properties (see also [**19, 18**]). Numerical examples can be found in [**9**] (see also [**4**]). Meanwhile convergence properties are given in the recent papers [**2**–**4**].

The paper is organized as follows. In Section 2 we recall some results concerning Gauss-type quadrature formulas and derive some technical results to be used in subsequent sections. In Section 3 we characterize certain weight functions which can be used to make the set of polynomials (or Laurent polynomials) dense in the set of continuous functions with respect to the uniform norm, weighted by this weight function. The main result is some fundamental property of the functions $Q(z) = \sum_{k=0}^{\infty} z^k/c_k$ and $M(z) = \sum_{k=0}^{\infty} z^k/c_{-k}$. These play a prominent role in Section 4 where we characterize a class of functions for which the Gauss-type quadrature formulas converge. In Section 5, error estimates for these quadratures are given when the integrand f satisfies certain continuity conditions. Finally, in Section 6 such estimates are illustrated when applied to the family of distributions $d\varphi(x) = x^{\alpha} \exp\{-(x^{\gamma_1} + x^{-\gamma_2})\}\, dx$, $\gamma_1, \gamma_2 \geq 1/2$, $\alpha \in \mathbf{R}$.

This paper is devoted to Gauss-type quadrature formulas. In a subsequent paper we shall look at the convergence properties of more general interpolatory quadrature formulas, and we shall look at more general (possibly complex) distribution.

2. Gauss-type quadrature formulas. As a consequence of (1.1), we can define an inner product over Λ

$$\langle f, g \rangle = \int_0^{\infty} f(x)\overline{g(x)}\, d\varphi(x), \quad \forall\, f, g \in \Lambda.$$

By applying the Gram-Schmidt orthogonalization process to the basis $\{x^j : -p(n) \leq j \leq q(n)\}$ of $\mathcal{L}_n$ an orthogonal basis $\{L_0, L_1, \dots, L_n\}$ can be obtained so that $L_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ and $L_n \perp \mathcal{L}_{n-1}$ ($\mathcal{L}_{-1} = \varnothing$). Observe that L_n is uniquely defined up to a multiplicative constant factor.

The sequence $\{L_n\}_{n=0}^{\infty}$ will be called a sequence of orthogonal Laurent polynomials with respect to the distribution function φ and the "ordering" induced by the sequence $\{p(n)\}_{n=0}^{\infty}$ (or $\{q(n)\}_{n=0}^{\infty}$). Paralleling the polynomial structure [**14**] rather closely, the following can be easily proved.

Theorem 2.1. *Let $\{L_n\}_{n=0}^{\infty}$ be a sequence of orthogonal Laurent polynomials as defined above. Then*

(1) *For each $n \geq 1$, $L_n(x)$ has exactly n distinct zeros $x_{1,n}, \dots, x_{n,n}$ on $(0, \infty)$.*

(2) *Let $I_n(f) = \sum_{i=1}^{n} \lambda_{i,n} f(x_{i,n})$ be the interpolatory quadrature formula based upon the zeros of $L_n(x)$, then*

(a) *$\lambda_{i,n} > 0$, $i = 1, \dots, n$,*

(b) *$I_n(R) = I(R)$ for all $R \in \Lambda_{-a,b}$, $a = p(n) + p(n-1)$ and $b = q(n) + q(n-1)$.*

Remark 2.2. Observe that $\operatorname{diam}(\Lambda_{-a,b}) = 2n$. For this reason, such a quadrature formula will be called of Gauss-type. However, it should be remarked that, unlike in the polynomial case, now it does not hold in general that $I(f) = I_n(f)$ for all $f \in \mathcal{L}_{2n-1}$. This happens if and only if the sequence $\{p(n)\}$ satisfies $p(n) + p(n-1) = p(2n-1)$. Thus if we take $p(n) = 0$ (or equivalently $q(n) = n$) for all n, then $\mathcal{L}_n = \Pi_n$ and the classical Gaussian quadratures are recovered.

Also, when considering $p(n) = E\left[\frac{n+1}{2}\right]$ or $p(n) = E\left[\frac{n}{2}\right]$, where $E[x]$ denotes the integer part of x, we see that $p(n) + p(n-1) = p(2n-1)$. These are the cases that have been basically studied in the literature on orthogonal Laurent polynomials. See e.g. [**6**] and the references mentioned there. As far as we know, the situation involving a general sequence $\{p(n)\}$ still remains to be studied.

Positivity of the weights $\lambda_{j,n}$, $j = 1, \dots, n$, $n = 1, 2, \dots,$ turns out to be essential when considering the convergence of the sequence $I_n(f) = \sum_{j=1}^n f(x_{j,n})\lambda_{j,n}$ to $I(f)$. The positivity is clearly displayed in the following.

Theorem 2.3. *Let $\{\lambda_{j,n}\}_{j=1}^n$ denote the weights in the Gauss-type quadrature formula defined above. Then*

$$\sum_{j=1}^n \lambda_{j,n} = c_0 \quad \text{and} \quad \lambda_{j,n} = I(L_{j,n}^2), \quad j = 1, \dots, n$$

where the L-polynomials $L_{j,n} \in \mathcal{L}_{n-1}$ satisfy $L_{j,n}(x_{k,n}) = \delta_{j,k}$, $1 \le j, k \le n$ and $c_0 = \int_0^\infty d\varphi(x)$.

Proof. Since $L_{j,n} \in \mathcal{L}_{n-1}$, we have $L_{j,n}^2 \in \Lambda_{-2p(n-1),2q(n-1)} \subset \Lambda_{-a,b}$ with $a = p(n) + p(n-1)$ and $b = q(n) + q(n-1)$. Hence,

$$I(L_{j,n}^2) = I_n(L_{j,n}^2) = \sum_{k=1}^n \lambda_{k,n} L_{j,n}^2(x_{k,n}) = \lambda_{j,n},$$

and

$$\sum_{j=1}^n \lambda_{j,n} = I\left(\sum_{j=1}^n L_{j,n}\right) = I(1) = c_0. \qquad \square$$

Remark 2.4. The L-polynomials $\{L_{j,n}\}_{j=1}^n$ represent an orthogonal set, i.e.,

$$\langle L_{j,n}, L_{k,n} \rangle = \lambda_{j,n}\delta_{j,k}, \quad 1 \le j, k \le n.$$

On the other hand, the authors prove in [**4**] that the above Gauss-type quadrature formulas are actually Riemann-Stieltjes sums providing the following result on convergence for such formulas which will be crucial for our results in Section 4 in terms of the behavior of the weights $\lambda_{j,n}$. Thus one has [**4**, Theorem 3.1].

Theorem 2.5. *Let* $\{\lambda_{j,n}\}_{j=1}^{n}$ *be the weights of the Gauss-type quadrature formula with either* $p = \infty$ *or* $q = \infty$*. Then* $\lim_{n\to\infty} I_n(f) = I(f)$ *for any function* f*, Riemann-Stieltjes integrable with respect to* $d\varphi$*, if and only if* $\lim_{n\to\infty} \lambda_{j,n} = 0$ *uniformly in* j*.*

Furthermore, as a consequence of the results given by López-Lagomasino et al. [**10, 15**], the following general result can be deduced.

Theorem 2.6. *Let* φ *be a distribution function on* $\mathbf{R}^+$ *so that the moments* $c_n = \int_0^\infty x^n \, d\varphi(x)$ *exist for any integer* n*. Assume that either*

$$\lim_{n\to\infty} [n - p(n)] = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} c_n^{-1/2n} = \infty \tag{2.1}$$

or

$$\lim_{n\to\infty} p(n) = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} c_{-n}^{-1/2n} = \infty \tag{2.2}$$

hold. Then the sequence $\{I_n(f)\}$ *of Gauss-type quadrature formulas converges to* $I(f)$ *for any function that is Riemann-Stieltjes integrable with respect to* $d\varphi$*.*

Proof. Denote $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. In [**15**] it can be seen that conditions (2.1)–(2.2) are sufficient to ensure the convergence of the sequence of rational functions $F_n(z) = I_n\left(\frac{1}{z-x}\right)$, $z \in \hat{\mathbf{C}} \setminus \mathbf{R}^+$ to $F_\varphi(z) = \int_0^\infty \frac{d\varphi(x)}{z-x}$ on compacts of $\hat{\mathbf{C}} \setminus \mathbf{R}^+$. On the other hand, in [**10**] the authors establish that the convergence of such rational approximants, called two-point Padé approximants to $F_\varphi(z)$ can be characterized in terms of the convergence of the sequence $\{I_n(f)\}$ to $I(f)$ for any Riemann-Stieltjes integrable function with respect to $d\varphi$. $\square$

By using similar arguments as in the paper by Uspensky [**22**], we get as a consequence of the last two theorems:

Corollary 2.7. *Let f be a function satisfying one of the following conditions.*

(1) *f is locally Riemann integrable on $\mathbf{R}^+$ and there exist constants M and m such that*

$$|f(x)| \leq x^m, \quad x \geq M \geq 0 \text{ and } m \in \mathbf{N}$$

and conditions (2.1) *hold.*

(2) *f is Riemann integrable on any interval $[b, \infty) \subset \mathbf{R}^+$ and there exist constants K and k such that*

$$|f(x)| \leq x^{-k}, \quad x \leq K, \quad K \geq 0 \text{ and } k \in \mathbf{N}$$

and conditions (2.2) *hold.*

Then $\lim_{n\to\infty} I_n(f) = I(f)$ *where* $\{I_n(f)\}$ *is the sequence of Gauss-type quadrature formulas.*

It seems quite natural that convergence can be assured for integrands f exhibiting a behavior similar to Laurent polynomials around the origin and at infinity because of the existence of the moments $c_n = \int_0^\infty x^n \, d\varphi(x)$ for all integers n.

However, when convergence is intended in a larger class of functions with a more general behavior than the one displayed in Corollary 2.7, then conditions (2.1) or (2.2) should be more deeply analyzed. In the forthcoming section the necessary ingredients will be given.

3. Density results. In order to establish an estimation for the rate of convergence of the sequence of Gauss-type quadrature formulas, an essential role will be played by the two functions associated with the moments sequence, namely $\{c_n\}_{-\infty}^{\infty}$

$$Q(z) = \sum_{k=0}^{\infty} \frac{z^k}{c_k}, \quad z \in \mathbf{C}, \tag{3.1}$$

$$M(z) = \sum_{k=0}^{\infty} \frac{z^k}{c_{-k}}, \quad z \in \mathbf{C}. \tag{3.2}$$

Take into account that, since φ is a distribution function that has infinitely many points of increase on any $[a,b] \subset \mathbf{R}^+$, it follows that for $k \geq 1$ and for any $a > 0$

$$c_k = \int_0^\infty x^k \, d\varphi(x) \geq \int_a^\infty x^k \, d\varphi(x) \geq a^k \int_a^\infty \, d\varphi(x).$$

Since $0 < \int_a^\infty d\varphi(x) < c_0$, we get $\underline{\lim}_{k\to\infty} [\int_a^\infty d\varphi(x)]^{1/k} = 1$. Hence, $\underline{\lim}_{k\to\infty} (c_k)^{1/k} \geq a$ and, since a can be arbitrarily large, the limit has to be infinite.

Similarly, using the interval $[0,a]$ for $a > 0$ arbitrary (small), it can be derived in a similar way that also $\underline{\lim}_{k\to\infty} (c_{-k})^{1/k} = \infty$. Therefore, $\underline{\lim}_{n\to\infty} 1/\sqrt[n]{c_{\pm n}} = 0$ and by Lemma 3.1 below we conclude that $\lim_{n\to\infty} 1/\sqrt[n]{c_{\pm n}} = 0$, and hence both functions $Q(y)$ and $M(y)$ are entire.

It is our intention to show that these functions can be used to construct weight functions for the uniform norm, such that the polynomials or the L-polynomials can approximate any continuous function to an arbitrary precision in $(0,\infty)$. In other words, that the polynomials or L-polynomials are dense in the set of continuous functions with respect to this weighted uniform norm.

For our purposes, we will start with the following.

Lemma 3.1. *Let $\{c_n\}_{n=0}^\infty$ be the sequence of positive moments for $d\varphi$ and assume that $c_0 = 1$. Then*

(1) $c_n^2 \leq c_{n-1} \cdot c_{n+1}$, $n = 1, 2, \dots$.

(2) $\sqrt[n]{c_n} \leq \sqrt[n+1]{c_{n+1}}$, $n = 1, 2, \dots$.

(3) $c_j \cdot c_{n-j} \leq c_n$ *with* $n \in \mathbf{N}$ *and* $1 \leq j \leq n$.

Proof. (1) This follows immediately from the Cauchy-Schwarz inequality.

(2) This follows from Hölder's inequality with $f(t) = t^n$, $g(t) = 1$ and $p = (n+1)/n$, $q = n+1$.

(3) This follows directly from (2) since for $0 \leq j \leq n$ we have $c_j \leq c_n^{j/n}$ and $c_{n-j} \leq c_n^{(n-j)/n}$. $\square$

Remark 3.2. The above result is also valid for the sequence $\{c_n\}$ with $n \leq 0$.

Definition 3.3. Let $\mathbf{A}$ be an unbounded subset of $\mathbf{R}$ and h a positive bounded continuous function defined on $\mathbf{A}$. Consider the space

$$C_h(\mathbf{A}) = \{f : \mathbf{A} \to \mathbf{C} : f \text{ is a continuous function and } \lim_{|x|\to\infty} f(x)h(x) = 0\}$$

and the norm $|f|_h = \sup_{x\in\mathbf{A}} |f(x)|h(x)$. Then h will be called a *weight* with respect to the uniform norm in $\mathbf{A}$ if Π is dense in $C_h(\mathbf{A})$ in the sense that, for any $f \in C_h(\mathbf{A})$ and any $\varepsilon > 0$, there is a polynomial P such that $|f - P|_h < \varepsilon$. We shall refer to $|\cdot|_h$ as the weighted uniform norm in $\mathbf{A}$.

Remark 3.4. Note that if f is a weight in $\mathbf{A}$, then $C_h(A)$ is a Banach space with this weighted uniform norm. See [**16**, p. 28] and [**1**].

As a consequence of the density of the polynomial set Π, an interesting question could be to find an estimate of the error of the best "weighted" uniform approximation of degree at most $k \geq 0$, i.e., to find an estimate for

$$\varepsilon_k(f; h, \mathbf{A}) = \inf_{P\in\Pi_k} \|f - P\|_h = \inf_{P\in\Pi_k} \sup_{x\in\mathbf{A}} |[f(x) - P(x)]h(x)|. \tag{3.3}$$

A solution can be found in the following (see [**17**, p. 102]).

Theorem 3.5. *Let $\tilde{h} > 0$ be a weight function in $\mathbf{R}$ such that $p(x) = -\log \tilde{h}(x)$ is even, continuous and strictly increasing on $\mathbf{R}^+$. If f admits a derivative of order d which is uniformly continuous on $\mathbf{R}$ with modulus of continuity $\omega_f(\delta)$ as defined below, then*

$$\varepsilon_k(f; \tilde{h}, \mathbf{R}) \leq B\nu_k^{-d}\omega_f(\nu_k^{-1})$$

where $q(t)$ is the inverse of $p(t)$, $\nu_k = \int_1^k dt/q(t)$,

$$\omega_f(\delta) = \sup\left\{|f(x) - f(y)| : \frac{|x - y|}{(1 + |x|)\cdot(1 + |y|)} < \delta,\ x, y \in \mathbf{R}\right\}$$

and B a positive constant independent of f.

Now we can immediately prove the following

Proposition 3.6. *Let $h(x)$ be defined in $\mathbf{R}^+$ and set $\tilde{h}(x) = h(x^2)$, $x \in \mathbf{R}$. If $\tilde{h}(x)$ is a weight in $\mathbf{R}$, then $h(x)$ is a weight in $\mathbf{R}^+$ and if $f \in C_h(\mathbf{R}^+)$, then the weighted minimax errors satisfy*

$$\varepsilon_n(f; h, \mathbf{R}^+) = \varepsilon_{2n}(\tilde{f}; \tilde{h}, \mathbf{R}),$$

where $\tilde{f}(x) = f(x^2)$.

Proof. Let $P_n \in \Pi_n$ and $Q_{2n} \in \Pi_{2n}$ be the weighted minimax polynomials which are the solution to the problems

$$\inf_{P \in \Pi_n} \left\{ \sup_{x \geq 0} |[f(x) - P(x)]h(x)| \right\}$$

and

$$\inf_{Q \in \Pi_{2n}} \left\{ \sup_{x \in \mathbf{R}} |[\tilde{f}(x) - Q(x)]\tilde{h}(x)| \right\},$$

respectively. Because

$$\begin{aligned} \sup_{x \in \mathbf{R}} [\tilde{f}(x) - Q_{2n}(x)]\tilde{h}(x) &= \sup_{x \in \mathbf{R}} [f(x^2) - Q_{2n}(x)]h(x^2) \\ &= \sup_{x \in \mathbf{R}} [f(x^2) - Q_{2n}(-x)]h(x^2) \\ &= \sup_{x \in \mathbf{R}} [\tilde{f}(x) - Q_{2n}(-x)]\tilde{h}(x), \end{aligned}$$

it follows from the unicity of the minimax solution that $Q_{2n}(x) = Q_{2n}(-x)$, so that there should exist a polynomial $S \in \Pi_n$ such that $Q_{2n}(x) = S(x^2)$. Thus

$$\begin{aligned} \varepsilon_{2n}(\tilde{f}; \tilde{h}, \mathbf{R}) &= \sup_{x \in \mathbf{R}} |[f(x^2) - S(x^2)]h(x^2)| \\ &= \sup_{x \geq 0} |[f(x) - S(x)]h(x)| \\ &\geq \sup_{x \geq 0} |[f(x) - P_n(x)]h(x)| = \varepsilon_n(f; h, \mathbf{R}^+). \end{aligned}$$

On the other hand,

$$\begin{aligned}\varepsilon_n(f;h,\mathbf{R}^+) &= \sup_{x\geq 0}|[f(x)-P_n(x)]h(x)| \\ &= \sup_{x\in\mathbf{R}}|[f(x^2)-P_n(x^2)]h(x^2)| \\ &= \sup_{x\in\mathbf{R}}|[\tilde{f}(x)-Q(x)]\tilde{h}(x)| \geq \varepsilon_{2n}(\tilde{f};\tilde{h},\mathbf{R})\end{aligned}$$

since $Q(x) := P_n(x^2) \in \Pi_{2n}$. This proves the proposition. □

Definition 3.7. Consider a positive bounded continuous function H defined on $(0,\infty)$, on the space

$$C_H((0,\infty)) = \{f:(0,\infty)\to\mathbf{C} : f \text{ continuous and } \lim_{x\to\infty} f(x)H(x) = \lim_{x\to 0} f(x)H(x) = 0\},$$

and the norm $\|f\|_H = \sup_{x\in(0,\infty)}|f(x)|H(x)$. We shall call H an *L-weight* on $(0,\infty)$ with respect to the uniform norm if the space Λ of all Laurent polynomials is dense in $C_H((0,\infty))$ in the sense that for all $f \in C_H((0,\infty))$ and any $\varepsilon > 0$, there is an $L \in \Lambda$ such that $\|f - L\|_H < \varepsilon$.

Remark 3.8. We note that also here, if H is an L-weight in $(0,\infty)$, then $C_H((0,\infty))$ is a Banach space with the weighted uniform norm $|f|_{\infty^H}$.

Finally let us see how to find L-weights in $(0,\infty)$ from weights in $\mathbf{R}^+$. The following holds.

Proposition 3.9. *Let h and p be two weights in $\mathbf{R}^+$. Set $H(x) = h(x)p(1/x)$; then $H(x)$ is an L-weight in $(0,\infty)$.*

Proof. Let $f \in C_H((0,\infty))$. For a given $\varepsilon > 0$, we have to find a Laurent polynomial $L(x)$ so that

$$|[f(x)-L(x)]H(x)| < \varepsilon, \quad \forall\, x \in (0,\infty).$$

Define $f_0(x)$ and $f_\infty(x)$ as follows:

$$f_0(x) = \begin{cases} f(x), & x \in (0,1] \\ f(1), & x \in (1,\infty) \end{cases} \quad ; \quad f_\infty(x) = \begin{cases} f(1), & x \in (0,1] \\ f(x), & x \in (1,\infty). \end{cases}$$

Clearly $f(x) = f_0(x) + f_\infty(x) - f(1)$. Since $f_0(1/x) \in C_p(\mathbf{R}^+)$, there exists a polynomial $P_0(x)$ such that

$$\|[f_0(1/x) - P_0(x)]p(x)\|_\infty = \|[f_0(x) - P_0(1/x)]p(1/x)\|_\infty < \delta_0. \tag{3.4}$$

On the other hand, $f_\infty \in C_h(\mathbf{R}^+)$. Therefore,

$$\|[f_\infty(x) - P_1(x)]h(x)\|_\infty < \delta_\infty. \tag{3.5}$$

Now, for any $x \in (0,\infty)$, one has

$$\begin{aligned} &|[f(x) - (P_0(1/x) + P_1(x) - f(1))]h(x)p(1/x)| \\ &\qquad = |[f_0(x) - P_0(1/x) + f_\infty(x) - P_1(x)]h(x)p(1/x)| \\ &\qquad \le C_1|[f_0(x) - P_0(1/x)]p(1/x)| + C_2|[f_\infty(x) - P_1(x)]h(x)| \end{aligned}$$

where $C_1 = \|h\|_\infty < \infty$ and $C_2 = \|p\|_\infty < \infty$. From (3.4)–(3.5) and taking $L(x) = P_0(1/x) + P_\infty(x) - f(1)$, the proof follows for $\delta_0 = \varepsilon/(2C_1)$ and $\delta_\infty = \varepsilon/(2C_2)$. □

Corollary 3.10. *If for $f \in C_H((0,\infty))$ we denote by*

$$M_n(f;H) = \inf_{L \in \mathcal{L}_n} \|f - L\|_H,$$

the L-weighted minimax error, then

$$M_n(f;H) \le C_1 \varepsilon_{p(n)}(\hat{f}_0; p, \mathbf{R}^+) + C_2 \varepsilon_{q(n)}(f_\infty; h, \mathbf{R}^+),$$

where $\hat{f}_0(x) = f_0(1/x)$, C_1 and C_2 are constants and $\varepsilon_k(f;h,\mathbf{A})$ is as in (3.3).

Next a theorem due to Bernstein (see e.g. [**17**]) will be stated in order to characterize a weight in $\mathbf{R}$.

Theorem 3.11. *Let* $\{a_k\}_0^\infty$ *be a sequence of nonnegative real numbers with* $a_0 > 0$. *Let* $\omega(x) = \sum_{k=0}^\infty a_k x^{2k}$ *and* $\tilde{h}(x) = 1/\omega(x)$. *Then* $\tilde{h}$ *is a weight in* $\mathbf{R}$ *if and only if* $\int_1^\infty \frac{\log \omega(x)}{1+x^2}\,dx = \infty$.

Before stating the main theorem of this section, we need the following classical result concerning quasi-analytic functions (see [**21**]).

Theorem 3.12 (Denjoy-Carleman). *Let* $\{M_n\}_{n=0}^\infty$ *be a sequence of positive numbers satisfying* $M_0 = 1$ *and* $M_n^2 \leq M_{n-1}M_{n+1}$ *for* $n \geq 1$. *Define the function* $F(x) = \sum_{k=0}^\infty (x^k/M_k)$, $x > 0$. *Then* $\int_0^\infty \frac{\log F(x)}{1+x^2}\,dx = \infty$ *if and only if* $\sum_{k=1}^\infty \frac{1}{\sqrt[k]{M_k}} = \infty$.

Now we can prove the main result of this section.

Theorem 3.13. *Let* Q *be the function defined by* (3.1), *i.e.,* $Q(x) = \sum_{k=0}^\infty x^k/c_k$, $x \in \mathbf{C}$ *with* $c_k = \int_0^\infty x^k\,d\varphi(x)$, $k = 0, 1, 2, \ldots$ *satisfying* (2.1). *Set* $\tilde{h}(x) = 1/Q(x^2)$. *Then* $\tilde{h}(x)$ *is a weight in* $\mathbf{R}$.

Proof. Observe that $Q(x^2) = \sum_{k=0}^\infty x^{2k}/c_k$; then by Theorem 3.11 it suffices to check that

$$\int_1^\infty \frac{\log(Q(x^2))}{1+x^2}\,dx = \infty.$$

For this, let us consider the sequence $\{\log c_k\}$. By (1) of Lemma 3.1 we know that $\{(\log c_k - \log c_{k-1})/2\}$ is a nondecreasing sequence. This fact enables us to construct a convex polynomial $l(x)$ verifying $l(2k) = \log(c_k)$ for any $k \in \mathbf{N}$. Thus, one can immediately see that

$$l(k) \leq \frac{1}{2}l(k-1) + \frac{1}{2}l(k+1), \quad k \in \mathbf{N}.$$

Let us consider a new sequence $\{\sigma(k)\}_0^\infty$ given by $\sigma(k) = e^{l(k)}$ and the function

$$F(x) = \sum_{k=0}^\infty \frac{x^k}{\sigma(k)}.$$

By Theorem 3.12, $\int_0^\infty \frac{\log F(x)}{1+x^2}\,dx = \infty$ if and only if $\sum_{k=0}^\infty \frac{1}{\sqrt[k]{\sigma(k)}} = \infty$. Now

$$\begin{aligned}\sum_{k=0}^\infty \frac{1}{\sqrt[k]{\sigma(k)}} &= \sum_{k=0}^\infty \frac{1}{\sqrt[2k]{\sigma(2k)}} + \sum_{k=0}^\infty \frac{1}{\sqrt[2k+1]{\sigma(2k+1)}} \\ &= \sum_{k=0}^\infty \frac{1}{\sqrt[2k]{c_k}} + \sum_{k=0}^\infty \frac{1}{\sqrt[2k+1]{\sigma(2k+1)}}.\end{aligned}$$

We are assuming that $\sum_{k=0}^\infty \frac{1}{\sqrt[2k]{c_k}} = \infty$ (condition (2.1)). Therefore, $\int_0^\infty \frac{\log F(x)}{1+x^2}\,dx = \infty$. On the other hand,

$$F(x) = Q(x^2) + \sum_{k=0}^\infty \frac{x^{2k+1}}{\sigma(2k+1)} \le \left(1 + \frac{x}{\sqrt{c_1}}\right) Q(x^2), \quad x \ge 0.$$

By the definition of $l(x)$ and (3) of Lemma 3.1, we have $\sigma(2k+1) = \sqrt{c_k}\sqrt{c_{k+1}} \ge \sqrt{c_1} \cdot c_k$. Hence,

$$\int_0^\infty \frac{\log\left[(1 + (x/\sqrt{c_1}))Q(x^2)\right]}{1+x^2}\,dx = \infty$$

and thus

$$\int_1^\infty \frac{\log Q(x^2)}{1+x^2}\,dx = \infty. \qquad \square$$

Remark 3.14. Making use of similar arguments and starting from condition (2.2), it can be proved that $1/M(x^2)$ is a weight in $\mathbf{R}$ with M defined by (3.2), i.e., $M(x) = \sum_{k=0}^\infty (x^k/c_{-k})$, $x \in \mathbf{C}$ and $c_k = \int_0^\infty x^k\,d\varphi(x)$, $k = 0, -1, -2, \dots$.

Corollary 3.15. *The functions $h(x) = 1/Q(x)$ and $p(x) = 1/M(x)$ are weight functions in $\mathbf{R}^+$.*

Corollary 3.16. *The function $H(x) = \dfrac{1}{Q(x)\cdot M(1/x)}$ is an L-weight function in $(0, \infty)$.*

4. Convergence. The results of this section complete those given by the present authors in [**4**]. As usual, $I_n(f) = \sum_{j=1}^n \lambda_{j,n} f(x_{j,n})$ will denote the n-point Gauss-type quadrature formula which is exact in $\Lambda_{-(p(n)+p(n-1)),(q(n)+q(n-1))}$ with $\{p(n)\}$ and $\{q(n)\}$ two nondecreasing sequences of nonnegative integers satisfying $p(n)+q(n)=n$, $n=0,1,2,\dots$. Recall also that the nodes $\{x_{j,n}\}$ are the zeros of the nth Laurent polynomial orthogonal with respect to $d\varphi$ and the ordering induced by $\{p(n)\}$ (or $\{q(n)\}$). Under these conditions, we have our first result (compare with Theorem 3.6 in [**4**]).

Theorem 4.1. *Let φ be a distribution function on $\mathbf{R}^+$ whose moments satisfy*

$$\sum_{n=1}^{\infty} c_n^{-1/2n} = \infty, \tag{4.1}$$

and assume that $\lim_{n\to\infty} q(n)=\infty$. Let $\{I_n(f)\}$ be the sequence of Gauss-type formulas introduced above. Then, for any locally integrable function f on $\mathbf{R}^+$ satisfying for sufficiently large x

$$|f(x)| \le c\cdot Q(sx), \quad 0<s<1, \quad c>0 \tag{4.2}$$

(Q *as given by* (3.1)) *it holds that*

$$\lim_{n\to\infty} I_n(f) = I(f).$$

Proof. Our main goal will be to first show that the integral $I(Q(sx))$ exists.

Take $b>0$; then because of the uniform convergence of the series (3.1) on $[0,b]$, we have

$$\begin{aligned}\int_0^b Q(sx)\,d\varphi(x) &= \lim_{n\to\infty}\sum_{l=0}^{n}\frac{s^l}{c_l}\int_0^b x^l\,d\varphi(x) = \sum_{l=0}^{\infty}\frac{s^l}{c_l}\int_0^b x^l\,d\varphi(x)\\ &\le \sum_{l=0}^{\infty}\frac{s^l}{c_l}\int_0^\infty x^l\,d\varphi(x) = \sum_{l=0}^{\infty} s^l < \infty.\end{aligned}$$

Thus we conclude that $I(Q(sx))$ exists for all $s: 0<s<1$. Now, making use of Theorem 2.5 and Theorem 2.6 and Upenski's arguments [**22**], the proof of the theorem is easily completed. $\square$

Remark 4.2 In case the moments $\{c_k\}_{k=0}^{\infty}$ satisfy another condition, for instance $\sum_{k=1}^{\infty} c_{k-1}/c_k < \infty$, the class of functions where convergence holds can be enlarged. This implies that the integral $I(Q(x)/(1+x))$ exists. Indeed,

$$\begin{aligned}\int_0^b \frac{Q(x)}{1+x}\, d\varphi(x) &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{c_k} \int_0^b \frac{x^k}{1+x}\, d\varphi(x) \\ &\le \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{c_k} \int_0^b x^{k-1}\, d\varphi(x) \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{c_{k-1}}{c_k} = 1 + \sum_{k=0}^{\infty} \frac{c_{k-1}}{c_k}.\end{aligned}$$

Therefore, Theorem 4.1 is also true for any locally integrable function f on $\mathbf{R}^+$ such that

$$|f(x)| \le c\, \frac{Q(x)}{1+x}, \quad x > T, \quad c > 0,$$

for T sufficiently large. For instance, take $d\varphi(x) = \frac{\exp\{-\sqrt{x}\}}{2\sqrt{x}}\, dx$. Then $c_n = (2n)!$ and $Q(x) = \cosh(\sqrt{x})$.

If, instead of condition (4.1), we have that

$$\sum_{n=1}^{\infty} c_{-n}^{-1/2n} = \infty,$$

a similar result can be proved in an analogous way, but now replacing the function $Q(x)$ by $M(x)$ given in (3.2).

Theorem 4.3. *Let φ be a distribution function on $\mathbf{R}^+$ whose moments satisfy*

$$\sum_{n=1}^{\infty} c_{-n}^{-1/2n} = \infty \tag{4.3}$$

and assume that $\lim_{n\to\infty} p(n) = \infty$. Finally, let $\{I_n(f)\}$ be the sequence of Gauss-type quadrature formulas as discussed above. Then, *for any integrable function f on $[a,\infty)$, $a>0$, satisfying for sufficiently small x,*

$$|f(x)| \le \tilde{c}M(r/x), \quad 0<r<1, \quad \tilde{c}>0, \tag{4.4}$$

(M as given by (3.1)), it holds that

$$\lim_{n\to\infty} I_n(f) = I(f).$$

Remark 4.4. Theorem 4.1 and Theorem 4.3 are also valid for any locally integral function on $(0,\infty)$ with both local behavior (4.2) and (4.4). In this case, both conditions (4.1) and (4.3) are required.

Remark 4.5. Note that, for $p(n)=0$, $n=1,2,\dots$, the classical Gaussian formulas are recovered. Assuming $c_n \le CR^{2n}(2n+1)!$ with $C,R>0$, the convergence results given by Uspensky [**22**] can be obtained. Other results of convergence for Gauss-type formulas were given in [**8**] concerning the "balanced" situation, i.e., $p(n)=E\left[\frac{n+1}{2}\right]$ and assuming $c_n \le CR^{2n}(2n+1)!$. In a more general setting, if we assume $c_n \le CR^{n\gamma}\Gamma(\gamma(n+1+\theta))$ with $C,R>0$, $0<\gamma\le 2$, and $\theta>-1$, the results given by the present authors in [**4**] are deduced.

5. Error estimates. In this part we give error estimates for the n-point Gauss-type quadrature formula $I_n(f)$. More precisely certain upper bounds for $|E_n(f)| = |I(f)-I_n(f)|$ will be given when f satisfies certain continuity conditions.

Theorem 5.1. *Let $\{I_n(f)\}$ be the sequence of Gauss-type formulas introduced above and f in $C_h(\mathbf{R}^+)$ with $h(x)=1/Q(sx)$ for some s such that $0<s<1$ and Q given by* (3.1). *Then*

$$|E_n(f)| \le 2I(Q(sx))\varepsilon_b\left(f;\frac{1}{Q(sx)},\mathbf{R}^+\right)$$

where $b=q(n)+q(n-1)$ and $\varepsilon_k(f;h,\mathbf{A})$ as defined in (3.3).

Proof. Taking into account that

$$E_n(f) = I(f) - I_n(f) = \int_0^\infty f(x)\, d\varphi(x) - \sum_{j=1}^n \lambda_{j,n} f(x_{j,n})$$

and recalling that the n-point Gauss-type formula is exact in Π_b, one can write

$$\begin{aligned} E_n(f) &= I(f) - I_n(f) \\ &= \int_b^\infty [f(x) - P(x)]\, d\varphi(x) - \sum_{j=1}^n \lambda_{j,n} [f(x_{j,n}) - P(x_{j,n})], \quad \forall\, P \in \Pi_b. \end{aligned}$$

Then

$$\begin{aligned} |E_n(f)| &\leq \int_0^\infty |f(x) - P(x)|\, d\varphi(x) + \sum_{j=1}^n \lambda_{j,n} |f(x_{j,n}) - P(x_{j,n})|, \\ &\qquad \forall\, P \in \Pi_b \\ &\leq \|f - P\|_h \cdot \left(I(Q(sx)) + \sum_{j=1}^n \lambda_{j,n} Q(sx_{j,n}) \right). \end{aligned}$$

On the other hand, it is known that (see [**3**])

$$E_n(Q(sx)) = \frac{[x^a Q(sx)]^{(2n)}_{x=p}}{(2n)!} \cdot \gamma_n, \quad \rho > 0$$

where $a = p(n) + p(n-1)$ and $\gamma_n = \int_0^\infty Q_n(x)^2 x^{-a}\, d\varphi(x) > 0$ with Q_n the nth monic orthogonal polynomial for the distribution function $x^{-a}\, d\varphi(x)$. Since

$$[x^a Q(sx)]^{(k)} > 0, \quad \forall\, x : x > 0, \quad k \in \mathbf{N},$$

we have

$$\sum_{j=1}^n \lambda_{j,n} Q(sx_{j,n}) \leq I(Q(sx)),$$

so that

$$|E_n(f)| \leq 2I(Q(sx)) \|f - P\|_h, \quad \forall\, P \in \Pi_b.$$

This proves the theorem. □

Remark 5.2. It is immediately clear that $1/Q(sx)$ with $0 < s < 1$ is a weight function on $\mathbf{R}^+$, since $\{c_n/s^n\}_{n=0}^\infty$ satisfies the same properties as $\{c_n\}_{n=0}^\infty$.

Observe that in Theorem 5.1 nothing has been said about the sequence $\{p(n)\}$. Thus when taking $p(n) = 0$ ($q(n) = n$) then $b = 2n - 1$ and an estimate for error in the n-point classical Gauss formula can be obtained. Now, when dealing with a proper "Laurent polynomial" situation, we have the following.

Theorem 5.3. *Let* $\{I_n(f)\}$ *be the sequence of Gauss-type formulas introduced above with* f *belonging to* $C_H((0,\infty))$ *where* $H(x) = \frac{1}{Q(sx)\cdot M(r/x)}$ *for some* s, r *such that* $0 < s < 1$ *and* $0 < r < 1$, *while* Q *and* M *are defined in* (3.1) *and* (3.2). *Then*

$$|E_n(f)| \leq 2I(Q(sx)M(r/x))[C_1\varepsilon_a(f_0(1/x); 1/M(rx), \mathbf{R}^+) \\ + C_2\varepsilon_b(f_\infty, 1/Q(sx), \mathbf{R}^+)]$$

where $a = p(n) + p(n-1)$, $b = q(n) + q(n-1)$, f_0, f_∞, C_1 *and* C_2 *as defined in the proof of Proposition* 3.9 *and* $\varepsilon_k(f; h, \mathbf{A})$ *as in* (3.3).

Proof. We can proceed in a similar way as in the proof of Theorem 5.1 but now replacing $P \in \Pi_b$ by $L \in \Lambda_{-a,b}$, h by H and taking into account that

$$x^a Q(sx)M(r/x) = T(1/x) + \tilde{T}(x)$$

where $T(1/x) = \sum_{k=1}^\infty t_k x^{-k}$ and $\tilde{T}(x) = \sum_{k=0}^\infty \tilde{t}_k x^k$ with $t_k, \tilde{t}_k > 0$, we have that

$$[x^a Q(sx)M(r/x)]_{x=\tilde{\rho}}^{(2n)} > 0, \quad \tilde{\rho} > 0.$$

Thus, we obtain

$$|E(f)| \leq 2I(Q(sx)M(r/x))M_n(f; H),$$

with $M_n(f; H)$ as defined in Corollary 3.10. Now the proof follows by the statement of Corollary 3.10. □

6. An illustrative example. In this section the results given in Sections 3 and 5 will be applied to the family of distribution functions

$$d\varphi(x) = \varphi'(x)\,dx, \quad \varphi'(x) = x^{\alpha}\exp\{-(x^{\gamma_1} + x^{-\gamma_2})\}, \quad \gamma_1, \gamma_2 \geq 1/2, \quad \alpha \in \mathbf{R}. \tag{6.1}$$

In this respect, it should be remarked that in [**15**] asymptotic properties for certain sequences of orthogonal Laurent polynomials associated with the distribution (6.1) were studied. It is easy to prove that the moments of this family of distributions satisfy the conditions (2.1) and (2.2).

Now our purpose will be to give upper bounds for $|E_n(f)| = |I(f) - I_n(f)|$ where

$$I(f) = \int_0^{\infty} f(x)x^{\alpha}\exp\{-x^{\gamma_1} + x^{-\gamma_2})\}\,dx$$

when f satisfies certain continuity conditions. More precisely, f is a continuous function in $(0,\infty)$, verifying

$$\lim_{x\to\infty} f(x)\varphi'(x) = \lim_{x\to 0} f(x)\varphi'(x) = 0.$$

In this case, our problem clearly reduces to finding an estimate for

$$I(g) = \int_0^{\infty} g(x)\exp\{-(x^{\gamma_1} + x^{-\gamma_2})\}\,dx$$

with g a continuous function in $(0,\infty)$ satisfying

$$\lim_{x\to\infty} g(x)\mu(x) = \lim_{x\to 0} g(x)\mu(x) = 0,$$

where $\mu(x) = \exp\{-(x^{\gamma_1} + x^{-\gamma_2})\}$, $\gamma_1, \gamma_2 \geq 1/2$.

Define the auxiliary Mittag-Leffer function

$$E_{\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\gamma k + 1)}, \quad z \in \mathbf{C}, \quad \gamma > 0,$$

which satisfies (see [**7**])

$$E_{\gamma}(z) = \frac{1}{\gamma}\exp(z^{1/\gamma}) + O\left(\frac{1}{|z|}\right), \quad z \to \infty, \quad |\mathrm{Arg}\,(z)| \leq \frac{\gamma\cdot\pi}{2},$$

and thus, for real x,

$$\lim_{x\to\infty}\left|\frac{\gamma\cdot E_\gamma(x)}{\exp(x^{1/\gamma})}\right| = 1.$$

Now, proceeding as in [**4**], we have

Theorem 6.1. *Let* $\{I_n(f)\}$ *be the sequences of Gauss-type formulas introduced above and* f *belonging to* $C_H((0,\infty))$ *where* $H(x) = \exp\{-((sx)^{\gamma_1} + (x/r)^{-\gamma_2})\}$ *for some* s, r *such that* $0 < s < 1$ *and* $0 < r < 1$. *Then*

$$\begin{aligned}|E_n(f)| \le 2\chi\big[&\varepsilon_a(f_0(1/x); \exp\{-(rx)^{\gamma_2}\}, \mathbf{R}^+)\\ &+ \varepsilon_b(f_\infty(x); \exp\{-(sx)^{\gamma_1}\}, \mathbf{R}^+)\big]\end{aligned}$$

where $a = p(n) + p(n-1)$, $b = q(n) + q(n-1)$, $\gamma_1, \gamma_2 \ge 1/2$,

$$\chi = \int_0^\infty \exp\left\{-\left((1-s^{\gamma_1})x^{\gamma_1} + \frac{1-r^{\gamma_2}}{x^{\gamma_2}}\right)\right\}dx,$$

and f_0 *and* f_∞ *are as in the proof of Proposition* 3.9. *and* $\varepsilon_k(f; h, \mathbf{A})$ *as in* (3.3).

Let us next see how we can estimate the minimax error $\varepsilon_k(f; e^{-(\sigma x)^\gamma}, \mathbf{R}^+)$, $\sigma > 0$, $\gamma \ge 1/2$, $k \in \mathbf{N}$.

For this purpose set $h(x) = e^{-(\sigma x)^\gamma}$, $\gamma \ge 1/2$, $x \in \mathbf{R}^+$ and define $\tilde h(x) = h(x^2) = e^{-(\sigma'\cdot x)^{2\gamma}}$, $\sigma' = \sqrt{\sigma}$.

Since we have that $\varepsilon_k(f; h, \mathbf{R}^+) = \varepsilon_{2k}(\tilde f; \tilde h, \mathbf{R})$ where $\tilde f(x) = f(x^2)$. Using Theorem 3.5 for this family of weight functions, $\tilde h(x) = e^{-(\sigma' x)^{2\gamma}}$, $\sigma' > 0$, $\gamma \ge 1/2$, we find that the following holds.

Theorem 6.2. *Let* g *be a real function defined on* $\mathbf{R}$ *such that* $g \in C^m(\mathbf{R})$ *and* $g^{(m)}$ *is uniformly continuous on* $\mathbf{R}$ *with modulus of continuity* $\omega_g(\varepsilon)$. *Then*

$$\varepsilon_n(g; \tilde h, \mathbf{R}) \le C\nu_n^{-m}(\sigma', \gamma)\omega_g(\nu_n^{-1}(\sigma', \gamma))$$

where C *is a positive constant independent of* g *and* $\nu_n(\sigma', \gamma) = \sigma'\int_1^n x^{-1/2\gamma}\,dx$.

Note that

$$\nu_n(\sigma',\gamma) = \begin{cases} \dfrac{2\gamma\sigma'}{2\gamma-1}\big[n^{\frac{2\gamma-1}{2\gamma}}-1\big], & 2\gamma>1\\ \sigma'\log n & 2\gamma=1. \end{cases}$$

Thus, from the above theorem, it follows that

(6.2)

$$\varepsilon_n(g;\tilde{h},\mathbf{R}) \leq C\begin{cases} \left(\dfrac{2\gamma-1}{2\gamma\sigma'}\dfrac{1}{n^{\frac{2\gamma-1}{2\gamma}}-1}\right)^m \omega_g\left(\dfrac{2\gamma-1}{2\gamma\sigma'}\dfrac{1}{n^{\frac{2\gamma-1}{2\gamma}}-1}\right), & 2\gamma>1\\ \dfrac{1}{\sigma'\log n} \qquad \omega_g\left(\dfrac{1}{\sigma'\log n}\right), & 2\gamma=1. \end{cases}$$

Finally, by (6.2) and Proposition 3.6, we have

Corollary 6.3. *Under the same assumptions as in Theorem* 6.1 *with f uniformly continuous on $\mathbf{R}^+$ with modulus of continuity $\omega_f(\varepsilon)$, it holds*

$$|E_n(f)| \leq C(\Omega_0+\Omega_\infty)$$

where in the case $\gamma_1,\gamma_2>1/2$,

$$\Omega_0 = \omega_{f_0(1/x^2)}\left(\frac{2\gamma_2-1}{2\gamma_2\sqrt{r}}\cdot\frac{1}{(2a)^{\frac{2\gamma_2-1}{2\gamma_2}}-1}\right)$$

and

$$\Omega_\infty = \omega_{f_\infty}(x^2)\left(\frac{2\gamma_1-1}{2\gamma_1\sqrt{s}}\cdot\frac{1}{(2b)^{\frac{2\gamma_1-1}{2\gamma_1}}-1}\right)$$

and, in the case of $\gamma_1=\gamma_2=1/2$,

$$\Omega_0 = \omega_{f_0(1/x^2)}\left(\frac{1}{\sqrt{r}\log 2a}\right) \quad \text{and} \quad \Omega_\infty = \omega_{f_{\infty(x^2)}}\left(\frac{1}{\sqrt{s}\log 2b}\right)$$

where f_0 and f_∞ are as in the proof of Proposition 3.9*, $a=p(n)+p(n-1)$ and $b=q(n)+q(n-1)$.*

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REFERENCES

1. A. Borichev, *On weighted polynomial approximation with monotone weights*, Proc. Amer. Math. Soc. **128** (2000), 3613–3619.

2. A. Bultheel, C. Díaz-Mendoza, P. González-Vera and R. Orive, *Quadrature on the half line and two-point Padé approximants to Stieltjes functions*. Part II: *Convergence*, J. Comput. Appl. Math. **77** (1997), 53–76.

3. ———, *Quadrature on the half line and two-point Padé approximants to Stieltjes functions*. Part III: *The unbounded case*, J. Comput. Appl. Math. **87** (1997), 95–117.

4. ———, *On the convergence of certain Gauss-type quadrature formulas for unbounded intervals*, Math. Comp. **69** (2000), 721–747.

5. A. Bultheel, P. González-Vera and R. Orive, *Quadrature on the half line and two-point Padé approximants to Stieltjes functions*, Part I: *Algebraic aspects*, J. Comput. Appl. Math. **65** (1995), 57–72.

6. L. Cochran and S.C. Cooper, *Orthogonal Laurent polynomials on the real line*, in *Continued fractions and orthogonal functions* (S.C. Cooper and W.J. Thron, eds.), Marcel Dekker, New York, 1994, pp. 47–100.

7. A. Erdélyi, ed., *Higher transcendental functions*, Vol. 3, McGraw-Hill, New York, 1953.

8. C. González-Concepción, P. González-Vera and L. Casasús, *On the convergence of certain quadrature formulas defined on unbounded intervals*, in *Orthogonal polynomials and their applications* (J. Vinuesa, ed.), Lecture Notes in Pure and Appl. Math., vol. 117, Dekker, New York, 1989, pp. 147–151.

9. P.E. Gustafson and B.A. Hagler, *Gaussian quadrature rules and numerical examples for strong extensions of mass distribution function*, J. Comput. Appl. Math. **105** (1-2) (1999), 317–326.

10. J. Illan and G. López-Lagomasino, *Sobre los metodos interpolatorios de integración numérica y su conexion con la approximación racional*, Rev. Cienc. Mat. **8** (1987), 31–44.

11. W.B. Jones, O. Njåstad and W.J. Thron, *Two-point Padé expansions for a family of analytic functions*, J. Comput. Appl. Math. **9** (1983), 105–124.

12. W.B. Jones and W.J. Thron, *Orhtogonal Laurent polynomials and Gaussian quadrature*, in *Quantum mechanics in mathematics, chemistry and physics* (K. Gustafson and W.P. Reinhardt, eds.), Plenum, New York, 1984, pp. 449–455.

13. W.B. Jones, W.J. Thron and H. Waadeland, *A strong Stieltjes moment problem*, Trans. Amer. Math. Soc. **206** (1980), 503–528.

14. V.J. Krylov, *Approximate calculation of integrals*, MacMillan, New York, 1962.

15. G. López-Lagomasino and A. Martínez-Finkelshtein, *Rate of convergence of two-point Padé approximants and logarithmic asymptotics of Laurent-type orthogonal polynomials*, Constr. Approx. **11** (1995), 255–286.

16. G. Lorentz, M.V. Golitschek and Y. Makovoz, *Constructive approximation, advanced topics*, Grundlehren der Mathematische Wissenschaften, vol. 304, Springer, Berlin, 1996. isbn: 0-540-57028-4

17. S.N. Mergelyan, *Weighted approximations by polynomials*, Amer. Math. Soc. Transl. Ser. 2 **10** (1958), 59–106.

18. A.S. Ranga, *On a recurrence formula associated with strong distributions*, SIAM J. Math. Anal. **21** (1990), 1335–1348.

19. ———, *Another quadrature rule of highest algebraic degree*, Numer. Math. **28** (1994), 283–294.

20. A.S. Ranga and J.H. McCabe, *On the extensions of some classical distributions*, Proc. Edinburgh Math. Soc. **34** (1991), 12–29.

21. W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, New York, 1987.

22. J.V. Uspensky, *On the convergence of quadrature formulas related to an infinite interval*, Trans. Amer. Math. Soc. **30** (1928), 542–554.

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