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ON THE SOLVABILITY OF TWO SIMULTANEOUS SYMMETRIC CUBIC DIOPHANTINE EQUATIONS WITH APPLICATIONS TO SEXTIC DIOPHANTINE EQUATIONS

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ABSTRACT. This paper provides a necessary and sufficient condition for the solvability of the simultaneous diophantine equations $C_1(x,y) = C_1(u,v)$ and $C_2(x,y) = C_2(u,v)$ where $C_i(x,y)$, $i = 1,2$, are arbitrary binary cubic forms. If the forms $C_i(x,y)$ have a common factor, we obtain the complete solution of these equations; otherwise, we obtain infinitely many solutions provided the condition of solvability is satisfied. The method has been used to solve some diophantine problems such as finding triads of cubes with equal sums and equal products, finding two arithmetic progressions of six terms each with equal products of terms, as well as for solving certain sextic diophantine equations of the type $f(x,y) = f(u,v)$.

This paper is concerned with the solvability of the simultaneous diophantine equations

$$C_1(x,y) = C_1(u,v), \tag{1.1}$$

$$C_2(x,y) = C_2(u,v), \tag{1.2}$$

where $C_i(x,y)$, $i = 1,2$, are two distinct binary cubic forms defined by

$$C_1(x,y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3, \tag{1.3}$$

$$C_2(x,y) = b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3, \tag{1.4}$$

where the coefficients a_j, b_j, $j = 0,1,2,3$, are integers. As both equations (1.1) and (1.2) are homogeneous, any rational solution of these equations may be multiplied by a suitable constant to obtain a solution in integers. A solution in integers, say (x_1, y_1, u_1, v_1), will be said to be primitive if $\gcd(x_1, y_1, u_1, v_1) = 1$. Further, any solution other than the trivial solution $x = u$, $y = v$ will be said to be nontrivial.

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We give in Section 2 the complete solution of equations (1.1) and (1.2) in certain simple cases. In Section 3 we obtain a necessary and sufficient condition for the solvability of these equations when the forms $C_i(x, y)$ are arbitrary. In Section 4 we illustrate the method of solving equations (1.1) and (1.2) and we also obtain triads of cubes with equal sums and equal products. In Section 5 we apply the method to solve certain equations of the type

$$f(x, y) = f(u, v)$$

where $f(x, y)$ is a binary sextic form and obtain inter alia infinitely many examples of two arithmetic progressions of six terms each with equal products of terms.

2. In this section we give the complete solution in integers of equations (1.1) and (1.2) in certain simple cases.

2.1. It is easy to find all solutions of equations (1.1) and (1.2) with both sides of one or both the equations equal to zero. When the two cubic forms $C_i(x, y)$, $i = 1, 2$, have a common linear factor, say $(mx + ny)$, infinitely many primitive nontrivial solutions are given by the parametric solution $(x, y, u, v) = (n\alpha, -m\alpha, n\beta, -m\beta)$. We, however, note that there can only be finitely many primitive nontrivial integer solutions of the simultaneous equations $C_1(x, y) = C_1(u, v) = 0$ and $C_2(x, y) = C_2(u, v) \neq 0$.

When the two forms $C_i(x, y)$, $i = 1, 2$, have a common quadratic factor, it is readily seen that for nontrivial solutions, both sides of equations (1.1) and (1.2) must be zero, and thus the complete solution may be found.

2.2. We will now find the complete solution of equations (1.1) and (1.2) when the two forms $C_i(x, y)$ have a common linear factor, say y. Without loss of generality, we may write the two equations as follows:

$$y(x^2 + my^2) = v(u^2 + mv^2), \tag{2.1}$$

$$y(px^2 + qxy + ry^2) = v(pu^2 + quv + rv^2). \tag{2.2}$$

We now write $y = tv$, when these equations give $u = t^2x + q^{-1}s(t^3 - 1)v$ where $s = r - mp$. Now (2.1) leads to a quadratic equation in x and

v and we get a rational solution when $t = \alpha^2(mq^2 + s^2)$, where α is rational. This readily gives the following solution of equations (2.1) and (2.2):

$$\begin{aligned} x &= -k\{s(mq^2+s^2)\alpha^3 - 1\}, & y &= kq(mq^2+s^2)\alpha^3, \\ u &= \ \ k\{(mq^2+s^2)^2\alpha^4 - s\alpha\}, & v &= kq\alpha, \end{aligned} \tag{2.3}$$

where k and α are arbitrary rational parameters. This rational solution may be suitably rewritten to give integer solutions. In addition, there are the solutions given by $y = 0$, $v = 0$, with x and u arbitrary.

As an example, the complete integer solution of the equations

$$\begin{aligned} (x+y)(x^2+xy-y^2) &= (u+v)(u^2+uv-v^2), \\ (x+y)(x^2-xy-y^2) &= (u+v)(u^2-uv-v^2), \end{aligned} \tag{2.4}$$

which may be reduced to equations of type (2.1) and (2.2) by a non-singular linear transformation, is given by $(x, y, u, v) = (\alpha, -\alpha, \beta, -\beta)$ and

$$\begin{aligned} x &= k(512\alpha^3\beta - \beta^4), & y &= k(512\alpha^3\beta + \beta^4), \\ u &= -k(4096\alpha^4 + 8\alpha\beta^3), & v &= k(4096\alpha^4 - 8\alpha\beta^3), \end{aligned} \tag{2.5}$$

where α and β are arbitrary integral parameters and k is a suitable rational number.

3. In this section we will obtain a necessary and sufficient condition for the solvability of equations (1.1) and (1.2).

Lemma 1. *A nontrivial solution of the simultaneous equations*

$$C_1(x, y) = C_1(u, v), \tag{3.1}$$

$$xy(x+y) = uv(u+v), \tag{3.2}$$

where $C_1(x, y)$ *is an arbitrary cubic form defined by* (1.3) *is given by*

$$\begin{aligned} x &= a_0 - a_1 + a_2 - 2a_3, & y &= a_0 + a_3, \\ u &= a_0 + a_3, & v &= -2a_0 + a_1 - a_2 + a_3, \end{aligned} \tag{3.3}$$

except when $C_1(x,y)$ *is a linear combination of the cubic forms* $x^3 - 3xy^2 - y^3$ *and* $xy(x+y)$, *in which case a nontrivial parametric solution is given by* $x = v$, $y = -u - v$ *where* u, v *are arbitrary.*

Proof. The truth of the lemma is readily verified by direct computation.

Lemma 2. *Let* $L_i(x,y)$, $i = 1,2,3$, *be three rational linear forms in* x *and* y *such that no two of these forms are linearly dependent, and* $C_1(x,y)$ *be a cubic form defined by* (1.3). *A nontrivial solution of the simultaneous equations*

$$C_1(x,y) = C_1(u,v), \tag{3.4}$$

$$L_1(x,y)L_2(x,y)L_3(x,y) = L_1(u,v)L_2(u,v)L_3(u,v), \tag{3.5}$$

can be obtained by an application of Lemma 1.

Proof. It is easily seen that there exist integers p, q, r such that $pL_1(x,y) + qL_2(x,y) + rL_3(x,y)$ vanishes identically. The nonsingular linear transformation determined by $pL_1(x,y) = X$, $qL_2(x,y) = Y$, $pL_1(u,v) = U$, $qL_2(u,v) = V$, reduces the above equations to two equations of type (3.1) and (3.2) in variables X, Y, U, V whereupon Lemma 1 gives a nontrivial solution.

To obtain a necessary and sufficient condition for the solvability of equations (1.1) and (1.2), we first define the following four functions in terms of the coefficients of the forms $C_i(x,y)$ and a variable ξ:

$$\begin{aligned}
\phi_1(\xi) &= a_0(b_1\xi^2 + b_2\xi + b_3) - b_0(a_1\xi^2 + a_2\xi + a_3),\\
\phi_2(\xi) &= (a_0b_2 - a_2b_0)\xi^2 + (a_0b_3 + a_1b_2 - a_2b_1 - a_3b_0)\xi\\
&\qquad + a_1b_3 - a_3b_1,\\
\phi_3(\xi) &= -a_3(b_0\xi^2 + b_1\xi + b_2) + b_3(a_0\xi^2 + a_1\xi + a_2),\\
f(\xi) &= \phi_2^2(\xi) - 4\phi_1(\xi)\phi_3(\xi).
\end{aligned} \tag{3.6}$$

We then have the following theorem:

Theorem. *Let* $C_1(x,y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3$ *and* $C_2(x,y) = b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3$ *be two arbitrary cubic forms. A necessary* *and sufficient condition that the simultaneous equations*

$$C_1(x,y) = C_1(u,v), \tag{3.7}$$
$$C_2(x,y) = C_2(u,v), \tag{3.8}$$

have infinitely many primitive nontrivial solutions in integers is that either the quartic function $f(\xi)$ is identically a perfect square for all values of ξ, or the quartic equation in ξ and η given by

$$\eta^2 = f(\xi)$$

represents an elliptic curve over $\mathbf{Q}$ of positive rank.

Proof. We first assume that the forms $C_1(x,y)$ and $C_2(x,y)$ do not have a common linear or quadratic factor and show that the condition stated in the theorem is both sufficient and necessary.

The condition is sufficient. We write

$$\begin{aligned} C(x,y) = {} & (b_0\xi^3 + b_1\xi^2 + b_2\xi + b_3)(a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3) \\ & - (a_0\xi^3 + a_1\xi^2 + a_2\xi + a_3)(b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3) \end{aligned} \tag{3.9}$$

and observe that

$$C(x,y) = (x - \xi y)\{\phi_1(\xi)x^2 + \phi_2(\xi)xy + \phi_3(\xi)y^2\}, \tag{3.10}$$

where $\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)$ are defined by (3.6).

We now show that the quartic function $f(\xi)$ cannot vanish identically. Assuming that $f(\xi) = 0$ identically, it follows from (3.9) and (3.10) that

$$\begin{aligned} & C_2(\xi,1)C_1(x,y) - C_1(\xi,1)C_2(x,y) \\ & \qquad = \psi_0(\xi)(x - \xi y)\{\psi_1(\xi)x + \psi_2(\xi)y\}^2 \end{aligned} \tag{3.11}$$

where $\psi_0(\xi), \psi_1(\xi), \psi_2(\xi)$ are suitable functions of ξ. Let ξ_1 and ξ_2 be any two roots, not necessarily rational, of $C_1(\xi,1) = 0$. Substituting in turn $\xi = \xi_1$ and $\xi = \xi_2$ in the identity (3.11), we get the identities:

$$\begin{aligned} C_2(\xi_1,1)C_1(x,y) &= \psi_0(\xi_1)(x - \xi_1 y)\{\psi_1(\xi_1)x + \psi_2(\xi_1)y\}^2, \\ C_2(\xi_2,1)C_1(x,y) &= \psi_0(\xi_2)(x - \xi_2 y)\{\psi_1(\xi_2)x + \psi_2(\xi_2)y\}^2. \end{aligned} \tag{3.12}$$

As the forms $C_i(x,y)$, $i=1,2$, do not have a common factor, $C_2(\xi_1)\neq 0$ and $C_2(\xi_2)\neq 0$, and hence we obtain from (3.12) the following identity:

$$\begin{aligned} C_1(x,y) &= C_2^{-1}(\xi_1,1)\psi_0(\xi_1)(x-\xi_1 y)\{\psi_1(\xi_1)x+\psi_2(\xi_1)y\}^2 \\ &= C_2^{-1}(\xi_2,1)\psi_0(\xi_2)(x-\xi_2 y)\{\psi_1(\xi_2)x+\psi_2(\xi_2)y\}^2. \end{aligned} \tag{3.13}$$

As the above two factorizations of $C_1(x,y)$ must be identical, it follows that ξ_1 and ξ_2 must be equal. In other words, $C_1(\xi,1)$ cannot have two distinct roots, and so we must have

$$C_1(x,y) = L_1^3(x,y) \tag{3.14}$$

where $L_1(x,y)$ is an integral linear form in x and y. Similarly, we must have

$$C_2(x,y) = L_2^3(x,y) \tag{3.15}$$

where $L_2(x,y)$ is another integral linear form in x and y. Using (3.14) and (3.15), we now obtain from (3.11) the following identity:

$$\begin{aligned} &\{L_2(\xi,1)L_1(x,y)\}^3 - \{L_1(\xi,1)L_2(x,y)\}^3 \\ &\qquad = \psi_0(\xi)(x-\xi y)\{\psi_1(\xi)x+\psi_2(\xi)y\}^2. \end{aligned} \tag{3.16}$$

The lefthand side is the difference of cubes of two distinct linear forms and hence cannot have the square of a linear form as a factor. This contradiction shows that our assumption must be false, that is, $f(\xi)$ cannot vanish identically.

We will now choose a suitable rational value of ξ such that $f(\xi)$ becomes a perfect square and none of the following four relations is satisfied:

$$\begin{aligned} f(\xi) &= 0, \\ \phi_1(\xi)\xi^2+\phi_2(\xi)\xi+\phi_3(\xi) &= 0, \\ a_0\xi^3+a_1\xi^2+a_2\xi+a_3 &= 0, \\ b_0\xi^3+b_1\xi^2+b_2\xi+b_3 &= 0. \end{aligned} \tag{3.17}$$

The lefthand side of each of these equations does not vanish identically, and accordingly these are four equations in ξ, of degrees at most 4, 4, 3, 3, respectively, and thus have at most 14 rational roots.

When $f(\xi)$ is identically a perfect square, we can readily choose, in infinitely many ways, a rational numerical value of ξ different from the possible 14 rational roots for which any one of the relations (3.17) may be satisfied. If $f(\xi)$ is not identically a perfect square, but the equation $\eta^2 = f(\xi)$ represents an elliptic curve over $\mathbf{Q}$ of positive rank, there exist infinitely many rational points on this curve, and hence we can again choose in infinitely many ways a suitable numerical value of ξ such that $f(\xi)$ is a perfect square while none of the relations (3.17) is satisfied. With such a value of ξ, say ξ_0, the discriminant of the quadratic form $\phi_1(\xi_0)x^2+\phi_2(\xi_0)xy+\phi_3(\xi_0)y^2$, namely, $f(\xi_0)$ is a nonzero perfect square, and hence this quadratic form has two distinct factors, say $L_1(x, y)$ and $L_2(x, y)$. Moreover, since $\phi_1(\xi_0)\xi_0^2 + \phi_2(\xi_0)\xi_0 + \phi_3(\xi_0) \neq 0$, therefore, $(x - \xi_0 y)$ is not a factor of the quadratic form $\phi_1(\xi_0)x^2+\phi_2(\xi_0)xy+\phi_3(\xi_0)y^2$. Thus, the three linear forms $(x-\xi_0 y)$, $L_1(x, y)$ and $L_2(x, y)$ are such that no two of them are linearly dependent. Thus, when we take the value of ξ as ξ_0 in (3.10), we get

$$C(x, y) = (x - \xi_0 y)L_1(x, y)L_2(x, y).$$

Similarly,

$$C(u, v) = (u - \xi_0 v)L_1(u, v)L_2(u, v),$$

and we can solve the simultaneous equations (3.7) and

$$C(x, y) = C(u, v), \tag{3.18}$$

by an application of Lemma 2. Since we had chosen ξ_0 such that $a_0\xi^3+a_1\xi^2+a_2\xi+a_3 \neq 0$ and also $b_0\xi^3+b_1\xi^2+b_2\xi+b_3 \neq 0$, it follows from (3.7), (3.9) and (3.18) that the solution so obtained also satisfies equation (3.8). We now let ξ_0 run through the infinitely many possible values such that $f(\xi_0)$ is a perfect square and none of the conditions (3.17) is satisfied. We thus obtain a number of solutions of equations (3.7) and (3.8). The solutions (x, y, u, v) thus obtained will be rational functions of ξ_0. Any solution (x, y, u, v) will be trivial if $x(\xi_0) = u(\xi_0)$ and $y(\xi_0) = v(\xi_0)$. These last two conditions which lead to a trivial solution may be considered as two polynomial equations in ξ_0, and these equations have a finite number of roots. We can choose ξ_0 in infinitely many ways such that the earlier requirements for ξ_0 are fulfilled and, at the same time, ξ_0 is not equal to anyone of the finite number of values that satisfy the conditions which lead to trivial solutions. Thus, we obtain infinitely many nontrivial solutions of equations (3.7) and (3.8). This shows that the condition stated in the theorem is sufficient for the existence of infinitely many nontrivial solutions.

The condition is necessary: Let equations (3.7) and (3.8) have infinitely many primitive nontrivial solutions.

We have already seen in Section 2 that there can only be a finite number of primitive nontrivial solutions of equations (3.7) and (3.8) with $C_1(x,y) = 0$ or $C_2(x,y) = 0$. Further, it is easy to see that there can only be finitely many primitive nontrivial solutions of these equations with $y = 0$ or with $v = 0$.

We also note that there can only be finitely many solutions of (3.7) and (3.8) with $x = ky$ where k is a nonzero constant. If $x = ky$, it follows from equation (3.7) that $y^3C_1(k,1) = C_1(u,v)$ and similarly, from equation (3.8) we get $y^3C_2(k,1) = C_2(u,v)$. Thus, we must have $C_1(k,1)C_2(u,v) - C_2(k,1)C_1(u,v) = 0$, and this equation has at most three primitive nontrivial solutions which, in turn, lead to at most three primitive nontrivial solutions of equations (3.7) and (3.8).

Excluding the finitely many primitive nontrivial solutions of (3.7) and (3.8) with $C_1(x,y) = 0$ or $C_2(x,y) = 0$ or $y = 0$ or $v = 0$, we choose from the remaining infinitely many nontrivial solutions a solution $(x,y,u,v) = (\alpha,\beta,\gamma,\delta)$ so that $\beta \neq 0$ and $\delta \neq 0$, and

$$C_1(\alpha,\beta) = C_1(\gamma,\delta) \tag{3.19}$$

$$C_2(\alpha,\beta) = C_2(\gamma,\delta). \tag{3.20}$$

The assumption $\alpha/\beta = \gamma/\delta$ implies that $\alpha = \beta m$, $\gamma = \delta m$ for some rational number m, and so it follows from (3.19) that $\beta^3C_1(m,1) = \delta^3C_1(m,1)$. Since $C_1(\alpha,\beta) \neq 0$, we also have $C_1(m,1) \neq 0$, and hence it follows that $\beta = \delta$, so that $\alpha = \gamma$, contradicting the assumption that $(\alpha,\beta,\gamma,\delta)$ was a nontrivial solution. Thus, we must have $\alpha/\beta \neq \gamma/\delta$ and hence also $\gamma - \alpha\beta^{-1}\delta \neq 0$.

We now rewrite the identities (3.9) and (3.10) as follows:

$$\begin{aligned}(x-\xi y)\{\phi_1(\xi)x^2 + \phi_2(\xi)xy + \phi_3(\xi)y^2\} &\\ = C(x,y)&\\ = C_2(\xi,1)C_1(x,y) - C_1(\xi,1)C_2(x,y).&\end{aligned} \tag{3.21}$$

Substituting $\xi = \alpha\beta^{-1}$, $x = \gamma$, $y = \delta$ in (3.21), we get

$$\begin{aligned}(\gamma - \alpha\beta^{-1}\delta)\{\phi_1(\alpha\beta^{-1})\gamma^2 + \phi_2(\alpha\beta^{-1})\gamma\delta + \phi_3(\alpha\beta^{-1})\delta^2\}\\ &= C_2(\alpha\beta^{-1}, 1)C_1(\gamma, \delta) - C_1(\alpha\beta^{-1}, 1)C_2(\gamma, \delta)\\ &= \beta^{-3}\{C_2(\alpha, \beta)C_1(\gamma, \delta) - C_1(\alpha, \beta)C_2(\gamma, \delta)\}\\ &= 0.\end{aligned}$$

As $(\gamma - \alpha\beta^{-1}\delta) \neq 0$, it follows that $\phi_1(\alpha\beta^{-1})\gamma^2 + \phi_2(\alpha\beta^{-1})\gamma\delta + \phi_3(\alpha\beta^{-1})\delta^2$ must be zero. Thus, the quadratic equation

$$\phi_1(\alpha\beta^{-1})\theta^2 + \phi_2(\alpha\beta^{-1})\theta + \phi_3(\alpha\beta^{-1}) = 0$$

has a rational root $\theta = \gamma\delta^{-1}$, and hence its discriminant, i.e., $f(\alpha\beta^{-1})$ must be a perfect square. We can choose the solution $(\alpha, \beta, \gamma, \delta)$ in infinitely many ways, and since there can only be finitely many solutions of (3.7) and (3.8) with $x = ky$ for some constant k, these infinitely many solutions $(\alpha, \beta, \gamma, \delta)$ provide infinitely many rational values $\alpha\beta^{-1}$ for ξ such that $f(\xi)$ is a perfect square. Thus, either $f(\xi)$ is identically a perfect square for all values of ξ, or these infinitely many rational values of ξ provide infinitely many rational points on the quartic curve $\eta^2 = f(\xi)$, and hence this curve must represent an elliptic curve over $\mathbf{Q}$ of positive rank. This proves that the condition of the theorem is necessary.

When the forms $C_1(x, y)$ and $C_2(x, y)$ have a common linear or quadratic factor, the complete solution has already been obtained and the theorem is readily verified to be true in this case as well.

4. We now illustrate the method of Section 3 by two examples.

4.1. We consider the equations

$$x^3 + 9x^2y + 24xy^2 + 2y^3 = u^3 + 9u^2v + 24uv^2 + 2v^3, \tag{4.1}$$

$$2x^3 + 24x^2y + 9xy^2 + y^3 = 2u^3 + 24u^2v + 9uv^2 + y^3. \tag{4.2}$$

Here,

$$f(\xi) = 9(177\xi^4 + 4368\xi^3 + 27198\xi^2 + 4368\xi + 177), \tag{4.3}$$

and we find by trial that $f(-2) = 783^2$. With $\xi = -2$, using (3.10) we get

$$C(x, y) = 9(x + 2y)(x + 8y)(11x + y).$$

We will solve equation (4.1) together with the equation

$$(x + 2y)(x + 8y)(11x + y) = (u + 2v)(u + 8v)(11u + v). \tag{4.4}$$

We now write

$$\begin{aligned} x &= (56X + 58Y), \qquad & y &= -(7X + 29Y), \\ u &= (56U + 58V), \qquad & v &= -(7U + 29V), \end{aligned} \tag{4.5}$$

so that equations (4.1) and (4.4) are transformed respectively to the equations

$$\begin{aligned} &2401X^3 - 4263X^2Y + 17661XY^2 + 24389Y^3 \\ &\qquad = 2401U^3 - 4263U^2V + 17661UV^2 + 24389V^3, \end{aligned} \tag{4.6}$$

and

$$XY(X + Y) = UV(U + V), \tag{4.7}$$

which, by Lemma 1 of Section 3, have the solution $X = -429$, $Y = 470$, $U = 470$, $V = -41$. Using (4.5) we get $x = 3236$, $y = -10627$, $u = 23942$, $v = -2101$ as a solution of equations (4.1) and (4.2). It is readily verified using APECS (a package written in MAPLE for working with elliptic curves) that the rank of the elliptic curve $\eta^2 = f(\xi)$ is 1. The infinitely many rational points on this curve will yield infinitely many solutions of the simultaneous equations (4.1) and (4.2). For instance, $f(\xi)$ becomes a perfect square when $\xi = -9223/56333$ and, with this value of ξ, using (3.10), we get

$$C(x, y) = \frac{504(56333x + 9223y)(2101x + 23942y)(10627x + 3236y)}{178767530278037},$$

which leads to the following solution of equations (4.1) and (4.2):

$$\begin{aligned} x &= -1818723620411622197402572828639890176, \\ y &= 2690254822190548794838236148105093327, \\ u &= -7909806737769958021098875076075 10682, \\ v &= 7067060890566162810402511361474789881. \end{aligned}$$

4.2. Guy [**4**, p. 142] has mentioned the problem of finding triads of cubes with equal sums and equal products, that is, of solving the simultaneous equations

$$\begin{aligned} a^3+b^3+c^3&=d^3+e^3+f^3,\\ abc&=def. \end{aligned} \tag{4.8}$$

On writing

$$\begin{aligned} a&=px, \quad b=qy, \quad c=x+y,\\ d&=pu, \quad e=qv, \quad f=u+v, \end{aligned} \tag{4.9}$$

the above equations reduce to the following two cubic equations

$$\begin{aligned} p^3x^3+q^3y^3+(x+y)^3&=p^3u^3+q^3v^3+(u+v)^3,\\ xy(x+y)&=uv(u+v). \end{aligned} \tag{4.10}$$

We solve the equations (4.10) by applying Lemma 1 of Section 3 and using (4.9), we get a solution of (4.8) which on substituting $p=\alpha/\gamma$, $q=\beta/\gamma$ and multiplying throughout by γ^4, may be written as follows:

$$\begin{aligned} a&=\alpha(\alpha^3-2\beta^3-\gamma^3), & d&=\alpha(\alpha^3+\beta^3+2\gamma^3),\\ b&=\beta(\alpha^3+\beta^3+2\gamma^3), & e&=\beta(-2\alpha^3+2\beta^3-\gamma^3),\\ c&=\gamma(2\alpha^3-\beta^3+\gamma^3), & f&=\gamma(-\alpha^3+2\beta^3+\gamma^3). \end{aligned}$$

5. We now apply the results of Section 3 to solve certain equations of the type

$$f(x,y)=f(u,v), \tag{5.1}$$

where $f(x,y)$ is a binary sextic form. If $f(x,y)$ is expressible as the product of two cubic factors $C_1(x,y)$ and $C_2(x,y)$, a solution of (5.1) may be obtained by solving the simultaneous equations (1.1) and (1.2). More generally, if we can write $f(x,y)$ as a quadratic combination of two cubic forms, that is, as $\{pC_1(x,y)^2+qC_1(x,y)C_2(x,y)+rC_2(x,y)^2\}$ where p,q,r are rational numbers, then also a solution of (5.1) may be obtained by solving equations (1.1) and (1.2).

5.1. Let us consider the equation

$$x^6 + 2x^5y - 2x^4y^2 - 6x^3y^3 - 2x^2y^4 + 2xy^5 + y^6 = u^6 + 2u^5v - 2u^4v^2 - 6u^3v^3 - 2u^2v^4 + 2uv^5 + v^6. \tag{5.2}$$

Here we observe that

$$x^6 + 2x^5y - 2x^4y^2 - 6x^3y^3 - 2x^2y^4 + 2xy^5 + y^6 = (x+y)^2(x^2+xy-y^2)(x^2-xy-y^2).$$

It follows that equation (5.2) may be solved by solving the equations (2.4), and thus a parametric solution of equation (5.2) is given by (2.5).

5.2. We now consider the equation

$$x^6 + 2158x^3y^3 + y^6 = u^6 + 2158u^3v^3 + v^6. \tag{5.3}$$

We note that

$$\begin{aligned} &3(x^6 + 2158x^3y^3 + y^6) \\ &\quad = -7(x^3 + 9x^2y + 24xy^2 + 2y^3)^2 - 7(2x^3 + 24x^2y + 9xy^2 + y^3)^2 \\ &\qquad + 19(x^3 + 9x^2y + 24xy^2 + 2y^3)(2x^3 + 24x^2y + 9xy^2 + y^3). \end{aligned}$$

It now follows that the infinitely many solutions of equations (4.1) and (4.2) also satisfy (5.3).

5.3. We now consider the problem, posed by Gabovich [**3**], of finding two arithmetic progressions with n terms each such that the products of the terms in the two arithmetic progressions are equal. When $n = 3$ or 4 or 5, infinitely many solutions are known [**2**], [**3**]. There are also two solutions known for arbitrary n [**2**], [**5**]. We now show how infinitely many solutions to this problem may be obtained when $n = 6$.

The two arithmetic progressions $x, x + y, \dots, x + 5y$ and $u, u + v, \dots, u + 5v$ will have equal products if

$$x(x+y)(x+2y)(x+3y)(x+4y)(x+5y) = u(u+v)(u+2v)(u+3v)(u+4v)(u+5v). \tag{5.4}$$

To solve equation (5.4), we solve, as described earlier, the equations

$$x(x+y)(x+2y) = u(u+v)(u+2v), \tag{5.5}$$

$$(x+3y)(x+4y)(x+5y) = (u+3v)(u+4v)(u+5v), \tag{5.6}$$

to get the solution $x = 6$, $y = 8$, $u = -28$, $v = 17$. We thus get two arithmetic progressions, namely, 6, 14, 22, 30, 38, 46 and −28, −11, 6, 23, 40, 57 such that the products of their terms are equal. Moreover, in view of equation (5.5), the products of the first three terms of both the arithmetic progressions are also equal, and similarly, the products of their last three terms are also equal.

We may obtain more solutions of equations (5.5) and (5.6) by the method already described. Here $f(\xi) = -9(15\xi^4 + 150\xi^3 + 399\xi^2 + 120\xi - 400)$. The curve $\eta^2 = f(\xi)$ represents an elliptic curve of rank 1, as may be verified using APECS. Since $f(0) = 3600 = 60^2$, we readily obtain another rational point on this curve with $\xi = -55/31$. This leads to the following two arithmetic progressions with equal products of all six terms, equal products of first three terms as well as equal products of the last three terms:

(i) − 17148021631332, −7988284710581, 1171452210170, 10331189130921, 19490926051672, 28650662972423;

(ii) − 11240729728109, −3681544447240, 3877640833629, 1143682611498, 18996011395367, 26555196676236.

As the curve $\eta^2 = f(\xi)$ represents an elliptic curve of positive rank, infinitely many examples of such arithmetic progressions may be obtained.

We could also obtain solutions of equation (5.4) by choosing the two cubic equations in several other ways.

5.4. We finally note that the above method does not yield solutions of the sextic equation $x^6 - y^6 = u^6 - v^6$, or of the equation $xy(x^4 - y^4) = uv(u^4 - v^4)$ which has been mentioned by Bremner and Guy [**1**] as a difficult diophantine problem. This, however, does not disprove the existence of integer solutions of either of these equations.

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