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SPHERE-FOLIATED MINIMAL AND CONSTANT MEAN CURVATURE HYPERSURFACES IN SPACE FORMS AND LORENTZ-MINKOWSKI SPACE

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ABSTRACT. We prove that a sphere-foliated minimal or constant mean curvature hypersurface in hyperbolic space of dimension ≥ 5 is one of the following: hypersurface of rotation around a geodesic, geodesic hyperplane, horosphere, equidistant hypersurface, or a geodesic sphere in the upper half-space model. And we show that a sphere-foliated minimal or constant mean curvature hypersurface in sphere of dimension ≥ 5 is either a hypersurface of rotation or a hypersphere.

We also show that a hypersurface of nonzero constant mean curvature in Lorentz-Minkowski space foliated by spheres in space-like hyperplanes is either a hypersurface of rotation or a pseudo-hyperbolic space and that maximal space-like hypersurfaces foliated by spheres in hyperplanes are rotational if the ambient space has dimension ≥ 4.

1. Introduction. A hypersurface M of $\mathbf{R}^{n+1}$ is said to be *sphere-foliated* if there is a one-parameter family of hyperplanes that meet M in round $(n-1)$-spheres. A circle-foliated surface is called *cyclic*.

Examples of cyclic constant mean curvature (CMC) surfaces are the Delaunay's surfaces and the spheres. While Delaunay's surfaces are rotational, spheres admit plenty of nonrotational foliations by circles. Nitsche claimed that *all cyclic surfaces of nonvanishing constant mean curvature are surfaces of rotation* [**12**]. Though his claim is right, his proof is incomplete. Our first aim in Section 2 is to give a complete proof of the following modified form of Nitsche's claim.

Theorem 1. *If M is a cyclic surface of nonzero constant mean curvature, then it is either a surface of rotation or a sphere.*

As a consequence of this theorem, we see that there is no cyclic surface of nonzero constant mean curvature spanning two *non-coaxial* circles

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in parallel planes, while there is a minimal one (Riemann's minimal surface).

Jagy showed that a similar result holds in higher dimensional Euclidean space [**7**]. If M is a sphere-foliated CMC hypersurface of $\mathbf{R}^{n+1}$, $n \geq 3$, then M is either rotational or spherical.

There are analogous results for cyclic or sphere-foliated minimal hypersurfaces. In this case the generating circles or spheres must by necessity lie in parallel hyperplanes [**13**], [**6**]. When M is a surface, the circles may not be co-axial (Riemann's minimal surfaces), whereas the spheres are co-axial whenever $n \geq 3$. Hence M is a hypersurface of rotation if $n \geq 3$ [**6**].

Our main goal in this paper is to generalize Jagy's theorem in various ambient spaces: Lorentz-Minkowski space, hyperbolic space and sphere.

Our second aim in Section 2 is to give a new proof of Jagy's theorem. The argument in Section 2 will be repeatedly used in the rest of this paper.

In Section 3 we study the properties of space-like maximal or CMC hypersurfaces in Lorentz-Minkowski space $\mathbf{L}^{n+1}$ foliated by spheres in space-like hyperplanes (*spacelike spheres*). López et al. showed that a space-like maximal surface foliated by space-like circles in $\mathbf{L}^3$ is either a Lorentzian catenoid or a Lorentzian Riemann's maximal surface where the generating circles lie in parallel space-like planes [**11**]. We show that space-like maximal hypersurfaces foliated by space-like spheres are rotational when $n \geq 3$. And if M has nonzero constant mean curvature, then it is either a hypersurface of rotation or a pseudohyperbolic space, which admits nonrotational foliations as spheres do in Euclidean space.

In Section 4 we study sphere-foliated minimal or CMC hypersurfaces in hyperbolic space $\mathbf{H}^{n+1}$ or in sphere $\mathbf{S}^{n+1}$ for $n \geq 4$. We say that a hypersurface M in $\mathbf{H}^{n+1}$ (or in $\mathbf{S}^{n+1}$) is *sphere-foliated* if there is a one-parameter family of geodesic hyperplanes (or totally geodesic n-spheres) which meet M in geodesic $(n-1)$-spheres.

In the *upper half-space model* for $\mathbf{H}^{n+1}$, one may view M as a sphere-foliated hypersurface with or without boundary in $\mathbf{R}^{n+1}$. Using this observation and Jagy's computation of the mean curvature of a sphere-foliated hypersurface in $\mathbf{R}^{n+1}$ [**7**], we show that a sphere-foliated minimal or CMC hypersurface in $\mathbf{H}^{n+1}$, when $n \geq 4$, is either a hypersurface of rotation around a geodesic or (part of) a (Euclidean) sphere in $\mathbf{R}_+^{n+1}$.

The examples of (Euclidean) spheres in the upper-half space model are geodesic hyperplanes, horospheres, equidistant hypersurfaces and geodesic spheres, which have constant mean curvature. These examples admit foliations by spheres which are not rotational around a geodesic.

We use, as a model for sphere, $\mathbf{R}^{n+1}$ with a conformal metric induced by a stereographic projection, see Section 4 for the precise definition. Therefore, we can view a sphere-foliated hypersurface in sphere as a sphere-foliated hypersurface in Euclidean space. We show that a sphere-foliated minimal or CMC hypersurface of $\mathbf{S}^{n+1}$ for $n \geq 4$ is either a hypersurface of rotation around a geodesic or a hypersphere.

When $n = 2$ or 3, there are only partial results. López and Jagy obtained similar results with various restrictions on the geodesic hyperplanes or the totally geodesic n-spheres of the foliation [**7**], [**9**].

The referee pointed out that the results are local.

2. Preliminaries and the proofs of Theorem 1 and Jagy's theorem. A smooth hypersurface in $\mathbf{R}^{n+1}$ can be locally written as the level set of a smooth function f. We define its mean curvature by

$$H = -\frac{1}{n}\operatorname{div}\left(-\frac{\nabla f}{|\nabla f|}\right). \tag{1}$$

Let M be a sphere-foliated hypersurface in $\mathbf{R}^{n+1}$. We construct a local coordinate system on M as in [**6**]. (We sketch the outline and recall the results.)

Let e_0 be the unit vector field normal to the hyperplanes of the foliation and $\gamma(t)$ an integral curve of e_0. We label the hyperplane containing $\gamma(t)$ by π_t and the center and the radius of the $(n-1)$-sphere on Π_t by $c(t)$ and $r(t)$. Since $\gamma(t)$ is a unit speed curve, we have orthonormal vector fields $e_0, e_1, \dots, e_n$ along $\gamma(t)$ that satisfy Frenet equations: $\gamma'(t) = e_0(\gamma(t))$, $e_0' = \kappa_0 e_1$, $e_1' = -\kappa_0 e_0 + \kappa_1 e_2, \dots$ and $e_n' = -\kappa_{n-1} e_{n-1}$, where $'$ denotes $\partial/\partial t$. Since $c(t)$ is a smooth curve, there are smooth functions $\alpha_0, \dots, \alpha_n$ such that $c'(t) = \sum_{i=0}^n \alpha_i e_i$.

We define a map $X : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ by

$$X(t, v_1, \dots, v_n) = c(t) + r(t) \sum_{i=1}^{n} v_i e_i(t). \tag{2}$$

Therefore, M is the zero set of $f = v_1^2 + \cdots + v_n^2 - 1$. If the differential $dX : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is nondegenerate at some point (t, v) with $|v| = 1$, then $X|_{|v|=1}$ gives a local coordinate system on M.

By a straightforward computation, we have

$$\partial X / \partial v_i = r e_i$$

and

$$\frac{\partial X}{\partial t} = (\alpha_0 - r\kappa_0 v_1) e_0 + \sum_{j=1}^{n} (\alpha_j + r' v_j + r v_{j-1} \kappa_{j-1} - r v_{j+1} \kappa_j) e_j.$$

The induced metric $g_{ij} = \langle \partial X/\partial v_i, \partial X/\partial v_j \rangle$, $v_0 = t$, satisfies [**6**]

$$\det (g_{ij}) = r^{2n} (\alpha_0 - r\kappa_0 v_1)^2.$$

Hence the differential dX is identically degenerate if and only if we have $\alpha_0 \equiv 0$ and $\kappa_0 \equiv 0$, which implies that M is a hyperplane.

From now on, we assume that $g = \det (g_{ij}) \neq 0$.

Proof of Theorem 1. When M is a surface, we have an explicit representation of X in terms of trigonometric functions:

$$X(t, \theta) = c(t) + r(t)(\cos\theta e_1(t) + \sin\theta e_2(t)).$$

Let E, F and G denote the coefficients of the first fundamental form: $E = \langle X_t, X_t \rangle$, $F = \langle X_t, X_\theta \rangle$ and $G = \langle X_\theta, X_\theta \rangle$. Then the mean curvature H of M satisfies

$$\begin{aligned} 2H(EG - F^2)^{3/2} &= G\langle X_{tt}, X_t \times X_\theta \rangle - 2F \langle X_{t\theta}, X_t \times X_\theta \rangle \\ &\quad + E \langle X_{\theta\theta}, X_t \times X_\theta \rangle \\ &\equiv \mathcal{P}. \end{aligned} \tag{3}$$

By a straightforward computation, we have

$$
\begin{aligned}
EG-F^2 &= \frac{1}{2}r^2(r^2\kappa_0^2+\alpha_1^2-\alpha_2^2)\cos(2\theta)+r^2\alpha_1\alpha_2\sin(2\theta)\\
&\quad + 2r^2(r'\alpha_1-r\alpha_0\kappa_0)\cos\theta+2r^2r'\alpha_2\sin\theta\\
&\quad + \frac{1}{2}r^2(\alpha_1^2+\alpha_2^2+2\alpha_0^2+2r'^2+r^2\kappa_0^2)
\end{aligned}
\tag{4}
$$

and

(5)

$$
\begin{aligned}
\mathcal{P} &= \frac{1}{2}r^3\kappa_0(\alpha_2^2-r^2\kappa_0^2-\alpha_1^2)\cos(3\theta)-r^3\alpha_1\alpha_2\kappa_0\sin(3\theta)\\
&\quad+\frac{1}{2}(-r^4\alpha_1\kappa_0'+5r^4\alpha_0\kappa_0^2+r^4\alpha_1'\kappa_0-6r^3r'\alpha_1\kappa_0)\cos(2\theta)\\
&\quad+\frac{1}{2}(r^4\alpha_2'\kappa_0-6r^3r'\alpha_2\kappa_0-r^4\alpha_2\kappa_0')\sin(2\theta)\\
&\quad+\frac{1}{2}(-3r^5\kappa_0^3-2r^4r'\kappa_0'-2r^3\alpha_0\alpha_1'-3r^3\alpha_1^2\kappa_0-6r^3r'^2\kappa_0+2r^4r''\kappa_0\\
&\qquad+2r^3\alpha_0\alpha_2\kappa_1-8r^3\alpha_0^2\kappa_0-3r^3\alpha_2^2\kappa_0+2r^3\alpha_0'\alpha_1+4r^2r'\alpha_0\alpha_1)\cos\theta\\
&\quad+(-r^3\alpha_0\alpha_2'-r^3\alpha_0\alpha_1\kappa_1+r^3\alpha_0'\alpha_2+2r^2r'\alpha_0\alpha_2-r^4r'\kappa_0\kappa_1)\sin\theta\\
&\quad+r^3r'\alpha_0'-r^3r''\alpha_0-\frac{1}{2}r^4\alpha_1\kappa_0'+\frac{1}{2}r^4\alpha_1'\kappa_0+\frac{5}{2}r^4\alpha_0\kappa_0^2\\
&\quad+r^2\alpha_0\alpha_1^2+r^2r'^2\alpha_0+r^2\alpha_0\alpha_2^2+r^2\alpha_0^3-2r^3r'\alpha_1\kappa_0-r^4\alpha_2\kappa_0\kappa_1.
\end{aligned}
$$

Without loss of generality, we may assume that $H^2=1$. We define a trigonometric polynomial $\mathcal{Q}$ by

$$\mathcal{Q}=\mathcal{P}^2-4(EG-F^2)^3.$$

(We will find conditions for $\mathcal{Q}$ to vanish identically.) Let us denote by $c_{\mathcal{Q},i}$ and $s_{\mathcal{Q},i}$ the coefficients of $\cos(i\theta)$ and $\sin(i\theta)$ in the Fourier series expansion of $\mathcal{Q}$. Similarly, we define $c_{\mathcal{P},i}$ and $s_{\mathcal{P},i}$.

It follows from $c_{\mathcal{Q},6}=0$ and $s_{\mathcal{Q},6}=0$ that

$$r^6A^3/8-3r^6AB^2/2=r^6\kappa_0^2A^2/8-r^6B^2/2$$

and

$$3r^6A^2B/4-r^6B^3=r^6\kappa_0^2AB/2,$$

where $A = r^2\kappa_0^2 + \alpha_1^2 - \alpha_2^2$ and $B = \alpha_1\alpha_2$. Therefore, one of the following holds: i) $A = B = \kappa_0 = 0$ or ii) $A = \kappa_0^2$, $B = 0$ and $\kappa_0 \neq 0$ or iii) $A = B = 0$ and $\kappa_0 \neq 0$.

If i) holds, then the generating circles lie in parallel planes and $\alpha_1 = \alpha_2 = 0$. Hence, M is a surface of rotation.

When ii) holds, we have $\alpha_1\alpha_2 = 0$. First, we suppose that $\alpha_1 = 0$. By a straightforward computation, we see that $c_{\mathcal{Q},6} = r^6(\alpha_2^2 - r^2\kappa_0^2)^2 \times (\kappa_0^2 + \alpha_2^2 - r^2\kappa_0^2)/8 = 0$.

When $\alpha_2^2 = r^2\kappa_0^2$ (this is case iii)), it follows that

$$EG - F^2 = -2r^3\alpha_0\kappa_0\cos\theta + 2r^2r'\alpha_2\sin\theta + \frac{1}{2}r^2(\alpha_2^2 + 2\alpha_0^2 + 2r'^2 + r^2\kappa_0^2).$$

Hence we have $c_{\mathcal{Q},4} = (c_{\mathcal{P},2}^2 - s_{\mathcal{P},2}^2)/2$ and $s_{\mathcal{Q},4} = c_{\mathcal{P},2}s_{\mathcal{P},2}/2$. Since these are both zero and $c_{\mathcal{P},2} = 5r^4\alpha_0\kappa_0^2/2$ and $s_{\mathcal{P},2} = r^4\kappa_0^2(\alpha_2/\kappa_0)'/2$, we have $\alpha_0 = r' = 0$. Substituting these into (4) and (5), we find that

$$EG - F^2 = r^2\alpha_2^2$$

and

$$\mathcal{P} = -3r^5\kappa_0^5\cos\theta - r^4\alpha_2\kappa_0\kappa_1.$$

Hence we must have $\kappa_0 = 0$. This contradicts our assumption that $\kappa_0 \neq 0$.

Now we suppose that $\kappa_0^2 + \alpha_2^2 = r^2\kappa_0^2$. Then we have $c_{\mathcal{Q},5} = 5r^7\alpha_0\kappa_0^3 \times (r^2\kappa_0^2 - \alpha_2^2)/4$ and $s_{\mathcal{Q},5} = r^7\kappa_0^5(\alpha_2/\kappa_0)'/4$. Therefore we have $\alpha_0 = 0$ and α_2/κ_0 is constant. The radius r satisfying $r^2 = 1 + (\alpha_2/\kappa_0)^2$ is also constant. Therefore we have

$$EG - F^2 = \frac{1}{2}r^2\kappa_0^2\cos(2\theta) + \frac{1}{2}r^2(\alpha_2^2 + r^2\kappa_0^2)$$

and

$$\mathcal{P} = -\frac{1}{2}r^3\kappa_0^3\cos(3\theta) - \frac{5}{2}r^5\kappa_0^3\cos(2\theta) - r^4\alpha_2\kappa_0\kappa_1.$$

Since there is no $\cos\theta$ term in $EG - F^2$, we have $\alpha_2 = 0$. Hence, $X(t, \theta)$ is a parametrization of a unit sphere.

Now we consider the second case $\alpha_2 = 0$. Since $\mathbf{C}_{\mathcal{Q},6} = r^6(\alpha_1^2 + r^2\kappa_0^2)^2 \times (\kappa_0^2 - \alpha_1^2 - r^2\kappa_0^2)/8 = 0$, it follows that $\kappa_0^2 = \alpha_1^2 + r^2\kappa_0^2$. Differentiating this with respect to t, we have $(\alpha_1/\kappa_0)(\alpha_1/\kappa_0)' + rr' = 0$.

Using the above two equations, we find that $c_{\mathcal{Q},5} = r^7\kappa_0^5((\alpha_1/\kappa_0)' - \alpha_0)/4$. Hence we have $\alpha_0 = (\alpha_1/\kappa_0)'$. From the definition of α_is, we have

$$c'(t) = (\alpha_1/\kappa_0)'e_0 + (\alpha_1/\kappa_0)\kappa_0 e_1 = ((\alpha_1/\kappa_0)e_0)'.$$

Hence $X(t,\theta) = C_0 + (\alpha_1/\kappa_0)e_0 + r(t)(\cos\theta e_1 + \sin\theta e_2)$ for some constant vector C_0. Since we have

$$|X(t,\theta) - C_0|^2 = r^2 + (\alpha_1/\kappa_0)^2 = 1,$$

M is the unit sphere centered at C_0. $\square$

Remark 1. Equation (3) in [**12**] is wrong (we must have $\overline{A}_{-n} = \overline{A}_n$ and $\bar{a}_{-n} = \bar{a}_n$). This led to the wrong conclusion that all the circles in the circle-foliation of a sphere are great circles. But there are plenty of cyclic foliations of a sphere consisting of nongeodesic circles.

To deal with sphere-foliated CMC hypersurfaces in $\mathbf{R}^{n+1}$, we recall some results about the mean curvature of a sphere-foliated hypersurface in $\mathbf{R}^{n+1}$ from [**6**] and [**7**].

Lemma 1. *Let M be a sphere-foliated hypersurface of $\mathbf{R}^{n+1}$ and $X(t,v)$ the parametrization of M introduced at the beginning of this section. Then the mean curvature H of M, which is the zero set of $f = v_1^2 + \cdots + v_n^2 - 1$, is given by*

$$H = \frac{1}{n\sqrt{g}}\left\{\frac{\partial}{\partial t}\left(\frac{T_0}{\sqrt{D}}\right) + \sum_{i=1}^{n}\frac{\partial}{\partial v_i}\left(\frac{T_i}{\sqrt{D}}\right)\right\},$$

where

$$\begin{aligned}
D &= (\alpha_0 - r\kappa_0 v_1)^2(v\cdot v) + (\alpha\cdot v + r'(v\cdot v))^2,\\
T_0 &= -r^n(\alpha\cdot v + r'(v\cdot v)),\\
T_i &= r^{n-1}(\alpha_0 - r\kappa_0 v_1)^2 v_i + r^{n-2}(\alpha\cdot v + r'(v\cdot v))g_{0i}
\end{aligned}$$

and $\alpha\cdot v = \sum_{i=1}^{n}\alpha_i v_i$.

Multiplying $2\sqrt{g}D^{3/2}$ on both sides of the above equation, we have

$$(6)\quad 2nH\sqrt{g}D^{3/2} = \left(2\frac{\partial T_0}{\partial t}D - T_0\frac{\partial D}{\partial t}\right) + \sum_{i=1}^{n}\left(2\frac{\partial T_i}{\partial v_i}D - T_i\frac{\partial D}{\partial v_i}\right) \equiv P.$$

Substituting $v \cdot v = 1$ into P after the necessary differentiations, we have [**7**]

(7)

$$\begin{aligned}
P &= 2nr^{n+1}\kappa_0^2 v_1^2\{r^2\kappa_0^2 v_1^2 + (\alpha\cdot v)^2\} \\
&\quad + 2r^n\kappa_0 v_1 \left\{\begin{aligned}
&(1-4n)r^2\alpha_0\kappa_0^2 v_1^2 + (2n+2)rr'\kappa_0 v_1(\alpha\cdot v)\\
&+ (2-2n)\alpha_0(\alpha\cdot v)^2 + r^2\kappa_0' v_1(\alpha\cdot v)\\
&- r^2\kappa_0 v_1(\alpha'\cdot v) + r^2\kappa_0\kappa_1 v_2(\alpha\cdot v)\\
&+ r\kappa_0 v_1(\alpha\cdot s)
\end{aligned}\right\}\\
&\quad + 2r^{n-1}\left\{\begin{aligned}
&(6n-3)r^2\alpha_0^2\kappa_0^2 v_1^2 - 4nrr'\alpha_0\kappa_0 v_1(\alpha\cdot v)\\
&+ (n+1)r^2r'^2\kappa_0^2 v_1^2 - 2r\alpha_0\kappa_0 v_1(\alpha\cdot s)\\
&+ (n-2)\alpha_0^2(\alpha\cdot v)^2 - r^2\alpha_0\kappa_0' v_1(\alpha\cdot v)\\
&+ r^3r'\kappa_0^2\kappa_1 v_1 v_2 - r^2\alpha_0\kappa_0\kappa_1 v_2(\alpha\cdot v) + r^3r'\kappa_0\kappa_0' v_1^2\\
&- r^2\alpha_0'\kappa_0 v_1(\alpha\cdot v) - r^3r''\kappa_0^2 v_1^2 - r^2\alpha_1\kappa_0^2 v_1(\alpha\cdot v)\\
&+ r^2\kappa_0^2(\alpha\cdot\alpha)v_1^2 + 2r^2\alpha_0\kappa_0 v_1(\alpha'\cdot v)
\end{aligned}\right\}\\
&\quad + 2r^{n-1}\left\{\begin{aligned}
&(3-4n)r\alpha_0^3\kappa_0 v_1 - 2nrr'^2\alpha_0\kappa_0 v_1 + 2r^2r''\alpha_0\kappa_0 v_1\\
&- r^2r'\alpha_1\kappa_0^2 v_1 - 2r\alpha_0(\alpha\cdot\alpha)\kappa_0 v_1 - r^2r'\alpha_0'\kappa_0 v_1\\
&+ (2n-2)r'\alpha_0^2(\alpha\cdot v) - r^2r'\alpha_0\kappa_0\kappa_1 v_2 + \alpha_0^2(\alpha\cdot s)\\
&- r^2r'\alpha_0\kappa_0' v_1 + r\alpha_0\alpha_0'(\alpha\cdot v)\\
&- r\alpha_0^2(\alpha'\cdot v) + r\alpha_0\alpha_1\kappa_0(\alpha\cdot v)
\end{aligned}\right\}\\
&\quad + 2r^{n-1}\alpha_0\left\{\begin{aligned}
&(n-1)\alpha_0^3 + (n-1)r'^2\alpha_0 + rr'\alpha_1\kappa_0\\
&+ \alpha_0(\alpha\cdot\alpha) + rr'\alpha_0' - rr''\alpha_0
\end{aligned}\right\}
\end{aligned}$$

where $s_i = r\kappa_{i-1}v_{i-1} - r\kappa_i v_{i+1}$ with $\kappa_{-1} = \kappa_n = v_0 = v_{n+1} = 0$. One may view P as a polynomial of $v_1, \ldots, v_n$.

Theorem 2 (Jagy). *Let M be a sphere-foliated* CMC *hypersurface of $\mathbf{R}^{n+1}$. If $n \geq 3$, then M is either a hypersurface of rotation or a sphere.*

Proof. Let us define a polynomial Q by

(8)

$$\begin{aligned} Q &= 4n^2H^2gD^3 - P^2 \\ &= 4n^2H^2r^{2n}(\alpha_0 - r\kappa_0 v_1)^2\{(\alpha_0 - r\kappa_0 v_1)^2 + (\alpha \cdot v + r')^2\}^3 - P^2. \end{aligned}$$

(Whenever D is not differentiated with respect to v_i, we substitute $v \cdot v = 1$ into D.) We will find conditions for Q to vanish identically when $|v| = 1$.

First we assume that $\alpha_0 \neq 0$. Substituting $v_1 = 0$, we get a new polynomial $Q|_{v_1=0}$ which satisfies

$$Q|_{v_1=0} = 4n^2H^2r^{2n}\alpha_0^2\left\{\alpha_0^2 + \left(\sum_{i=2}^{n}\alpha_i v_i + r'\right)^2\right\}^3 - P^2|_{v_1=0}.$$

This new polynomial $Q|_{v_1=0}$ vanishes identically when $\sum_{i=2}^n v_i^2 = 1$. Since $P|_{v_1=0}$ has only the degree-2 , 1, 0 terms, see (7), we have $\alpha_i = 0$ for $i = 2, \ldots, n$. (To see this, it suffices to substitute $v_i = \cos\theta$, $v_j = \sin\theta$ and $v_k = 0$ into $Q|_{v_1=0}$ for mutually distinct $i, j, k \geq 2$.) Therefore, P is a polynomial of v_1 and v_2. Since $n \geq 3$, we may let $v_1 = \cos\theta$, $v_3 = \sin\theta$ and $v_i = 0$ for $i \neq 1, 3$. Hence, we may regard Q as a polynomial of v_1, which vanishes identically for $v_1 \in [-1, 1]$.

Lemma 2. *The polynomial identity*

(9) $$(\alpha x + \beta)^2(\gamma x^2 + \delta x + \varepsilon)^3 = (ax^4 + bx^3 + cx^2 + dx + e)^2,$$

$\alpha \neq 0$, $\gamma, \varepsilon > 0$, *holds only if*

$$\delta^2 = 4\gamma\varepsilon.$$

Proof. Comparing coefficients, we have

$$\begin{aligned}
a^2 &= \alpha^2\gamma^3,\\
2ab &= 2\alpha\beta\gamma^3 + 3\alpha^2\gamma^2\delta,\\
2ac + b^2 &= \beta^2\gamma^3 + 6\alpha\beta\gamma^2\delta + 3\alpha^2\gamma^2\varepsilon + 3\alpha^2\gamma\delta^2,\\
2ad + 2bc &= \alpha^2\delta^3 + 6\alpha^2\gamma\delta\varepsilon + 6\alpha\beta\gamma^2\varepsilon + 6\alpha\beta\gamma\delta^2 + 3\beta^2\gamma^2\delta,\\
2ce + d^2 &= \alpha^2\varepsilon^3 + 6\alpha\beta\delta\varepsilon^2 + 3\beta^2\gamma\varepsilon^2 + 3\beta^2\delta^2\varepsilon,\\
2de &= 2\alpha\beta\varepsilon^3 + 3\beta^2\delta\varepsilon^2,\\
e^2 &= \beta^2\varepsilon^3.
\end{aligned}$$

First we suppose that $a = \pm\alpha\gamma^{3/2}$ and $e = \pm\beta\varepsilon^{3/2}$. From the first and second equations, we have $b = \pm\gamma^{1/2}(\beta\gamma+3\alpha\delta/2)$ depending on the signature of $\alpha^2\gamma^3/a$. Similarly, we have $d = \mp\varepsilon^{1/2}(\alpha\varepsilon + 3\beta\delta/2)$. From the third and fifth equations, we find that $2ac = 3\alpha\beta\gamma^2\delta + 3\alpha^2\gamma^2\varepsilon + 3\alpha^2\gamma\delta^2/4$ and $2ce = 3\alpha\beta\delta\varepsilon^2 + 3\beta^2\gamma\varepsilon^2 + 3\beta^2\delta^2\varepsilon/4$. Substituting the values of a, e, ac, ce into $(2ac)e = a(2ce)$, we find that

$$(\alpha\varepsilon^{1/2} - \beta\gamma^{1/2})(\delta - 2\gamma^{1/2}\varepsilon^{1/2})^2 = 0.$$

Multiplying both sides of the fourth equation by a, we have $2a^2d + b(2ac) = a(\alpha^2\delta^3 + 6\alpha^2\gamma\delta\varepsilon + 6\alpha\beta\gamma^2\varepsilon + 6\alpha\beta\gamma\delta^2 + 3\beta^2\gamma^2\delta)$. From this, we find that

$$(\delta - 2\gamma^{1/2}\varepsilon^{1/2})^2(\alpha\delta - 6\beta\gamma + 4\alpha\gamma^{1/2}\varepsilon^{1/2}) = 0.$$

It is easy to see that $\alpha\varepsilon^{1/2} = \beta\gamma^{1/2}$ and $\alpha\delta - 6\beta\gamma + 4\alpha\gamma^{1/2}\varepsilon^{1/2} = 0$ imply $\delta = 2\gamma^{1/2}\varepsilon^{1/2}$.

When $a = \pm\alpha\gamma^{3/2}$ and $e = \mp\beta\varepsilon^{3/2}$, we have

$$(\alpha\varepsilon^{1/2} + \beta\gamma^{1/2})(\delta + 2\gamma^{1/2}\varepsilon^{1/2})^2 = 0$$

and

$$(\delta + 2\gamma^{1/2}\varepsilon^{1/2})^2(\alpha\delta - 6\beta\gamma - 4\alpha\gamma^{1/2}\varepsilon^{1/2}) = 0.$$

The equations $\alpha\varepsilon^{1/2} + \beta\gamma^{1/2} = 0$ and $\alpha\delta - 6\beta\gamma - 4\alpha\gamma^{1/2}\varepsilon^{1/2} = 0$ imply $\delta + 2\gamma^{1/2}\varepsilon^{1/2} = 0$. $\square$

Applying the above lemma to (8), we find that

$$\alpha_0\alpha_1 + rr'\kappa_0 = 0. \tag{10}$$

If $\kappa_0 \equiv 0$, then we have $\alpha_1 \equiv 0$. Hence M is a hypersurface of rotation.

Let us now suppose that $\kappa_0 \neq 0$. Since the degree-8 term of Q is $(r^2\kappa_0^2 + \alpha_1^2)^2(H^2(r^2\kappa_0^2 + \alpha_1^2) - \kappa_0^2)$, we have

$$r^2 + \left(\frac{\alpha_1}{\kappa_0}\right)^2 = \frac{1}{H^2}. \tag{11}$$

From (10) and (11) we have $\alpha_0 = (\alpha_1/\kappa_0)'$. As in the proof of Theorem 1, one can see that M is a sphere.

If $\alpha_0 = 0$, then we have

$$Q = v_1^2\{4n^2H^2r^{2n}r^2\kappa_0^2\{r^2\kappa_0^2v_1^2 + (\alpha \cdot v)^2\}^3 - (P/v_1)^2\} = v_1^2Q_1.$$

Using Q_1 instead of Q, we can derive the same conclusion. □

3. Maximal and CMC hypersurfaces in Lorentz-Minkowski space. The Lorentz-Minkowski space $\mathbf{L}^{n+1}$ is $\mathbf{R}^{n+1}$ endowed with the Lorentzian scalar product $\langle v, w\rangle_l = v_1w_1 + \cdots + v_nw_n - v_{n+1}w_{n+1}$ for $v, w \in \mathbf{R}^{n+1}$. We say that a vector $v \in \mathbf{L}^{n+1}$ is *spacelike, respectively, timelike or lightlike*, if $\langle v, v\rangle_l > 0$, respectively $\langle v, v\rangle_l < 0$ or $\langle v, v\rangle_l = 0$. A hyperplane is said to be *spacelike, respectively, timelike or lightlike*, if its Euclidean normal is timelike, respectively spacelike or lightlike. A *spacelike sphere* $S(c, r)$ is a sphere on a spacelike hyperplane Π centered at c with radius r defined by

$$S(c, r) = \{v \in \Pi : \langle v - c, v - c\rangle_l = r^2\}.$$

A smooth hypersurface M is said to be *spacelike* if the tangent plane T_pM is spacelike at each point p of M. This is equivalent to the requirement that the restriction of the Lorentzian scalar product $\langle , \rangle_l$ to TM be positive definite at each point. Timelike hypersurfaces are defined similarly.

Example 1. The *pseudohyperbolic space* $\mathcal{PH}_r$ defined by

$$\mathcal{PH}_r = \{x \in \mathbf{L}^{n+1} : \langle x, x\rangle_l = -r^2\}$$

is a spacelike hypersurface of constant mean curvature $1/r$. On the other hand, the *pseudo sphere* $\mathcal{PS}_r$ defined by

$$\mathcal{PS}_r = \{x \in \mathbf{L}^{n+1} : \langle x, x\rangle_l = r^2\}$$

is a timelike hypersurface of constant mean curvature $1/r$. We note that these hypersurfaces admit foliations by spacelike spheres in *nonparallel* hyperplanes.

The *gradient* and the *divergence* on a Lorentz manifold (M, h_{ij}) are defined as follows: for some smooth function f,

$$\nabla f = \sum_{i,j=1}^{n} h^{ij} \frac{\partial f}{\partial y^i} \frac{\partial}{\partial y^j}.$$

For a smooth vector field $Y = \sum_{i=1}^{n} \beta_i (\partial/\partial y^i)$,

$$\mathrm{div}\,(Y) = \frac{1}{\sqrt{|\det{(h_{ij})}|}} \sum_{i=1}^{n} \frac{\partial}{\partial y^i} \Big(\sqrt{|\det{(h_{ij})}|}\beta_i\Big).$$

If M is locally written as the level set of a smooth function f on $\mathbf{L}^{n+1}$, then the mean curvature H is defined by

$$H = -\frac{1}{n}\mathrm{div}\left(-\frac{\nabla f}{\sqrt{|\langle \nabla f, \nabla f\rangle_l|}}\right). \tag{12}$$

We construct a coordinate system on a hypersurface M of $\mathbf{L}^{n+1}$ foliated by spacelike spheres as in Section 2. Let l_0 be the (timelike) unit vector field normal to the hyperplanes of the foliation. We choose an integral curve $\gamma(t)$ of l_0, and we label the hyperplane of the foliation by Π_t if it contains $\gamma(t)$. There are orthonormal vector fields $l_0, l_1, \dots, l_n$ which satisfy the following (Lorentzian) Frenet equations: $\gamma'(t) = l_0(\gamma(t))$, $l_0' = \kappa_0 l_1$, $l_1'(t) = \kappa_0 l_0 + \kappa_1 l_2$, $l_2' = -\kappa_1 l_1 + \kappa_2 l_3, \dots,$ and $l_n' = -\kappa_{n-1} l_{n-1}$, where $'$ denotes $\partial/\partial t$. We define $c(t)$ and $r(t)$ as the center and the radius of the sphere of the foliation on Π_t. And we find smooth functions $\beta_0, \beta_1, \dots, \beta_n$ which satisfy $c'(t) = \sum_{i=0}^{n} \beta_i(t) l_i(t)$.

We define a map $X : \mathbf{L}^{n+1} \to \mathbf{L}^{n+1}$ by

$$X(t, v_1, \dots, v_n) = c(t) + r(t) \sum_{i=1}^{n} v_i l_i(t).$$

Hence, M is the zero set of $f = v_1^2 + \cdots + v_n^2 - 1$. It is easy to see that

$$\frac{\partial X}{\partial v_i} = r l_i$$

and

$$\frac{\partial X}{\partial t} = (\beta_0 + r\kappa_0 v_1) l_0 + \sum_{j=1}^{n} (\beta_j + r' v_j + r v_{j-1}\kappa_{j-1} - r v_{j+1}\kappa_j) l_j.$$

Moreover, the induced metric $h_{ij} = \langle \partial X/\partial v_i, \partial X/\partial v_j \rangle$, $v_0 = t$, satisfies

$$h = \det(h_{ij}) = -r^{2n}(\beta_0 + r\kappa_0 v_1)^2.$$

Hence the identical degeneracy of the differential dX implies that M is a spacelike hyperplane. From now on, we assume that $h \neq 0$. Hence, $X|_{|v|=1}$ gives a local parametrization of M.

Theorem 3. *Let M be a spacelike hypersurface in Lorentz-Minkowski space $\mathbf{L}^{n+1}$ foliated by spacelike spheres.*

i) *If M is maximal and $n \geq 3$, then it is a hypersurface of rotation.*

ii) *If M has nonzero constant mean curvature, then it is either a hypersurface of rotation or a pseudohyperbolic space.*

Proof. Keeping in mind the signature changes in $\partial X/\partial t$, h and that ∇f is timelike, one may compute as in [**6**, pp. 261–265] to find that

(13)
$$2nH\sqrt{|h|}\mathfrak{D}^{3/2} = \left(2\frac{\partial \mathfrak{I}_0}{\partial t}\mathfrak{D} - \mathfrak{I}_0\frac{\partial \mathfrak{D}}{\partial t}\right) + \sum_{i=1}^{n}\left(2\frac{\partial \mathfrak{I}_i}{\partial v_i}\mathfrak{D} - \mathfrak{I}_i\frac{\partial \mathfrak{D}}{\partial v_i}\right) \equiv \mathfrak{B},$$

where

$$\mathfrak{D} = -(\beta_0 + r\kappa_0 v_1)^2 (v\cdot v) + (\beta\cdot v + r'(v\cdot v))^2,$$
$$\mathfrak{I}_0 = r^n(\beta\cdot v + r'(v\cdot v))$$

and

$$\mathfrak{I}_i = r^{n-1}(\beta_0 + r\kappa_0 v_1)^2 v_i - r^{n-2}(\beta\cdot v + r'(v\cdot v)) g_{0i}.$$

(As in Section 2 we substitute $v \cdot v = 1$ after the necessary differentiations.)

A long calculation shows that

$$\mathfrak{B} = 2nr^{n+1}\kappa_0^2 v_1^2((\beta\cdot v)^2 - r^2\kappa_0^2 v_1^2)$$

$$(14)\qquad + \left\{\begin{array}{l} v_1\{(4n-4)r^n\beta_0\kappa_0(\beta\cdot v)^2 + 2r^{n+2}\kappa_0^2\kappa_1 v_2(\beta\cdot v)\} \\ \quad + v_1^2\left\{\begin{array}{l}(4+4n)r^{n+1}r'\kappa_0^2(\beta\cdot v) + 2r^{n+2}\kappa_0^2(\beta\cdot s) \\ -\,2r^{n+2}\kappa_0^2(\beta'\cdot v) + r^n(r^2\kappa_0^2)'(\beta\cdot v)\end{array}\right\} \\ \quad + v_1^3\{(2-8n)r^{n+2}\beta_0\kappa_0^3\}\end{array}\right\}$$

$$+ \text{lower degree terms},$$

where $s_i = r\kappa_{i-1}v_{i-1} - r\kappa_i v_{i+1}$ with $\kappa_{-1} = \kappa_n = v_0 = v_{n+1} = 0$.

Proof of i). If M is maximal, then $\mathfrak{B}$ vanishes at $|v| = 1$. Substituting $v_1 = \cos\theta$, $v_i = \sin\theta$ and $v_j = 0$ for $i, j \geq 2$ into (14), we find that the coefficients of $\cos(4\theta)$ and $\sin(4\theta)$ in the Fourier series expansion of $\mathfrak{B}$ are $nr^{n+1}\kappa_0^2(\beta_1^2 - \beta_i^2 - r^2\kappa_0^2)/2$ and $nr^{n+1}\beta_1\beta_i\kappa_0^2$. Since these are both zero, we have either $\kappa_0 = 0$ or $\beta_i = 0$ for $2 \leq i \leq n$ and $\beta_1^2 - r^2\kappa_0^2 = 0$. (In Euclidean space, there is only one possibility $\kappa_0 \equiv 0$.)

When $\kappa_0 \equiv 0$, López showed that M is a hypersurface of rotation [**9**].

The second case cannot happen. To see this, assume that $\kappa_0 \neq 0$, $\beta_1^2 = r^2\kappa_0^2$ and $\beta_i = 0$ for $2 \leq i \leq n$. Now the coefficient of the degree-3 term of $\mathfrak{B}$ satisfies

$$\begin{aligned}\mathfrak{B}_3|_{\beta_i} = 0 (i\geq 2)/2r^{n-1} &= -6r^2\beta_1\kappa_0^2 + 3r'\beta_1^3 - 4r\beta_0\beta_1^2\kappa_0 + 7r^3\beta_0\kappa_0^3 \\ &\quad + r\beta_1(rr'\kappa_0^2 + r^2\kappa_0\kappa_0' - \beta_1\beta_1') \\ &\quad + 2(n+2)r^2\kappa_0^2(r'\beta_1 - r\beta_0\kappa_0) \\ &= (2n+1)r^2\kappa_0^2(r'\beta_1 - r\beta_0\kappa_0).\end{aligned}$$

(As in the proof of Theorem 2, we may let $v_1 = \cos\theta$, $v_3 = \sin\theta$ and $v_i = 0$ for $i \neq 1, 3$ to see that $\mathfrak{B}$ is a polynomial of v_1.) Since $\mathfrak{B}$ vanishes identically for $v_1 \in [-1, 1]$, we have $\beta_0 = (\beta_1/r\kappa_0)r'$. This and $\beta_1^2 - r^2\kappa_0^2 = 0$ imply that

$$c'(t) = \beta_0 l_0 + \beta_1 l_1 = (rl_0)' \quad \text{or} \quad -(rl_0)'.$$

Therefore, we have $|X(t, v) - C_0| = 0$ for some constant vector C_0. Hence $X(t, v)$ is a foliation of a lightcone contrary to the hypothesis.

Proof of ii). If $H \neq 0$ and $n \geq 3$, then we have $\beta_i = 0$ for $i \geq 2$ as in the proof of Theorem 2. Moreover, we have either $r^2\kappa_0^2 - \beta_1^2 = 0$ or $r^2\kappa_0^2 + (\kappa_0/H)^2 = \beta_1^2$ by comparing the degree-8 terms in the square of (13).

Therefore, we can apply Lemma 2 to (13) to get

$$\beta_0\beta_1 = rr'\kappa_0. \tag{15}$$

This with the above equalities implies that $\beta_0 = (\beta_1/\kappa_0)'$. Hence,

$$c'(t) = (\beta_1/\kappa_0)'l_0 + (\beta_1/\kappa_0)\kappa_0 l_1 = (\beta_1/\kappa_0 l_0)'.$$

And $c(t) = (\beta_1/\kappa_0)l_0 + C_0$ for some constant vector C_0. Therefore, $X(t, v)$ is a foliation of a *pseudohyperbolic space* (as in the case of maximal hypersurfaces, $r^2\kappa_0^2 - \beta_1^2 = 0$ is impossible), which has constant mean curvature H.

When M is a surface, the local parametrization of M is given by (like the cyclic surfaces in $\mathbf{R}^3$)

$$X(t, \theta) = c(t) + r(t)(\cos\theta l_1(t) + \sin\theta l_2(t)).$$

Let $\mathfrak{E}, \mathfrak{F}$ and $\mathfrak{G}$ be the coefficients of the first fundamental form and $\mathfrak{B} \equiv \mathfrak{G}\langle X_{tt}, X_t \times X_\theta\rangle_l - 2\mathfrak{F}\langle X_{t\theta}, X_t \times X_\theta\rangle_l + \mathfrak{E}\langle X_{\theta\theta}, X_t \times X_\theta\rangle_l$. Then the mean curvature H satisfies

$$4(\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2)^3 H^2 = \mathfrak{B}^2,$$

where

$$\begin{aligned}\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2 = &\frac{1}{2}r^2(\beta_1^2 - r^2\kappa_0^2 - \beta_2^2)\cos(2\theta) + r^2\beta_1\beta_2\sin(2\theta)\\ &- 2r^2(r\beta_0\kappa_0 - r'\beta_1)\cos\theta + 2r^2r'\beta_2\sin\theta\\ &- \frac{1}{2}r^2(2\beta_0^2 - 2r'^2 + r^2\kappa_0^2 - \beta_1^2 - \beta_2^2)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{B} = &\frac{1}{2}r^3\kappa_0(\beta_1^2 - r^2\kappa_0^2 - \beta_2^2)\cos(3\theta) + r^3\beta_1\beta_2\kappa_0\sin(3\theta)\\ &+ \frac{1}{2}(-r^4\beta_1'\kappa_0 - 5r^4\beta_0\kappa_0^2 + r^4\beta_1\kappa_0' + 6r^3r'\beta_1\kappa_0)\cos(2\theta)\\ &+ \frac{1}{2}(r^4\beta_2\kappa_0' + 6r^3r'\beta_2\kappa_0 - r^4\beta_2'\kappa_0)\sin(2\theta)\\ &+ \frac{1}{2}\big(-3r^5\kappa_0^3 + 2r^4r'\kappa_0' - 2r^3\beta_0\beta_1' + 3r^3\beta_1^2\kappa_0 + 6r^3r'^2\kappa_0\\ &\qquad - 2r^4r''\kappa_0 + 2r^3\beta_0\beta_2\kappa_1 - 8r^3\beta_0^2\kappa_0 + 3r^3\beta_2^2\kappa_0\\ &\qquad + 2r^3\beta_0'\beta_1 + 4r^2r'\beta_0\beta_1\big)\cos\theta\\ &+ \big(r^3\beta_0'\beta_2 - r^3\beta_0\beta_1\kappa_1 - r^3\beta_0\beta_2' + 2r^2r'\beta_0\beta_2 + r^4r'\kappa_0\kappa_1\big)\sin\theta\\ &+ r^3r'\beta_0' - r^3r''\beta_0 + \frac{1}{2}r^4\beta_1\kappa_0' - \frac{1}{2}r^4\beta_1'\kappa_0 - \frac{5}{2}r^4\beta_0\kappa_0^2\\ &+ r^2\beta_0\beta_1^2 + r^2r'^2\beta_0 + r^2\beta_0\beta_2^2 - r^2\beta_0^3 + 2r^3r'\beta_1\kappa_0 + r^4\beta_2\kappa_0\kappa_1.\end{aligned}$$

(Without loss of generality, one may assume that $H^2 = 1$.) We define a trigonometric polynomial $\mathcal{O}$ by

$$\mathcal{O} = \mathfrak{B}^2 - 4(\mathfrak{E}\mathfrak{B} - \mathfrak{F}^2)^3.$$

As in the proof of Theorem 1, we have either $\kappa_0 = 0$ and $\beta_1 = \beta_2 = 0$ or $\kappa_0 \neq 0$ and $\beta_1\beta_2 = 0$.

In the first case, M is rotational.

When $\kappa_0 \neq 0$ and $\beta_1 = 0$, we have $r^2\kappa_0^2 + \beta_2^2 = -\kappa_0^2$ from the $\cos(6\theta)$ term of $\mathcal{O}$. This contradicts our assumption that $\kappa_0 \neq 0$.

When $\kappa_0 \neq 0$ and $\beta_2 = 0$, we have $\beta_1^2 - r^2\kappa_0^2 = \kappa_0^2$ or $\beta_1^2 - r^2\kappa_0^2 = 0$ from the $\cos(6\theta)$ term of $\mathcal{O}$. And we have $\beta_0 = (\beta_1/\kappa_0)'$ from the $\cos(5\theta)$ term of $\mathcal{O}$. Hence, M is a *pseudohyperbolic* space (if $\beta_1^2 - r^2\kappa_0^2 = 0$, then M is a lightcone). □

Remark 2. 1. López et al. showed that a spacelike maximal surface foliated by spacelike circles is either a Lorentzian catenoid or a Lorentzian Riemann's maximal surface [**11**].

2. We have a similar result for a timelike constant mean curvature hypersurface foliated by spacelike spheres. Noting that ∇f is a spacelike

vector, we follow the proof of Theorem 3 to see that such a hypersurface is either a hypersurface of rotation or a *pseudo sphere*.

4. Minimal and CMC hypersurfaces in hyperbolic space and sphere. To investigate sphere-foliated minimal and constant mean curvature hypersurfaces in hyperbolic space or in sphere, we adopt the following models.

For hyperbolic space, we use the *upper half-space model* $(\mathbf{R}^{n+1}_+, ds^2_h)$, where $\mathbf{R}^{n+1}_+ = \{(x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} : x_{n+1} > 0\}$ and $ds^2_h = (dx_1^2 + \cdots + dx_{n+1}^2)/x_{n+1}^2$. A smooth hypersurface in the upper half-space model can be treated in view of two different metrics: the Euclidean metric ds^2_0 and the hyperbolic metric ds^2_h. Let H_h and H_0 be the mean curvatures of M with respect to ds^2_h and ds^2_0 and N the Euclidean normal vector of M. Then we have, at a point x in M,

$$H_h = x_{n+1}H_0 + N_{n+1}. \tag{16}$$

As for the sphere, we use the *stereographic sphere model*. It is $\mathbf{R}^{n+1}$ endowed with the conformal metric $ds^2_s = ds^2_0/((1+\langle x, x\rangle)/2)^2$ induced by the stereographic projection of a unit sphere from the north pole $(0, \dots, 0, 1)$ onto the hyperplane $x_{n+1} = 0$. (We denote by $\langle , \rangle$ the usual Euclidean inner product in $\mathbf{R}^{n+1}$.) We note that different choices of the north pole in the embedding of $\mathbf{S}^{n+1}$ into $\mathbf{R}^{n+2}$ induces an isometry in our model. The mean curvatures H_s and H_0 of M with respect to the metrics ds^2_s and ds^2_0 satisfy

$$H_s = \frac{1 + \langle x, x\rangle}{2} H_0 + \langle x, N\rangle. \tag{17}$$

Definition 1. A hypersurface in $\mathbf{H}^{n+1}$, or in $\mathbf{S}^{n+1}$, is said to be sphere-foliated if there is a one-parameter family of geodesic hyperplanes, or totally geodesic n-spheres, that meet M in $(n-1)$-spheres.

Using these models one may consider a sphere-foliated hypersurface M in hyperbolic space or in sphere as a sphere-foliated hypersurface (with or without boundary) in $\mathbf{R}^{n+1}$ satisfying (16) or (17).

Lemma 3. *Let M be a sphere-foliated hypersurface in $\mathbf{R}^{n+1}$. If the parametrization of M is given by* (2), *then we have*

$$N = -\frac{\nabla f}{|\nabla f|} = \frac{(\alpha \cdot v + r')e_0 - (\alpha_0 - r\kappa_0 v_1)\sum_{i=1}^n v_i e_i}{\sqrt{(\alpha \cdot v + r')^2 + (\alpha_0 - r\kappa_0 v_1)^2}}. \tag{18}$$

Proof. We note that N lies in the plane σ spanned by e_0 and $\sum_{i=1}^n v_i e_i$. Let $p : \mathbf{R}^{n+1} \to \sigma$ be the canonical projection. Then we have

$$\begin{aligned} p\left(\frac{\partial X}{\partial t}\right) &= \left\langle \frac{\partial X}{\partial t}, e_0 \right\rangle e_0 + \left\langle \frac{\partial X}{\partial t}, \sum_{i=1}^n v_i e_i \right\rangle \sum_{i=1}^n v_i e_i \\ &= (\alpha_0 - r\kappa_0 v_1) e_0 + (\langle \alpha, v\rangle + r') \sum_{i=1}^n v_i e_i. \end{aligned}$$

Since N is perpendicular to $p(\partial X/\partial t)$ in $\mathbf{S}$, the claim follows. □

We assume that $g \neq 0$ and use the notations in Section 2.

Theorem 4. *Let M be a sphere-foliated minimal or* CMC *hypersurface in hyperbolic space $\mathbf{H}^{n+1}$ with $n \geq 4$. Then M is either a hypersurface of rotation around a geodesic or* (*part of*) *a* (*Euclidean*) *sphere.*

Proof. (We prove this theorem for $n = 4$.) (i) First we assume that $H_h \neq 0$. From (2), (6) and (18), we have

(19)

$$\begin{aligned} 2nH_h\sqrt{g}\,D_1^{3/2} &= \left(\langle c, \vec{E}\rangle + r\sum_{i=1}^n v_i \langle e_i, \vec{E}\rangle\right) P \\ &\quad + 2n\sqrt{g}\,D_1 \left\{ (\alpha \cdot v + r')\langle e_0, \vec{E}\rangle - (\alpha_0 - r\kappa_0 v_1) \sum_{i=1}^n v_i \langle e_i, \vec{E}\rangle \right\} \\ &\equiv L, \end{aligned}$$

where $\vec{E} = (0, \dots, 0, 1)$ and $D_1 = (\alpha_0 - r\kappa_0 v_1)^2 + (\alpha \cdot v + r')^2$.

Let us suppose that $\alpha_0 > 0$. Substituting $v_1 = 0$ into (19), we have

(20)
$$2nH_h r^n \alpha_0 D_2^{3/2} = \left(\langle c, \vec{E} \rangle + r \sum_{i=2}^{n} v_i \langle e_i, \vec{E} \rangle \right) P|_{v_1=0} + 2nr^n \alpha_0 D_2 \left\{ (\alpha * v + r') \langle e_0, \vec{E} \rangle - \alpha_0 \sum_{i=2}^{n} v_i \langle e_i, \vec{E} \rangle \right\},$$

where $\alpha * v = \sum_{i=2}^n \alpha_i v_i$ and $D_2 = \alpha_0^2 + (\alpha * v + r')^2$. From (7), we see that the degree-3 term of $P|_{v_1=0}$ is $(2n-4)r^{n-1}\alpha_0^2(\alpha * v)^2 - 2r^{n+1}\alpha_0\kappa_0\kappa_1 v_2(\alpha * v)$.

Lemma 4. *A necessary condition for* (20) *when* $\sum_{i=2}^n v_i^2 = 1$ *is* $\alpha_i = 0$ *for* $i = 2, \dots, n$.

Proof. The degree-3 term of $L|_{v_1=0}$ is $2r^n\alpha_0(\alpha * v)\{n(\alpha * v)^2 \langle e_0, \vec{E}\rangle + r^2\kappa_0\kappa_1 v_2 \sum_{i=2}^n v_i \langle e_i, \vec{E}\rangle - 2\alpha_0(\alpha * v)\sum_{i=2}^n v_i\langle e_i, \vec{E}\rangle\}$. Let us substitute $v_2 = \cos\theta$, $v_3 = \sin\theta$ and $v_4 = 0$ into (20). From the $\cos(6\theta)$ and $\sin(6\theta)$ terms in the square of (20), we have either $\alpha_2 = \alpha_3 = 0$ or

$$\begin{pmatrix} \langle e_2, \vec{E}\rangle \\ \langle e_3, \vec{E}\rangle \end{pmatrix} = \Lambda \begin{pmatrix} 2\alpha_0\alpha_2(\alpha_2^2 + \alpha_3^2) - r^2\kappa_0\kappa_1(\alpha_2^2 - \alpha_3^2) \\ 2\alpha_0\alpha_3(\alpha_2^2 + \alpha_3^2) - 2r^2\kappa_0\kappa_1\alpha_2\alpha_3 \end{pmatrix},$$

where $\Lambda = n(\langle e_0, \vec{E}\rangle \pm H_h)/((2\alpha_0\alpha_2 - r^2\kappa_0\kappa_1)^2 + 4\alpha_0^2\alpha_3^2)$.

Substituting $v_2 = 0$, $v_3 = \cos\theta$ and $v_4 = \sin\theta$, we have either $\alpha_3 = \alpha_4 = 0$ or

$$\begin{pmatrix} \langle e_3, \vec{E}\rangle \\ \langle e_4, \vec{E}\rangle \end{pmatrix} = \frac{n(\langle e_0, \vec{E}\rangle \pm H_h)}{2\alpha_0} \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

Hence, we have three possibilities:

i) $\alpha_i = 0$ for $i = 2, 3, 4$, or

ii) $\langle e_2, \vec{E}\rangle = n\alpha_2(\langle e_0, \vec{E}\rangle \pm H_h)/(r^2\kappa_0\kappa_1 + 2\alpha_0\alpha_2)$ and $\alpha_3 = \alpha_4 = 0$, or

iii) $\alpha_3 \neq 0$ and $\kappa_1 = 0$.

We show that ii) implies i). It is easy to see that the polynomial identity $(\alpha x^2 + \beta x + \gamma)^3 = (ax^3 + bx^2 + cx + d)^2$ with $\alpha, \gamma > 0$ holds if and only if $\beta^2 = 4\alpha\gamma$. Applying this observation to (20) with ii), we find that $\alpha_2 = 0$.

The case iii) cannot happen. Otherwise, one may choose e_4 so that $\langle e_4, \vec{E}\rangle = n\alpha_4(\langle e_0, \vec{E}\rangle \pm H_h)/2\alpha_0 = 0$. Applying the above observation with $v_2 = \alpha_4 = 0$ to (20), we get $\alpha_3 = 0$. □

Applying Lemma 2 to (19), we find that $\alpha_0\alpha_1 + rr'\kappa_0 = 0$ and $r\kappa_0\sqrt{\alpha_0^2 + r'^2} = -\alpha_0\sqrt{r^2\kappa_0^2 + \alpha_1^2}$. Hence, $\kappa_0 \equiv 0$ and $\alpha_0 > 0$ implies that M is a hypersurface of rotation.

The following lemma holds for $n \geq 2$.

Lemma 5. *Let M be a hypersurface of rotation in $\mathbf{R}_+^{n+1}$ that has constant mean curvature when regarded as a submanifold of H^{n+1}. If the axis of the rotation is not perpendicular to $x_{n+1} = 0$, then M is a (Euclidean) sphere.*

Proof. We fix a coordinate system on $\mathbf{R}_+^{n+1}$ so that $e_0 = (a, 0, \dots, 0, b)$ with $a^2 + b^2 = 1$, $a \neq 0$ and $c'(t) = \alpha_0 e_0$. It is easy to see that

$$\begin{aligned}\overline{P} &\equiv P|_{\kappa_0 = \alpha_1 = \cdots = \alpha_n = 0} \\ &= 2r^{n-1}\alpha_0^2\left\{n(\alpha_0^2 + r'^2) - \left(\alpha_0 + \left(\frac{rr'}{\alpha_0}\right)'\right)\right\}.\end{aligned}$$

From (19), we see that

$$\begin{aligned}2nH_h r^n \alpha_0(\alpha_0^2 + r'^2)^{3/2} = {}&(\langle c, \vec{E}\rangle + rv_1\langle e_1, \vec{E}\rangle)\overline{P} \\ &+ 2nr^n(\alpha_0^2 + r'^2)\{r'\langle e_0, \vec{E}\rangle - \alpha_0 v_1\langle e_1, \vec{E}\rangle\}.\end{aligned}$$

Hence we have $\alpha_0 + (rr'/\alpha_0)' = 0$ and $\alpha_0\langle c, \vec{E}\rangle + rr'\langle e_0, \vec{E}\rangle = H_h r\sqrt{\alpha_0^2 + r'^2}$. Since $c(t) = \int_{t_0}^t \alpha_0 e_0$, the second equation becomes

$$b\alpha_0\int_{t_0}^t \alpha_0 + brr' = H_h r\sqrt{\alpha_0^2 + r'^2}.$$

Substituting $\alpha_0 + (rr'/\alpha_0)' = 0$ into this, we find that $H_h r\sqrt{\alpha_0^2 + r'^2} = \tilde{b}\alpha_0$ for some constant $\tilde{b}$. Hence we find that $r^2 + (rr'/\alpha_0)^2 = r^2 + \alpha_0^2$ is constant and that M is a sphere. □

Now we consider the case $\kappa_0 \neq 0$. Using $\alpha_0\alpha_1 + rr'\kappa_0 = 0$ and $r\kappa_0\sqrt{\alpha_0^2 + r'^2} = -\alpha_0\sqrt{r^2\kappa_0^2 + \alpha_1^2}$ to simplify the degree-4 term of L_4 of L, we find that

$$L_4/2r^{n+1}\kappa_0 = n(r^2\kappa_0^2 + \alpha_1^2)\Big(\kappa_0\langle c, \vec{E}\rangle - \alpha_1\langle e_0, \vec{E}\rangle\Big)v_1^4 + r^2\kappa_0^2\left(\alpha_0 - \left(\frac{\alpha_1}{\kappa_0}\right)'\right)\sum_{i=1}^{n}\langle e_i, \vec{E}\rangle v_1^3 v_i.$$

Since the $v_1^3 v_i$, $i \neq 1$, terms must vanish, we have either $\alpha_0 = (\alpha_1/\kappa_0)'$ or $\langle e_i, \vec{E}\rangle = 0$ for $i = 2, \dots, n$.

When $\alpha_0 = (\alpha_1/\kappa_0)'$, it follows from $\alpha_0\alpha_1 + rr'\kappa_0 = 0$ that $(\alpha_1/\kappa_0) \times (\alpha_1/\kappa_0)' + rr' = 0$. Hence $r^2 + (\alpha_1/\kappa_0)^2$ is a constant. As in the proof of Theorem 1, we conclude that M is spherical.

If the second case holds, we apply a Möbius transformation to obtain another (Euclidean) foliation of M. Hence we may assume that the second condition holds under any Möbius transformation and for all $\gamma(t)$ which corresponds to the new foliation. Let $\tilde{c}(t)$ be the curve determined by the hyperbolic centers of the spheres of the foliation. Then $\gamma(t)$ and $\tilde{c}(t)$ lie on the same plane which is perpendicular to $x_{n+1} = 0$ under any Möbius transformation. It is straightforward to see that $\tilde{c}(t)$ is a reparametrization of a geodesic. Moreover, the geodesic hyperplanes are perpendicular to $\tilde{c}(t)$. Therefore, M is a hypersurface of rotation unless $\tilde{c}(t)$ is a point curve. If $\tilde{c}(t)$ is a point, then M is a (hyperbolic) sphere.

If $\alpha_0 \equiv 0$, then v_1 divides the righthand side of (19). And the lefthand side of (19) is $2nH_h|r\kappa_0 v_1| \times \{r^2\kappa_0^2 v_1^2 + (r' + \alpha\cdot v)^2\}^{3/2}$. Therefore, we may formally divide (19) by v_1, and the resulting equation holds for $v_1 \geq 0$. Substituting $v_1 = 0$ into this new equation, we find that

$$\text{(21)} \quad n(\langle e_0, \vec{E}\rangle \pm H_h)(\alpha * v)^3 + r^2\kappa_0\kappa_1 v_2(\alpha * v)\sum_{i=2}^{n} v_i\langle e_i, \vec{E}\rangle + \text{lower degree terms} = 0.$$

Hence we have $\alpha_i = 0$ for $i = 2, \dots, n$ as in Lemma 4. We have only to repeat the argument for $\alpha_0 > 0$ to conclude that M is either a hypersurface of rotation or (part of) a sphere.

(ii) Let us now assume that $H_h = 0$. One may assume that $\kappa_0 \neq 0$ and $\alpha_0 > 0$ (the case $\alpha_0 \equiv 0$ can be proved as above). We have $\alpha_i = 0$ for $i = 2, \dots, n$ from (20) and Lemma 4. By a straightforward computation, we have

$$\begin{aligned} L_4/2r^{n+1}\kappa_0 &= n(r^2\kappa_0^2 + \alpha_1^2)(\kappa_0\langle c, \vec{E}\rangle - \alpha_1\langle e_0, \vec{E}\rangle)v_1^4 \\ &\quad + \Big(r^2\alpha_0\kappa_0^2 + 2rr'\alpha_1\kappa_0 + 2\alpha_0\alpha_1^2 \\ &\qquad + r^2\alpha_1\kappa_0' - r^2\kappa_0\alpha_1'\Big)\sum_{i=1}^{n}\langle e_i, \vec{E}\rangle v_1^3 v_i. \end{aligned}$$

If $\langle e_i, \vec{E}\rangle \neq 0$ for some $i \geq 2$, then we have $r^2\alpha_0\kappa_0^2 + 2rr'\alpha_1\kappa_0 + 2\alpha_0\alpha_1^2 + r^2\alpha_1\kappa_0' - r^2\kappa_0\alpha_1' = 0$. From the degree-4 and constant terms of L, we find that

$$\kappa_0\langle c, \vec{E}\rangle - \alpha_1\langle e_0, \vec{E}\rangle = 0$$

and

$$\alpha_0\langle c, \vec{E}\rangle + rr'\langle e_0, \vec{E}\rangle = 0.$$

Since $\langle c, \vec{E}\rangle > 0$, we have $\alpha_0\alpha_1 + rr'\kappa_0 = 0$ and $\alpha_0 = (\alpha_1/\kappa_0)'$. Hence M is a sphere.

When $\langle e_i, \vec{E}\rangle = 0$ for all $i \geq 2$, we can argue as in the case $H_h \neq 0$.

If $\kappa_0 \equiv 0$, then we have $L = 2nr^n\alpha_0(\alpha \cdot v)^3\langle e_0, \vec{E}\rangle - 4r^n\alpha_0^2(\alpha \cdot v)^2\sum_{i=1}^n v_i\langle e_i, \vec{E}\rangle +$ lower degree terms. Therefore we have either $\alpha_i = 0$ or $n\langle e_0, \vec{E}\rangle\alpha_i = 2\alpha_0\langle e_i, \vec{E}\rangle$ for $i \geq 2$. Since $\kappa_0 \equiv 0$ and $H_h = 0$, the second condition is equivalent to the first. Hence M is a hypersurface of rotation. $\square$

The following theorem and proof about the sphere-foliated minimal or CMC hypersurface in sphere are analogous to those of the hyperbolic space case.

Theorem 5. *Let M be a sphere-foliated minimal or* CMC *hypersurface in the unit sphere $\mathbf{S}^{n+1}$ with $n \geq 4$. Then it is either a hypersurface of rotation or a hypersphere.*

Proof. In the sphere, the mean curvature H_s of M satisfies

$$\begin{aligned} 2nH_s\sqrt{g}\,D_1^{3/2} &= \left(\frac{1+\langle c,c\rangle+r^2}{2}+r\sum_{i=1}^n v_i\langle e_i,c\rangle\right)P \\ &\quad + 2n\sqrt{g}\,D_1\Big\langle c+r\sum_{i=1}^n v_ie_i,(\alpha\cdot v+r')e_0 \\ &\qquad - (\alpha_0-r\kappa_0v_1)\sum_{i=1}^n v_ie_i\Big\rangle \\ &\equiv S. \end{aligned} \tag{22}$$

Substituting $v_1=0$ into (22), we find that the degree-3 term of $S|_{v_1=0}$ is $2r^n\alpha_0(\alpha{*}v)\{n(\alpha{*}v)^2\langle c,e_0\rangle+r^2\kappa_0\kappa_1v_2\sum_{i=2}^n v_i\langle c,e_i\rangle-2\alpha_0(\alpha{*}v)\sum_{i=2}^n v_i\langle c,e_i\rangle\}$. Hence we have $\alpha_i=0$ for $i=2,\dots,n$ for any value of H_s as in the proof of Theorem 4.

First we assume that $H_s\neq 0$. Applying Lemma 4 to (22), we have $\alpha_0\alpha_1+rr'\kappa_0=0$. If $\kappa_0\equiv 0$, then M is a hypersurface of rotation. Otherwise, the degree-4 term of S satisfies

$$\begin{aligned} S_4/2r^{n+1}\kappa_0 &= (r^2\kappa_0^2+\alpha_1^2)\left(\frac{1+\langle c,c\rangle-r^2}{2}n\kappa_0-n\alpha_1\langle c,e_0\rangle\right)v_1^4 \\ &\quad + r^2\kappa_0^2\left(\alpha_0-\left(\frac{\alpha_1}{\kappa_0}\right)'\right)\sum_{i=1}^n\langle e_i,c\rangle v_1^3v_i. \end{aligned}$$

Since the $v_1^3v_i$ terms vanish, we have either $\langle e_i,c\rangle\neq 0$ for some $i\geq 2$ and $\alpha_0=(\alpha_1/\kappa_0)'=0$ or $\langle e_i,c\rangle=0$ for all $i=2,\dots,n$. In the first case M is a hypersphere.

Let us now suppose that $\langle e_i,c\rangle=0$ for $i\geq 2$. Differentiating $\langle e_2,c\rangle=0$, we find that $\kappa_1\langle e_1,c\rangle=0$. Since $\kappa_0\neq 0$, we have $\kappa_1\equiv 0$. We postpone the proof for the case $\kappa_1\equiv 0$ after the discussion of the case $H_s=0$.

Let $H_s=0$. When $\alpha_0>0$ and $\kappa_0\neq 0$, we have

$$\begin{aligned} S_4/2r^{n+1}\kappa_0 &= (r^2\kappa_0^2+\alpha_1^2)\left(\frac{1+\langle c,c\rangle-r^2}{2}\,n\kappa_0-n\alpha_1\langle c,e_0\rangle\right)v_1^4 \\ &\quad + \big(r^2\alpha_0\kappa_0^2+2rr'\alpha_1\kappa_0+2\alpha_0\alpha_1^2 \\ &\qquad + r^2\alpha_1\kappa_0'-r^2\kappa_0\alpha_1'\big)\sum_{i=1}^n\langle e_i,c\rangle v_1^3v_i. \end{aligned}$$

If $\langle e_i, c\rangle \neq 0$ for some $i \geq 2$, then we have from the degree-4 and constant terms of S that

$$\frac{1+\langle c,c\rangle - r^2}{2}\kappa_0 - \langle c, e_0\rangle\alpha_1 = 0$$

and

$$\frac{1+\langle c,c\rangle - r^2}{2}\alpha_0 + \langle c, e_0\rangle rr' = 0.$$

Therefore we have either $\alpha_0\alpha_1 + rr'\kappa_0 = 0$ or $1 + \langle c, c\rangle - r^2 = 0$ and $\langle c, e_0\rangle = 0$. Since the second condition implies the first, M is either a hypersurface of rotation or a sphere.

When $\kappa_0 = 0$, we must have $\alpha_0 \neq 0$. Since γ is a line through the origin, we have $S = 2r^n\alpha_0(\alpha\cdot v)^2\{n(\alpha\cdot v)\langle c, e_0\rangle - 2\sum_{i=1^n} v_i\langle c, e_i\rangle\} +$ lower degree terms. Hence we have $\alpha_i = 0$ for all $i = 1, \ldots, n$, which implies that M is a hypersurface of rotation.

In short, there are two possibilities (for any value of H_s):

i) $\langle e_i, c\rangle \neq 0$ for some $i \geq 2$ and $\alpha_0\alpha_1 + rr'\kappa_0 = 0$. (In this case M is either a hypersurface of rotation or a sphere.)

ii) $\langle e_i, c\rangle = 0$ for all $i = 2, \ldots, n$ and $\kappa_1 \equiv 0$ for any choice of the north pole. (In this case, $\gamma(t)$ lies on a plane that contains the origin.)

When ii) holds, we define a new curve $\tilde{c}(t)$ connecting the spherical centers of the spheres of the foliation. It is easy to see that $\tilde{c}(t)$ is a reparametrization of a geodesic which meet orthogonally the totally geodesic n-spheres containing the spheres of the foliation. When $\tilde{c}(t)$ is not a point curve, M is a hypersurface of rotation. If $\tilde{c}(t)$ is a point curve, then M is a sphere. □

In $\mathbf{S}^3$, there are foliations that do not belong to any class discussed above.

Example 2. The Clifford torus is the intersection of $\mathbf{S}^3$ with a quadratic cone in $\mathbf{R}^4$ defined by the equation $x_1^2 + x_2^2 - x_3^2 - x_3^2 = 0$. Sterographic projection from the north pole (0,0,0,1) onto the hyperplane $x_4 = 0$ transforms it into the torus $T = \{x = (\sqrt{2} + \cos\theta)\cos\psi,\ y = (\sqrt{2} + \cos\theta)\sin\psi,\ z = \sin\theta : 0 \leq \theta, \psi \leq 2\pi\}$ [**13**]. The Clifford torus is a ruled minimal surface, that is, it admits a foliation by great circles. The planes which contain the origin and normal to $1/\sqrt{2}(-\cos\phi, -\sin\phi, 1)$, $0 \le \phi \le 2\pi$, meet T in two great circles (they intersect at two points and their center is the origin in the conformal metric). The smooth one-parameter family of these great circles does not belong to any class of foliations discussed above, while the Clifford torus is rotational. In fact, we have $e_0 = 1/\sqrt{2}(-\cos\phi, -\sin\phi, 1)$, $e_1 = (\sin\phi, -\cos\phi, 0)$, $e_2 = 1/\sqrt{2}(\cos\phi, \sin\phi, 1)$, $c(\phi) = (-\sin\phi, \cos\phi, 0)$ and $r = \sqrt{2}$. It is straightforward to see that $\kappa_0 = -1/\sqrt{2}$, $\langle c, e_2\rangle = 0$, $\alpha_2 = \langle c', e_2\rangle \ne 0$ and $\kappa_1 = 1/\sqrt{2}$.

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