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PARAQUATERNIONIC KÄHLER MANIFOLDS

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ABSTRACT. Paraquaternionic Kähler manifolds are studied with special attention to their curvatures. Since such manifolds are necessarily pseudo-Riemannian of neutral signature, the behavior of the Jacobi operators along spacelike, timelike and null geodesics is considered separately, for the purpose of study on Osserman problems on paraquaternionic Kähler manifolds.

1. Introduction. The study of Jacobi operators, as a basic part of the curvature tensor, is one of the central topics in Riemannian and pseudo-Riemannian geometry. Since Jacobi operators provide a way to measure the geodesic deviation, it is natural to expect that their properties strongly influence the geometry of manifolds. Osserman conjectured in his study [**30**] of those Riemannian manifolds, whose Jacobi operators have constant eigenvalues on the unit sphere bundle, that they must be locally flat or rank-one symmetric. This was shown by Chi in many cases, although the general problem still remains open. (See [**9**], [**10**], [**11**] and [**21**] for more details and further references). In the pseudo-Riemannian setting, the Osserman problem was firstly studied in the framework of Lorentzian geometry, showing that it is equivalent to constant curvature [**4**], [**17**], [**18**]. The situation is however much more complicated for pseudo-Riemannian metrics of other signatures. Indeed, in the pseudo-Riemannian setting, non-symmetric Osserman manifolds of signature (p,q), $p,q>1$, exist. See [**19**]. In [**5**] a systematic study of the Osserman problem for metrics of signature $(++--)$ is developed. If the Jacobi operators are assumed to be diagonalizable, such spaces correspond to the indefinite real and

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complex space forms and the paracomplex space forms. Recently, pseudo-Riemannian manifolds, whose Jacobi operators have a simple form, have been investigated in [**7**], where it is shown that the simplest pseudo-Riemannian spaces (from the point of view of their curvature) are, besides the spaces of constant curvature, indefinite complex and quaternionic space forms, and paracomplex and paraquaternionic space forms.

Since the geometry of paraquaternionic manifolds is not well developed, the purpose of this paper is to study such manifolds further and give special attention to their curvature. Section 2 introduces paraquaternionic manifolds. Also, some basic identities for the curvature tensor are pointed out with special attention to the case of dimension ≥ 8. In Section 3, the definition of the paraquaternionic sectional curvature is given and the expression of the curvature tensor of paraquaternionic space forms is obtained. In Section 4, Theorem 4.1 gives a characterization of constant paraquaternionic sectional curvature in terms of the existence of certain distinguished eigenspaces for the Jacobi operators along spacelike, timelike and null geodesics. Finally, since the paraquaternionic sectional curvature is not defined on degenerate paraquaternionic sections, the significance of the curvature tensor on such sections is investigated in Section 5.

2. Paraquaternionic Kähler manifolds.

In this section we will establish the definition of paraquaternionic Kähler structures and derive some curvature identities that we will use in this paper.

2.1. Definitions. Let M be a smooth manifold. A *paraquaternionic structure* $\mathbf{V}$ *on* M is defined to be a rank-3 subbundle of $\operatorname{End}(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ exists of sections of $\mathbf{V}$ satisfying

$$
\begin{aligned}
J_\tau^2 &= \varepsilon_\tau Id, \\
J_1 J_2 &= -J_2 J_1 = J_3 \\
\varepsilon_1 &= \varepsilon_2 = -\varepsilon_3 = 1.
\end{aligned}
\tag{2.1}
$$

A pseudo-Riemannian metric g is said to be *adapted* to the paraquaternionic structure $\mathbf{V}$ if it satisfies

$$
g(J_\tau X, J_\tau Y) = -\varepsilon_\tau g(X, Y), \quad \tau = 1, 2, 3, \tag{2.2}
$$

for all vector fields X, Y on M and any local basis of $\mathbf{V}$. (Note that any paraquaternionic manifold is of dimension $4m$ and any adapted metric is necessarily of neutral signature $(2m, 2m)$). Moreover, $(M, g, \mathbf{V})$ is said to be a *paraquaternionic Kähler manifold* if the bundle $\mathbf{V}$ is parallel with respect to the Levi-Civita connection of g. Equivalently, locally defined 1-forms p, q, r exist such that

$$\begin{aligned} \nabla_X J_1 &= q(X)J_2 - r(X)J_3, \\ \nabla_X J_2 &= -q(X)J_1 + p(X)J_3, \\ \nabla_X J_3 &= -r(X)J_1 + p(X)J_2. \end{aligned} \tag{2.3}$$

It is also worthwhile to characterize the paraquaternionic Kähler structures from a group theoretical point of view. In dimension four, the rotational group $SO(4)$ splits as $SO(4) \cong SO(3) \times SO(3)$, and, correspondingly, the pseudo-rotational group $SO_0(2,2)$ exhibits a similar splitting $SO_0(2,2) \cong SO_0(2,1) \times SO_0(2,1)$. From Lie algebra isomorphisms, we know the following

$$\begin{aligned} \mathfrak{so}\,(3) &= \mathfrak{su}\,(2) = \mathfrak{sp}\,(1) \\ \mathfrak{so}\,(2,1) &= \mathfrak{su}\,(1,1) = \mathfrak{sp}\,(1, \mathbf{R}). \end{aligned}$$

The fact that the Lie group $Sp(1)$ ($\cong SU(2)$) of the Lie algebra $\mathfrak{sp}\,(1)$ can be interpreted as a set of unit quaternions suggests that the group $Sp(1, \mathbf{R})$ ($\cong SU(1,1)$) of the Lie algebra $\mathfrak{sp}\,(1, \mathbf{R})$ must be a set of paraquaternions of unit norm. Here, by a paraquaternion, we mean a number

$$\mathfrak{q} = a + b\mathfrak{i} + c\mathfrak{j} + d\mathfrak{k},$$

where three kinds of imaginary units $\mathfrak{i}$, $\mathfrak{j}$ and $\mathfrak{k}$ satisfy the following properties

$$\begin{gathered} \mathfrak{i}^2 = \mathfrak{j}^2 = -\mathfrak{k}^2 = 1 \\ \mathfrak{i}\mathfrak{j} = \mathfrak{k}, \quad \mathfrak{k}\mathfrak{i} = -\mathfrak{j}, \quad \mathfrak{j}\mathfrak{k} = -\mathfrak{i}. \end{gathered}$$

It is clear that the set of paraquaternions $\tilde{\mathbf{H}}$ can be identified with the 4-dimensional pseudo-Euclidean space $\mathbf{R}^{2,2}$ (the vector space $\mathbf{R}^4$ with a norm of signature $(++--)$).

Observing an analogue of paraquaternions $\tilde{\mathbf{H}}$ to the quaternions $\mathbf{H}$, we are led to consider in higher dimensions a corresponding structure to the quaternionic Kähler structure in the Riemannian category as characterized by the linear holonomy group in $Sp(1)Sp(m)$ on a $4m$-dimensional manifold [**32**]. The symplectic group $Sp(m)$, a subgroup of $SO(4m)$, on $\mathbf{H}^m \cong \mathbf{R}^{4m}$ of signature $(+\overset{4m}{\cdots}+)$ corresponds to the real symplectic group $Sp(m, \mathbf{R})$, a subgroup of $SO(2m, 2m)$ on $\tilde{\mathbf{H}}^m \cong \mathbf{R}^{2m,2m}$ of neutral signature $(+\overset{2m}{\cdots}+-\overset{2m}{\cdots}-)$. Therefore, by characterizing the linear holonomy group as in a subgroup of $Sp(1, \mathbf{R})Sp(m, \mathbf{R})$, we naturally arrive at a notion of a paraquaternionic Kähler structure on a $4m$-dimensional pseudo-Riemannian manifold of neutral signature $(2m, 2m)$. (It should be noted [**23**], [**31**] that $Sp(m)$ and $Sp(m, \mathbf{R})$ appear as the real and imaginary parts of the complex symplectic group $Sp(m, \mathbf{C})$).

Remark 2.1. It is interesting to note that any oriented 4-manifold admitting a metric of signature $(++--)$ is paraquaternionic Kähler, since $Sp(1, \mathbf{R})Sp(1, \mathbf{R}) \cong SO_0(2, 2)$. This fact provides us with a large family of four-dimensional examples. (See [**27**] for necessary and sufficient conditions for a compact 4-manifold to admit such a neutral metric).

Remark 2.2. Note also that the tensor fields $\{J_\tau, \tau = 1, 2, 3\}$ of the paraquaternionic structure $\mathbf{V}$ are locally defined. If J_1, J_2, J_3 are globally defined satisfying (2.1) and g is an adapted metric, then the quadruple (g, J_1, J_2, J_3) is called a *neutral almost hyper-Hermitian structure* and it is called *neutral hyper-Kähler* if, in addition, J_τ, $\tau = 1, 2, 3$, are parallel with respect to the Levi-Civita connection of g, see [**23**], [**24**], [**25**]. It is clear that any neutral hyper-Kähler manifold is a paraquaternionic one. However, the converse is not true. In fact, taking account of Chern classes of J_3, one attains a large list of oriented 4-manifolds admitting neutral metrics which do not admit any neutral almost hyper-Hermitian structure (cf. [**27**]). Also, it is not difficult to show examples of paraquaternionic Kähler manifolds that do not admit any neutral hyper-Kähler structure on the basis of the results in [**14**].

2.2. Curvature consequences. Since our main purpose is to investigate the curvature of paraquaternionic Kähler manifolds, we must, first of all, derive some important curvature identities that are consequences of the Kähler identity (2.3). Let $(M, g, \mathbf{V})$ be a paraquaternionic Kähler manifold of $\dim M = 4m$ and $\{J_\tau; \tau = 1, 2, 3\}$ a canonical local basis of $\mathbf{V}$. Let ∇ be the Levi-Civita connection of g and R its curvature tensor, defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The Ricci tensor ρ is defined to be $\rho(X,Y) = \operatorname{trace}\{Z \mapsto R(X,Z)Y\}$ and the scalar curvature Sc is the trace of the Ricci tensor. Then a straightforward calculation from (2.3) shows that

$$\begin{aligned} [R(X,Y), J_1] &= C(X,Y)J_2 - B(X,Y)J_3, \\ [R(X,Y), J_2] &= -C(X,Y)J_1 + A(X,Y)J_3, \\ [R(X,Y), J_3] &= -B(X,Y)J_1 + A(X,Y)J_2, \end{aligned} \tag{2.4}$$

where A, B and C are locally defined 2-forms satisfying $A = 2(dp - q \wedge r)$, $B = 2(dr - p \wedge q)$ and $C = 2(dq - p \wedge r)$.

After some calculations, one gets $A(X,Y) = (1/(2m))\operatorname{trace} J_1 R(X,Y)$, $B(X,Y) = (1/(2m))\operatorname{trace} J_2 R(X,Y)$ and $C(X,Y) = (1/(2m))\operatorname{trace} J_3 \times R(X,Y)$. If $m > 1$, then

$$\begin{aligned} \rho(X,Y) &= (m+2)A(X, J_1Y) \\ &= (m+2)B(X, J_2Y) \\ &= -(m+2)C(X, J_3Y). \end{aligned} \tag{2.5}$$

Now, if M is assumed to be of dimension strictly greater than four, then by using (2.5) we may express the identities (2.4) in a suitable form for our purposes:

$$\begin{aligned} &R(X,Y,J_1Z,J_1W) + R(X,Y,Z,W) \\ &\qquad = \frac{1}{m+2}\{\rho(X,J_3Y)g(Z,J_3W) - \rho(X,J_2Y)g(Z,J_2W)\}, \\ &R(X,Y,J_2Z,J_2W) + R(X,Y,Z,W) \\ &\qquad = \frac{1}{m+2}\{\rho(X,J_3Y)g(Z,J_3W) - \rho(X,J_1Y)g(Z,J_1W)\}, \\ &R(X,Y,J_3Z,J_3W) - R(X,Y,Z,W) \\ &\qquad = \frac{1}{m+2}\{\rho(X,J_2Y)g(Z,J_2W) + \rho(X,J_1Y)g(Z,J_1W)\}. \end{aligned} \tag{2.6}$$

As an immediate application of (2.6), we obtain the following, which is a fundamental observation:

Theorem 2.1. *Any paraquaternionic Kähler manifold* $(M, g, \mathbf{V})$ *is Einstein, provided that* $\dim M > 4$.

Proof. Let X and Z be arbitrary vectors tangent to M at some point $p \in M$, and put $Y = J_1X$, $W = J_1Z$, where $\{J_1, J_2, J_3\}$ is some local basis of the paraquaternionic structure satisfying (2.1). From (2.6) it follows that

$$R(X, J_1X, J_2Z, J_3Z) - R(X, J_1X, Z, J_1Z) = \frac{1}{m+2}\rho(X, X)g(Z, Z). \tag{2.7}$$

Analogously, if we take X, Y, Z and W of the form J_2X, J_3X, Z and J_1Z, respectively, (2.6) also gives

$$R(J_2X, J_3X, J_2Z, J_3Z) - R(J_2X, J_3X, Z, J_1Z) = \frac{1}{m+2}\rho(J_2X, J_2X)g(Z, Z). \tag{2.8}$$

Now, note that $\rho(J_2X, J_2X) = -\rho(X, X)$ easily follows from (2.5), and hence (2.7) and (2.8) lead to

$$\begin{aligned} \frac{2}{m+2}\rho(X, X)g(Z, Z) = {} & R(X, J_1X, J_2Z, J_3Z) - R(X, J_1X, Z, J_1Z) \\ & - R(J_2X, J_3X, J_2Z, J_3Z) \\ & + R(J_2X, J_3X, Z, J_1Z). \end{aligned} \tag{2.9}$$

Next, change X with Z in (2.9) to obtain

$$\begin{aligned} \frac{2}{m+2}\rho(Z, Z)g(X, X) = {} & R(X, J_1X, J_2Z, J_3Z) - R(X, J_1X, Z, J_1Z) \\ & - R(J_2X, J_3X, J_2Z, J_3Z) \\ & + R(J_2X, J_3X, Z, J_1Z). \end{aligned} \tag{2.10}$$

It then follows from (2.9) and (2.10) that $\rho(X, X)g(Z, Z) = \rho(Z, Z) \times g(X, X)$ for all vectors X, Z. This shows that the Ricci tensor is bounded from above and from below on unit space-like and time-like vectors and, thus, the manifold is Einstein as a consequence of [**28**, Lemma A]. $\square$

We close this section with two remarks.

Remark 2.3. As an immediate consequence of the above theorem and (2.6), the curvature tensor of a $4m$-dimensional, $m > 1$, paraquaternionic Kähler manifold satisfies

$$\begin{aligned}
&R(X,Y,J_1Z,J_1W) + R(X,Y,Z,W)\\
&\qquad = \frac{Sc}{4m(m+2)}\{g(X,J_3Y)g(Z,J_3W) - g(X,J_2Y)g(Z,J_2W)\},\\
&R(X,Y,J_2Z,J_2W) + R(X,Y,Z,W)\\
&\qquad = \frac{Sc}{4m(m+2)}\{g(X,J_3Y)g(Z,J_3W) - g(X,J_1Y)g(Z,J_1W)\},\\
&R(X,Y,J_3Z,J_3W) - R(X,Y,Z,W)\\
&\qquad = \frac{Sc}{4m(m+2)}\{g(X,J_2Y)g(Z,J_2W) + g(X,J_1Y)g(Z,J_1W)\},
\end{aligned} \tag{2.11}$$

where $\{J_\tau, \tau = 1,2,3\}$ is any local basis of the paraquaternionic structure as in (2.1).

Remark 2.4. Let $\pi = \langle\{X,Y\}\rangle$ be a plane tangent to M at a point $p \in M$. The sectional curvature $K(\pi)$ is defined by

$$K(\pi) = K(X,Y) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

Clearly, this definition makes sense only for nondegenerate planes (i.e., those satisfying $g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0$. See [**13**], [**22**], [**26**]). As a consequence of (2.11), it follows that the sectional curvature of a $4m$-dimensional paraquaternionic Kähler manifold is constant if and only if it vanishes, provided that $m > 1$. This fact introduces a new curvature function: the paraquaternionic sectional curvature. Also note that a paraquaternionic Kähler manifold, $m > 1$ is locally conformally flat if and only if it is flat.

3. Paraquaternionic sectional curvatures. Let $(M, g, \mathbf{V})$ be a paraquaternionic Kähler manifold. Then any vector $X \in T_pM$ determines a four-dimensional subspace $\mathbf{V}(X) = \langle \{X, J_1X, J_2X, J_3X\} \rangle$ which remains invariant under the action of the paraquaternionic structure. We will refer to it as the $\mathbf{V}$-*section* determined by X. Note that the restriction of the metric g to any $\mathbf{V}$-section is indefinite of signature (2,2) or totally degenerate, in which the latter case occurs if and only if the $\mathbf{V}$-section is generated by a null vector.

Suppose now that $\mathbf{V}(X)$ is a $\mathbf{V}$-section spanned by a nonnull vector, X. If the sectional curvature of nondegenerate planes on $\mathbf{V}(X)$ is constant, $c(X)$, we will refer to it as the *paraquaternionic sectional curvature* of M with respect to X at $p \in M$. If $(M, g, \mathbf{V})$ is a paraquaternionic Kähler manifold of constant paraquaternionic sectional curvature, we will call it a *paraquaternionic space form*.

It is clear from the previous definition that a four-dimensional paraquaternionic Kähler manifold is of constant paraquaternionic sectional curvature if and only if the sectional curvature of M is constant. Therefore, we will concentrate our attention on the case of $\dim M \geq 8$.

Since our purpose in this section is to determine the forms of the curvature tensor of a paraquaternionic space form, we will reexamine the definition of a curvature-like function on a vector space E. A quadrilinear map $F : E \times E \times E \times E \to \mathbf{R}$ is said to be a curvature-like function if it satisfies

$$\begin{aligned} &F(X, Y, Z, W) = -F(Y, X, Z, W) = -F(X, Y, W, Z) \\ &F(X, Y, Z, W) = F(Z, W, X, Y) \\ &F(X, Y, Z, W) + F(Y, Z, X, W) + F(Z, X, Y, W) = 0, \end{aligned}$$

for all vectors $X, Y, Z, W \in E$. Moreover, if E is endowed with a paraquaternionic structure $\mathbf{V}$ and an adapted metric g, we will say that F is a *paraquaternionic curvature-like function* if it satisfies (2.11). The following lemma will be of interest for our purposes.

Lemma 3.1. *Let $(E^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic vector space and F a paraquaternionic curvature-like function on E. If $F(X, J_\tau X, X, J_\tau X) = 0$ for all $X \in E$ and some basis $\{J_\tau, \tau = 1, 2, 3\}$ of $\mathbf{V}$, then $F = 0$.*

Proof. Applying (2.11) to the paraquaternionic curvature-like function F, we get

$$\begin{aligned} F(X,J_1X,X,J_1X) &= F(X,J_1X,J_2X,J_3X) \\ &\quad - \frac{Sc^F}{4m(m+2)}g(X,X)^2 = 0, \\ F(X,J_2X,X,J_2X) &= F(X,J_2X,J_3X,J_1X) \\ &\quad - \frac{Sc^F}{4m(m+2)}g(X,X)^2 = 0, \\ F(X,J_3X,X,J_3X) &= -F(X,J_3X,J_1X,J_2X) \\ &\quad + \frac{Sc^F}{4m(m+2)}g(X,X)^2 = 0, \end{aligned}$$

where Sc^F denotes the scalar curvature of F. From these expressions we have

$$\begin{aligned} \frac{3Sc^F}{4m(m+2)}g(X,X)^2 &= F(X,J_1X,J_2X,J_3X) \\ &\quad + F(X,J_2X,J_3X,J_1X) \\ &\quad + F(X,J_3X,J_1X,J_2X) \end{aligned}$$

and, using the first Bianchi identity, we obtain $Sc^F = 0$. Therefore, (2.11) shows that F is a paraholomorphic curvature-like function on (E,g,J_1) satisfying $F(X,J_1X,X,J_1X) = 0$ for all vectors X. Now it follows from [**15**, Corollary 3.5] that $F = 0$. ◻

Next we will introduce a paraquaternionic curvature-like function which plays an important role in the study of the paraquaternionic sectional curvature. Let $(E,g,\mathbf{V})$ be a paraquaternionic vector space, and define F_0 by

$$\begin{aligned} F_0(X,Y,Z,W) &= g(X,Z)g(Y,W) - g(Y,Z)g(X,W) \\ &\quad + \sum_{\tau=1}^{3} \varepsilon_\tau \{ g(J_\tau Y,Z)g(J_\tau X,W) - g(J_\tau X,Z)g(J_\tau Y,W) \\ &\quad + 2g(X,J_\tau Y)g(J_\tau Z,W)\}. \end{aligned}$$

The following theorem shows the special significance of F_0.

Theorem 3.1. *Let* $(M^{4m}, g, \mathbf{V})$, $m > 1$, *be a paraquaternionic Kähler manifold. Then the paraquaternionic sectional curvature is constant* c *at a point* $p \in M$ *if and only if the curvature tensor* $R = (c/4)F_0$ *at* p.

Proof. First, suppose that the paraquaternionic sectional curvature is constant c at p, and consider the paraquaternionic curvature-like function $F = R - (c/4)F_0$. Then $R(X, J_\tau X, X, J_\tau X) = -\varepsilon_\tau c g(X,X)^2$, and it immediately follows from the definition of F_0 that $F_0(X, J_\tau X, X, J_\tau X) = -4\varepsilon_\tau g(X,X)^2$. Therefore, $F(X, J_\tau X, X, J_\tau X) = 0$ for all $X \in T_pM$, $\tau = 1, 2, 3$, and it follows from Lemma 3.1 that $R = (c/4)F_0$. Conversely, if $R = (c/4)F_0$ at p, a straightforward calculation shows that the sectional curvature of any nondegenerate plane in $\mathbf{V}(X)$ equals c for any $X \in T_pM$, which shows that the paraquaternionic sectional curvature is constant at p. $\square$

Remark 3.1. The model spaces of nonzero constant paraquaternionic sectional curvature are constructed by Blažić [**3**]. For the sake of completeness, we will reexamine their construction. Let $\tilde{\mathbf{H}}$ denote the algebra of paraquaternionic numbers generated by $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$ over $\mathbf{R}$. Let $\tilde{\mathbf{H}}_0 = \{\mathfrak{q} \in \tilde{\mathbf{H}} / \|\mathfrak{q}\| \neq 0\}$, where the norm of a paraquaternionic number is defined to be $\|\mathfrak{q}\|^2 = \mathfrak{q}\bar{\mathfrak{q}}$ and the conjugate of a paraquaternion $\mathfrak{q} = x_0 + x_1\mathfrak{i} + x_2\mathfrak{j} + x_3\mathfrak{k}$ is given by $\bar{\mathfrak{q}} = x_0 - x_1\mathfrak{i} - x_2\mathfrak{j} - x_3\mathfrak{k}$.

Now, the paraquaternionic projective space $P_m(\tilde{\mathbf{H}})$ associated with $\tilde{\mathbf{H}}^{m+1}$ is defined to be the equivalence classes of nonsingular paraquaternionic lines

$$P_m(\tilde{\mathbf{H}}) = \tilde{\mathbf{H}}_0^{m+1} / \tilde{\mathbf{H}}_0.$$

In a more geometrical way, the paraquaternionic projective space is represented as the quotient space

$$P_m(\tilde{\mathbf{H}}) = S^{4m+3}_{2m+1} / S^3_1$$

where the pseudosphere S^3_1 is identified with the group of unit paraquaternions. Moreover, it is shown in [**3**] that $P_m(\tilde{\mathbf{H}})$ is complete and simply connected, thus being the model space of paraquaternionic space forms of nonzero curvature. (Since the signature of the metric of any paraquaternionic Kähler manifold is neutral, the sign of the curvature is not essential).

Note that, from the expression of the curvature tensor in Theorem 3.1, any paraquaternionic space form is a locally symmetric space. (Since the function $c = (Sc/(4m(m+2)))$ and any paraquaternionic Kähler manifold of dimension > 4 is Einstein, c is necessarily constant, provided that M is connected). Moreover, since the holonomy group at each point $p \in M$ is generated by the endomorphisms of T_pM of the form $R(X,Y), X, Y \in T_pM$, it does not leave invariant any nontrivial subspace of T_pM, which shows that paraquaternionic space forms are locally symmetric irreducible spaces, provided that the paraquaternionic sectional curvature $c \neq 0$. Now a straightforward calculation from the expression of the curvature in Theorem 3.1 and the Kähler identities (2.11) shows that the dimension of the linear isotropy group of $P_m(\tilde{\mathbf{H}})$ is given by $2m^2 + m + 3$. Moreover, since it is an irreducible symmetric space, it can be identified with the symmetric space $Sp(m+1, \mathbf{R})/Sp(1, \mathbf{R})Sp(m, \mathbf{R})$ in Berger's list (cf. [**1**]).

4. Constancy of the paraquaternionic sectional curvatures. The study of the Jacobi operators is one of the central topics in pseudo-Riemannian geometry. Although it is very difficult to determine such operators explicitly, much interesting information can be derived from the properties of the eigenvalues and eigenspaces of the Jacobi operators. Since $(M, g, \mathbf{V})$ is a pseudo-Riemannian manifold equipped with a paraquaternionic Kähler structure $\mathbf{V}$, it is interesting to investigate the existence of distinguished eigenspaces for the Jacobi operators induced by the paraquaternionic structure. This is carried out in the next theorem, which provides a criterion for the constancy of the paraquaternionic sectional curvature. Such a criterion has proved its usefulness in the characterization of paraquaternionic space forms among the so-called special Osserman manifolds in [**7**].

Theorem 4.1. *Let $(M^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic Kähler manifold. Then the following conditions are equivalent to each other*

(i) *the paraquaternionic sectional curvature is constant*,

(ii) $R(X, J_\tau X)J_\tau X \sim X$ *for all spacelike vectors* X,

(iii) $R(X, J_\tau X)J_\tau X \sim X$ *for all timelike vectors* X,

(iv) $R(U, J_\tau U)J_\tau U = 0$ *for all null vectors* U, *for some local basis $\{J_\tau, \tau = 1, 2, 3\}$ of the paraquaternionic structure, where $\sim$ means "is proportional to."*

In order to prove Theorem 4.1, we need the following lemmas.

Lemma 4.1. *Let $(M^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic Kähler manifold and $X \in T_pM$ a unit vector. If a local basis $\{J_\tau; \tau = 1, 2, 3\}$ of the paraquaternionic structure exists such that*

(i) $K(X, J_1X) = K(X, J_2X) = K(X, J_3X)$,

(ii) $R(X, J_1X, X, J_2X) = R(X, J_1X, X, J_3X) = R(X, J_2X, X, J_3X) = 0$, *then the sectional curvature is constant $(Sc/(4m(m + 2)))$ on $\mathbf{V}(X)$.*

Proof. By a direct computation from (2.11) we obtain

$$K(X, J_1X) = -R(X, J_1X, J_2X, J_3X) + \frac{Sc}{4m(m+2)},$$
$$K(X, J_2X) = -R(X, J_2X, J_3X, J_1X) + \frac{Sc}{4m(m+2)},$$
$$K(X, J_3X) = -R(X, J_3X, J_1X, J_2X) + \frac{Sc}{4m(m+2)}.$$

Adding these expressions we get

$$\sum_{r=1}^{3} K(X, J_\tau X) = -R(X, J_1X, J_2X, J_3X) - R(X, J_2X, J_3X, J_1X) - R(X, J_3X, J_1X, J_2X) + \frac{3Sc}{4m(m+2)}.$$

Now, (i), together with the first Bianchi identity, implies $K(X, J_\tau X) = (Sc/(4m(m+2)))$, $\tau = 1, 2, 3$, and hence the sectional curvature of the planes $\{X, J_\tau X\}$ is $(Sc/(4m(m+2)))$. Finally, a straightforward calculation from (ii) shows that the sectional curvature of any nondegenerate plane in $\mathbf{V}(X)$ also equals $(Sc/(4m(m + 2)))$. □

Lemma 4.2. *Let $(M^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic Kähler manifold and $X \in T_pM$ a unit vector. If $K(Y, J_\tau Y)$ is constant for all* *unit space-like vectors* $Y \in \mathbf{V}(X)$ *and some local basis* $\{J_\tau; \tau = 1, 2, 3\}$, *then the sectional curvature is constant on* $\mathbf{V}(X)$.

Proof. Let X be a unit spacelike vector at T_pM and suppose the sectional curvatures are a constant c on $\mathbf{V}(X)$. Take λ, $\mu \in \mathbf{R}$ with $\lambda^2 + \mu^2 = 1$. Then $Y = \lambda X + \mu J_3 X$ is a unit spacelike vector in $\mathbf{V}(X)$ and, since $K(Y, J_1Y) = c = -R(Y, J_1Y, Y, J_1Y)$, one has

$$\begin{aligned}
-c = {} & \lambda^4 R(X, J_1X, X, J_1X) + \mu^4 R(J_3X, J_2X, J_3X, J_2X) \\
& + \lambda^2\mu^2\{R(X, J_2X, X, J_2X) + R(J_3X, J_1X, J_3X, J_1X) \\
& \qquad + 2R(X, J_1X, J_3X, J_2X) + 2R(X, J_2X, J_3X, J_1X)\} \\
& + 2\lambda^3\mu\{R(X, J_1X, X, J_2X) + R(X, J_1X, J_3X, J_1X)\} \\
& + 2\lambda\mu^3\{R(X, J_2X, J_3X, J_2X) + R(J_3X, J_1X, J_3X, J_2X)\}.
\end{aligned}$$

After some calculations, we get

$$\begin{aligned}
-c = {} & -(\lambda^2 - \mu^2)^2 K(X, J_1X) - 4\lambda^2\mu^2 K(X, J_2X) \\
& + 4\lambda\mu(\lambda^2 - \mu^2) R(X, J_1X, X, J_2X),
\end{aligned}$$

and, moreover, that

$$R(X, J_1X, X, J_2X) = 0 \tag{4.1}$$

and

$$K(X, J_2X) = K(X, J_1X).$$

We then take $\lambda, \mu \in \mathbf{R}$ with $\lambda^2 - \mu^2 = 1$. Then $Y = \lambda X + \mu J_2 X$ is a unit space-like vector in $\mathbf{V}(X)$, and therefore $K(Y, J_1Y) = c$. Proceeding as before, after some calculations,

$$\begin{aligned}
-a = {} & -(\lambda^2 + \mu^2)^2 K(X, J_1X) + 4\lambda^2\mu^2 K(X, J_3X) \\
& + 4\lambda\mu(\lambda^2 + \mu^2) R(X, J_1X, X, J_3X),
\end{aligned}$$

which shows that

$$R(X, J_1X, X, J_3X) = 0 \tag{4.2}$$

and

$$K(X, J_3X) = K(X, J_1X).$$

Finally, take $\lambda, \mu, \gamma \in \mathbf{R}$ such that $\lambda^2 + \mu^2 - \gamma^2 = 1$. Then $Y = \lambda X + \mu J_3X + \gamma J_2X$ is a unit spacelike vector in $\mathbf{V}(X)$, and thus $K(Y, J_1Y) = c$. Now, after a straightforward calculation, since the coefficient of $\lambda^2\mu\gamma$ must vanish, it follows that

$$R(X, J_2X, X, J_3X) = 0. \tag{4.3}$$

Then the constancy of the sectional curvature on $\mathbf{V}(X)$ follows from Lemma 4.1, together with (4.1)–(4.3). □

Lemma 4.3. *Let $(M^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic Kähler manifold. If $R(X, J_\tau X, J_\tau X, Y) = 0$ for some basis $\{J_\tau, \tau = 1, 2, 3\}$ of the paraquaternionic structure and all orthonormal vectors $X, Y \in T_pM$ with $g(X, X) = -g(Y, Y)$ and such that $\mathbf{V}(X) \perp \mathbf{V}(Y)$, then the paraquaternionic sectional curvature is constant.*

Proof. Let X and Y be orthonormal vectors with $g(X, X) = -g(Y, Y)$ such that $\mathbf{V}(X) \perp \mathbf{V}(Y)$, and take $\lambda, \mu \in \mathbf{R}$ satisfying $\lambda^2 - \mu^2 = 1$. Then $Z = \lambda X + \mu Y$ and $W = \mu X + \lambda Y$ are orthonormal vectors in T_pM with $g(Z, Z) = -g(W, W)$ and such that $\mathbf{V}(Z) \perp \mathbf{V}(W)$. By hypothesis, $R(Z, J_\tau Z, J_\tau Z, W) = 0$, linearizing this expression

$$\begin{aligned}
0 = \lambda^2\{&R(X, J_\tau X, J_\tau X, X) + R(Y, J_\tau X, J_\tau X, Y) \\
&+ R(X, J_\tau X, J_\tau Y, Y) + R(X, J_\tau Y, J_\tau X, Y)\} \\
+ \mu^2\{&R(X, J_\tau Y, J_\tau Y, X) + R(Y, J_\tau Y, J_\tau Y, Y) \\
&+ R(X, J_\tau X, J_\tau Y, Y) + R(X, J_\tau Y, J_\tau X, Y)\}.
\end{aligned}$$

Since the coefficients of λ^2 and μ^2 must vanish, we get

$$\begin{aligned}
R(X, J_\tau X, &J_\tau X, X) + R(Y, J_\tau X, J_\tau X, Y) \\
&= R(Y, J_\tau Y, J_\tau Y, Y) + R(X, J_\tau Y, J_\tau Y, X).
\end{aligned}$$

Then, using that $R(Y, J_\tau X, J_\tau X, Y) = R(X, J_\tau Y, J_\tau Y, X)$ for all $\tau = 1, 2, 3$, it follows that $R(X, J_\tau X, X, J_\tau X) = R(Y, J_\tau Y, Y, J_\tau Y)$, and thus

$$K(X, J_\tau X) = K(Y, J_\tau Y), \quad \tau = 1, 2, 3. \tag{4.4}$$

As a final stage of the proof, take a unit vector X and choose Z to be a unit time-like vector, $Z \in \mathbf{V}(X)^{\perp}$. If Y is a unit space-like vector in $\mathbf{V}(X)$, then Y and Z are orthonormal, with $g(Y,Y) = -g(Z,Z)$ and $\mathbf{V}(Y) \perp \mathbf{V}(Z)$. Thus, (4.4) implies $K(Y, J_\tau Y) = K(Z, J_\tau Z)$, $\tau = 1,2,3$. Therefore, $K(Y, J_\tau Y)$ is constant for all unit space-like vectors in $\mathbf{V}(X)$. Then the sectional curvature is constant on $\mathbf{V}(X)$ by Lemma 4.2 with value $(Sc/(4m(m+2)))$. □

Now we are ready to give the announced

Proof of Theorem 4.1. If the paraquaternionic sectional curvature is constant, it follows from Theorem 3.1 that $R(X, J_\tau X)J_\tau X = \varepsilon_\tau c g(X,X)X$ for all $\tau = 1,2,3$, and therefore (ii), (iii) and (iv) hold.

Next suppose that (ii) holds, and take X, Y to be orthonormal vectors, with $g(X,X) = -g(Y,Y)$ and such that $\mathbf{V}(X) \perp \mathbf{V}(Y)$. If X is space-like, then (ii) implies that $R(X, J_\tau X, J_\tau X, Y) = 0$. If X is time-like, then $J_1 X$ is space-like, and also from (ii) we get $R(J_1 X, J_\tau J_1 X, J_\tau J_1 X, J_1 Y) = 0$. Thus $R(X, J_\tau X, J_\tau X, Y) = 0$, and (i) follows from Lemma 4.3.

If (iii) holds, take X to be a space-like vector and Y orthogonal to X. Since X is space-like and $J_1 X$ is time-like, (iii) implies that $R(J_1 X, J_\tau J_1 X, J_\tau J_1 X, J_1 Y) = 0$, and therefore $R(X, J_\tau X, J_\tau X, Y) = 0$.

Suppose then that (iv) holds and take X, Y to be orthonormal vectors with $g(X,X) = -g(Y,Y)$ and $\mathbf{V}(X) \perp \mathbf{V}(Y)$. Then $X \pm Y$ are null vectors, and from (iv) we have

$$\begin{aligned} &R(X+Y, J_\tau X + J_\tau Y, J_\tau X + J_\tau Y, X - Y) \\ &\qquad - R(X-Y, J_\tau X - J_\tau Y, J_\tau X - J_\tau Y, X + Y) = 0, \end{aligned}$$

and thus

$$R(X, J_\tau X, J_\tau X, Y) - R(Y, J_\tau Y, J_\tau Y, X) = 0. \tag{4.5}$$

Similarly, from (iv) we have

$$\begin{aligned} &R(X+Y, J_\tau X + J_\tau Y, J_\tau X + J_\tau Y, X + Y) \\ &\qquad - R(X-Y, J_\tau X - J_\tau Y, J_\tau X - J_\tau Y, X - Y) = 0, \end{aligned}$$

and thus

$$R(X, J_\tau X, J_\tau X, Y) + R(Y, J_\tau Y, J_\tau Y, X) = 0. \tag{4.6}$$

Now (4.5) together with (4.6) implies $R(X, J_\tau X, J_\tau X, Y) = 0$, and the proof follows from Lemma 4.3. □

Let $X, Y \in T_pM$. If the $\mathbf{V}$-planes $\mathbf{V}(X)$ and $\mathbf{V}(Y)$ are orthogonal, we can say that $\{X, Y\}$ spans an *anti-paraquaternionic plane*.

Remark 4.1. For each null vector U, orthonormal space-like and time-like vectors X and Y exist such that $U = \alpha(X + Y)$, and moreover, X and Y may be chosen such that $\langle\{X, J_3X\}\rangle \perp \langle\{Y, J_3Y\}\rangle$. However, $\{X, Y\}$ does not necessarily span an anti-paraquaternionic plane.

Also note that (iv) in Theorem 4.1 remains valid if it is only checked for null vectors of the form $U = \alpha(X + Y)$, where $\{X, Y\}$ spans an anti-paraquaternionic plane.

5. Degenerate paraquaternionic sections. As well as for the sectional curvature, the definition of the paraquaternionic sectional curvature makes sense only for nondegenerate paraquaternionic sections. This fact has an important consequence; the paraquaternionic sectional curvature is not necessarily bounded at each point $p \in M$. As we will show at the end of this section, such boundedness may occur only when $(M, g, \mathbf{V})$ is a paraquaternionic space form. This fact is strongly related with the possibility of continuously extending the definition of the paraquaternionic sectional curvature to degenerate $\mathbf{V}$-sections. We recall that $\mathbf{V}(X) = \langle\{X, J_1X, J_2X, J_3X\}\rangle$ associated with a vector X is degenerate if and only if X is null. Moreover, the restriction of the metric g to degenerate $\mathbf{V}$-sections has signature $(0, 0, 0, 0)$, i.e., totally degenerate. Therefore, a necessary condition for the desired extension of the paraquaternionic sectional curvature is

$$R(X, Y, Z, W) = 0 \tag{5.1}$$

for all vectors $X, Y, Z, W \in \mathbf{V}(U)$, and for all null vectors U.

Note that the above condition implies

$$R(U, J_\tau U, J_\tau U, U) = 0, \quad \tau = 1, 2, 3 \tag{5.2}$$

for all null vectors U.

We start with the consideration of this last condition and its relation with the paraquaternionic sectional curvature. First of all, note that if (5.2) holds, then $R(U, J_\tau U)J_\tau U \in \langle U\rangle^\perp$. If $V \in \langle U\rangle^\perp$ is a null vector, it follows that $U + \lambda V$ is null for all real numbers λ, and hence $R(U + \lambda V, J_\tau(U + \lambda V), J_\tau(U + \lambda V), U + \lambda V) = 0$. Linearizing this expression and considering the coefficient corresponding to λ, we get $g(R(U, J_\tau U)J_\tau U, V) = 0$. Hence, $R(U, J_\tau U)J_\tau U$ lies in the direction of $\langle U\rangle$ and therefore (5.2) is equivalent to

$$R(U, J_\tau U)J_\tau U = c_U^\tau U, \quad \tau = 1, 2, 3. \tag{5.3}$$

According to Theorem 4.1, the eigenvalue c_U measures the failure of a paraquaternionic Kähler manifold satisfying (5.2) to be of constant paraquaternionic sectional curvature.

Lemma 5.1. *Let* $(M^{4m}, g, \mathbf{V})$*,* $m > 1$*, be a paraquaternionic Kähler manifold satisfying* (5.2)*. If* X *and* Y *are orthogonal vectors with* $g(X, X) = -g(Y, Y)$*, then*

(a) $R(X, J_\tau X, J_\tau X, Y) = -R(Y, J_\tau Y, J_\tau Y, X)$,

(b) $R(X, J_\tau X, J_\tau X, X) + R(Y, J_\tau Y, J_\tau Y, Y) = 2R(X, J_\tau X, Y, J_\tau Y) + 2R(X, J_\tau Y, Y, J_\tau X) + R(X, J_\tau Y, X, J_\tau Y) + R(Y, J_\tau X, Y, J_\tau X)$,

for some local basis $\{J_\tau; \tau = 1, 2, 3\}$*.*

Proof. Since X and Y are orthogonal unit vectors with $g(X, X) = -g(Y, Y)$, $U = X \pm Y$ are null vectors. Now the result follows from (5.2) after linearization. □

Lemma 5.2. *Let* $(M^{4m}, g, \mathbf{V})$*,* $m > 1$*, be a paraquaternionic Kähler manifold satisfying* (5.2)*. If* $U = \alpha(X + Y)$ *is a null vector and* $\mathbf{V}(X) \perp \mathbf{V}(Y)$*, then*

(5.4)

$$c_U^1 = 4\varepsilon_Z R(U, Z, U, Z) + \frac{3Sc}{4m(m+2)}\varepsilon_Z\{g(Z, J_2U)^2 - g(Z, J_3U)^2\}$$

for all unit vectors $Z \in \langle\{X, Y, J_1X, J_1Y\}\rangle^\perp$,

(5.5)

$$c_U^2 = 4\varepsilon_Z R(U, Z, U, Z) + \frac{3Sc}{4m(m+2)}\varepsilon_Z\{g(Z, J_1U)^2 - g(Z, J_3U)^2\}$$

for all unit vectors $Z \in \langle\{X, Y, J_2X, J_2Y\}\rangle^\perp$,

(5.6)

$$c_U^3 = -4\varepsilon_Z R(U, Z, U, Z) - \frac{3Sc}{4m(m+2)}\varepsilon_Z\{g(Z, J_1U)^2 + g(Z, J_2U)^2\}$$

for all unit vectors $Z \in \langle\{X, Y, J_3X, J_3Y\}\rangle^\perp$.

Proof. Suppose X is unit space-like, and consider the null vector $V = (1/(4\alpha))(Y - X)$. If Z is a unit vector in $\langle\{X, Y, J_1X, J_1Y\}\rangle^\perp$, for each $t > 0$, $w_t = (1/\sqrt{t})(U + t\varepsilon_Z V)$ and Z are orthogonal with $g(w_t, w_t) = -g(Z, Z)$. Therefore, Lemma 5.1(b) for $\tau = 1$ implies

$$\begin{aligned} R(Z, J_1Z, Z, J_1Z) &+ R(w_t, J_1w_t, J_1w_t, w_t) \\ &= 2R(Z, J_1Z, w_t, J_1w_t) + 2R(Z, J_1w_t, w_t, J_1Z) \\ &\quad + R(Z, J_1w_t, Z, J_1w_t) + R(w_t, J_1Z, w_t, J_1Z) \end{aligned}$$

and hence

$$\begin{aligned} tR(Z, J_1Z, Z, J_1Z) &+ \frac{1}{t}R(U + t\varepsilon_Z V, J_1(U + t\varepsilon_Z V), \\ &J_1(U + t\varepsilon_Z V), U + t\varepsilon_Z V) \\ &= 2R(Z, J_1Z, U + t\varepsilon_Z V, J_1(U + t\varepsilon_Z V)) \\ &\quad + 2R(Z, J_1(U + t\varepsilon_Z V), U + t\varepsilon_Z V, J_1Z) \\ &\quad + R(Z, J_1(U + t\varepsilon_Z V), Z, J_1(U + t\varepsilon_Z V)) \\ &\quad + R(U + t\varepsilon_Z V, J_1Z, U + t\varepsilon_Z V, J_1Z). \end{aligned}$$

Linearizing this expression and taking limits as $t \to 0$, we get

$$\begin{aligned} -4\varepsilon_Z R(U, J_1U, U, J_1V) &= 2R(Z, J_1Z, U, J_1U) \\ &\quad + 2R(Z, J_1U, U, J_1Z) \\ &\quad + R(Z, J_1U, Z, J_1U) \\ &\quad + R(U, J_1Z, U, J_1Z). \end{aligned}$$

Now from (2.11) the expression above reduces to

$$\begin{aligned} c_U^1 &= -\varepsilon_Z R(Z, J_1Z, U, J_1U) - 2\varepsilon_Z R(Z, J_1U, U, J_1Z) \\ &\quad - \frac{Sc}{4m(m+2)}\varepsilon_Z\{g(Z, J_2U)^2 - g(Z, J_3U)^2\}, \end{aligned}$$

and by the first Bianchi identity

$$\begin{aligned} c_U^1 &= \varepsilon_Z R(U, Z, J_1Z, J_1U) - 3\varepsilon_Z R(Z, J_1U, U, J_1Z) \\ &\quad - \frac{Sc}{4m(m+2)}\varepsilon_Z\{g(Z, J_2U)^2 - g(Z, J_3U)^2\}. \end{aligned} \tag{5.7}$$

Note at this point that if Z is a unit vector in $\langle\{X, Y, J_1X, J_1Y\}\rangle^\perp$, then so is J_1Z, and hence from (5.7),

$$\begin{aligned} c_U^1 &= 3\varepsilon_Z R(U, Z, J_1Z, J_1U) - \varepsilon_Z R(Z, J_1U, U, J_1Z) \\ &\quad - \frac{Sc}{4m(m+2)}\varepsilon_Z\{g(Z, J_2U)^2 - g(Z, J_3U)^2\}. \end{aligned} \tag{5.8}$$

Now from (5.7) and (5.8) we get $R(Z, J_1U, U, J_1Z) = -R(U, Z, J_1Z, J_1U)$ and thus (5.4) is obtained from (5.7).

The remaining identities (5.5) and (5.6) are derived in an analogous way. □

The previous lemma has two important corollaries:

Corollary 5.1. *Let $(M^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic Kähler manifold satisfying* (5.2). *Then*

$$c_U^1 = c_U^2 = -c_U^3$$

for all null vectors U of the form $U = \alpha(X + Y)$, where $\{X, Y\}$ spans an anti-paraquaternionic plane.

Proof. Let $U = \alpha(X + Y)$ be a null vector where $\{X, Y\}$ spans an anti-paraquaternionic plane. Since $Z = J_3X$ is a unit vector in $\langle\{X, Y, J_1X, J_1Y\}\rangle^\perp \cap \langle\{X, Y, J_2X, J_2Y\}\rangle^\perp$, from Lemma 5.2 we get

$$c_U^1 = 4\varepsilon_Z R(U, Z, U, Z) + \frac{3Sc}{4m(m+2)}\varepsilon_Z\{g(Z, J_2U)^2 - g(Z, J_3U)^2\}$$

and

$$c_U^2 = 4\varepsilon_Z R(U,Z,U,Z) + \frac{3Sc}{4m(m+2)}\varepsilon_Z\{g(Z,J_1U)^2 - g(Z,J_3U)^2\}.$$

Now, since $Z \in \langle\{J_1U, J_2U\}\rangle^\perp$, it follows that $c_U^1 = c_U^2$. The proof of $c_U^2 = -c_U^3$ is similar. □

Corollary 5.2. *Let $(M^{4m}, g, \mathbf{V})$, $m > 2$, be a paraquaternionic Kähler manifold satisfying* (5.2)*. If U is a null vector of the form $U = \alpha(X+Y)$, where $\{X,Y\}$ spans an anti-paraquaternionic plane, then*

$$c_U^\tau = 4\varepsilon_\tau\varepsilon_Z R(U,Z,U,Z), \quad \tau = 1,2,3,$$

Z being a unit vector such that $\mathbf{V}(X) \perp Z \perp \mathbf{V}(Y)$.

Proof. Note that $Z \in \langle\{X, Y, J_\tau X, J_\tau Y\}\rangle^\perp$ for all $\tau = 1,2,3$. Hence, the result follows from Lemma 5.2. □

Next we will relate the distinguished eigenvalue c_U with the Ricci tensor, which is a basic observation in the proof of Theorem 5.1.

Proposition 5.1. *Let $(M^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic Kähler manifold satisfying* (5.2)*. Then*

$$\rho(U,U) = \varepsilon_\tau(m+4)c_U^\tau, \quad \tau = 1,2,3, \tag{5.9}$$

for all null vectors U of the form $U = \alpha(X+Y)$, where $\{X,Y\}$ spans an anti-paraquaternionic plane.

Proof. Let $\{Z_1, Z_2, \dots, Z_{4m-8}\}$ be an orthonormal basis of $(\mathbf{V}(X) \oplus \mathbf{V}(Y))^\perp$. Then, if we assume X to be a unit space-like,

$$\begin{aligned} \rho(U,U) &= R(U,X,U,X) - R(U,Y,U,Y) \\ &\quad - \sum_{\tau=1}^{3}\varepsilon_\tau\{R(U,J_\tau X,U,J_\tau X) - R(U,J_\tau Y,U,J_\tau Y)\} \\ &\quad + \sum_{i=1}^{4m-8}\varepsilon_{Z_i} R(U,Z_i,U,Z_i), \end{aligned} \tag{5.10}$$

where $\varepsilon_{Z_i} = g(Z_i, Z_i)$, $i = 1, \dots, 4m-8$.

Now, since $U = \alpha(X+Y)$, one easily obtains $R(U,X,U,X) - R(U,Y,U,Y) = 0$ and $R(U, J_\tau X, U, J_\tau X) - R(U, J_\tau Y, U, J_\tau Y) = -2c_U^\tau$. Therefore, (5.10) becomes

$$\rho(U,U) = 2c_U^1 + 2c_U^2 - 2c_U^3 + \sum_{i=1}^{4m-8} \varepsilon_{Z_i} R(U, Z_i, U, Z_i),$$

and the desired result is obtained from Corollaries 5.1 and 5.2. □

Next we state the main result of this section.

Theorem 5.1. *Let $(M^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic Kähler manifold. Then* (5.2) *is equivalent to the constancy of the paraquaternionic sectional curvature.*

Proof. Since any paraquaternionic Kähler manifold of $\dim M > 4$ is Einstein (cf. Theorem 2.1), it follows from (5.9) that $c_u^\tau = 0$, $\tau = 1,2,3$, and for all null vectors U of the form $U = \alpha(X+Y)$, where $\{X, Y\}$ spans an anti-paraquaternionic plane. Hence, (5.3) implies that $R(U, J_\tau U)J_\tau U = 0$, $\tau = 1,2,3$, and for such null vectors U. Now the result follows from (iv) in Theorem 4.1 and Remark 4.1. □

It is necessary to immediately recognize from Theorem 3.1 that (5.1) is satisfied by any paraquaternionic space form. Therefore, $(M^{4m}, g, \mathbf{V})$, $m > 1$, is a *paraquaternionic space form if and only if the restriction of the curvature tensor to any degenerate paraquaternionic section vanishes identically.* (This result should be contrasted with a corresponding one for indefinite Kähler manifolds, which shows that the restriction of the curvature tensor to degenerate holomorphic sections vanishes if and only if the manifold is null holomorphically flat, a strictly weaker condition than constant holomorphic sectional curvature [**6**]).

Remark 5.1. The above result is no longer true if we are not in the category of paraquaternionic Kähler manifolds. Indeed, it is easy to show that the tangent bundle TM of any paraquaternionic

Kähler manifold $(M, g, \mathbf{V})$ inherits in a natural way a paraquaternionic structure $(g^H, \mathbf{V}^H)$, where g^H and $\mathbf{V}^H$ are the horizontal lifts of the metric and the paraquaternionic structure of M, respectively. (Horizontal lifts are taken with respect to the Levi-Civita connection of (M, g)). Moreover, after some calculations as in [**8**], it follows that the curvature tensor of $(TM, g^H, \mathbf{V}^H)$ satisfies (5.2) if and only if $(M, g, \mathbf{V})$ is a paraquaternionic space form. However, the paraquaternionic sectional curvature of $(TM, g^H, \mathbf{V}^H)$ is not constant, unless M is flat.

Remark 5.2. If $\dim M = 4$, (5.2) is equivalent to the vanishing of the Weyl tensor, and therefore equivalent to locally conformally flatness (cf. [**16**]).

Finally, as an application of Theorem 5.1, we will prove the announced result on the boundedness of the paraquaternionic sectional curvature.

Theorem 5.2. *Let $(M^{4m}, g, \mathbf{V})$, $m > 1$, be a paraquaternionic Kähler manifold. If the paraquaternionic sectional curvature is bounded from below and from above, then it is constant.*

Proof. Consider a null vector $U \in T_pM$ and approximate it by a sequence $\{Z_n\}_{n\in\mathbf{N}}$ of nonnull vectors, for instance, $Z_n = U + (Z/n)$, Z being a nonnull vector orthogonal to U. Since Z_n is nonnull and the paraquaternionic sectional curvature is assumed to be bounded,

$$|K(Z_n, J_\tau Z_n)| \leq A, \quad \tau = 1, 2, 3,$$

for some constant A. Hence

$$|R(Z_n, J_\tau Z_n, Z_n, J_\tau Z_n)| \leq A|g(Z_n, Z_n)|, \quad \tau = 1, 2, 3$$

and, taking limits when $n \to \infty$, one gets

$$R(U, J_\tau U, U, J_\tau U) = 0, \quad \tau = 1, 2, 3$$

for all null vectors U. Now the result follows from Theorem 5.1. $\square$

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