ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 2001

EQUAL SUMS OF LIKE POWERS

AJAI CHOUDHRY

ABSTRACT. This paper is concerned with four diophantine systems, namely, (i) $\sum_{i=1}^4 x_i^k = \sum_{i=1}^4 y_i^k$, $k = 1, 2, 4$; (ii) $\sum_{i=1}^4 x_i^k = \sum_{i=1}^4 y_i^k$, $k = 1, 3, 4$; (iii) $\sum_{i=1}^4 x_i^k = \sum_{i=1}^4 y_i^k$, $k = 2, 3, 4$; (iv) $\sum_{i=1}^3 x_i^k = \sum_{i=1}^3 y_i^k$, $k = 2, 3, 4$. Parametric solutions as well as numerical examples of solutions in positive integers of the first three diophantine systems have been obtained in the paper. For the fourth diophantine system, solutions do not exist in positive real numbers and a single numerical solution in integers has been obtained.

This paper is concerned with the following four diophantine systems relating to the problem of equal sums of like powers:

$$\begin{array}{lll} \text{I.} & x_1^k + x_2^k + x_3^k + x_4^k = y_1^k + y_2^k + y_3^k + y_4^k, & k = 1, 2, 4. \\ \text{II.} & x_1^k + x_2^k + x_3^k + x_4^k = y_1^k + y_2^k + y_3^k + y_4^k, & k = 1, 3, 4. \\ \text{III.} & x_1^k + x_2^k + x_3^k + x_4^k = y_1^k + y_2^k + y_3^k + y_4^k, & k = 2, 3, 4. \\ \text{IV.} & x_1^k + x_2^k + x_3^k = y_1^k + y_2^k + y_3^k, & k = 2, 3, 4. \end{array}$$

Solutions of these diophantine systems have not been published before. We will obtain two parametric solutions of the diophantine system I, one parametric solution of system II and one parametric solution of system III. We also give an additional method of generating infinitely many integer solutions of system III. As numerical examples we will obtain solutions in positive integers of these three diophantine systems. Finally, we obtain the following numerical solution of system IV:

$$\begin{aligned} 358^2 + (-815)^2 + 1224^2 &= (-410)^2 + (-776)^2 + 1233^2, \\ 358^3 + (-815)^3 + 1224^3 &= (-410)^3 + (-776)^3 + 1233^3, \\ 358^4 + (-815)^4 + 1224^4 &= (-410)^4 + (-776)^4 + 1233^4. \end{aligned} \tag{1}$$

This solution is particularly interesting since it has been proved earlier by Palama [**4**] that the diophantine system IV has no solutions in

Received by the editors on April 5, 1999, and in revised form on September 20, 1999.

positive real numbers. Further, according to a well-known theorem of Bastien (as quoted by Dickson [**1**, p. 712]), the set of simultaneous equations

$$x_1^k + x_2^k + \cdots + x_n^k = y_1^k + y_2^k + \cdots + y_n^k \tag{2}$$

where $k = 1, 2, \ldots, n$ has no nontrivial solutions. It has also been shown [**2**] that the diophantine system

$$\begin{aligned} x_1^2 + x_3^2 &= y_1^2 + y_2^2 \\ x_1^3 + x_2^3 &= y_1^3 + y_2^3 \end{aligned} \tag{3}$$

has no nontrivial integer solutions. Thus, (1) provides the first example of a solution of the equation (2) with the equality holding for n consecutive values of the exponent k.

As solutions of the system of equations

$$x_1^k + x_2^k + x_3^k = y_1^k + y_2^k + y_3^k, \quad k = 1, 2, 4 \tag{4}$$

have been known for a long time, diophantine system I given above may not seem to be of much interest. However, in contrast to diophantine system (4), solutions of system I can be obtained in positive integers. Moreover, it is useful to study system I since a solution of this system will be used to obtain solutions of system III which, in turn, would lead to the numerical solution of system IV already given above.

2. We now consider diophantine system I given by the following three equations:

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4, \tag{5}$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2, \tag{6}$$
$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = y_1^4 + y_2^4 + y_3^4 + y_4^4. \tag{7}$$

We will obtain a parametric solution of this system of equations by substituting

$$\begin{aligned} x_i &= (-a_i + b_i)u + \alpha_i v, \quad i = 1, 2, 3, 4, \\ y_i &= (a_i + b_i)u + \alpha_i v, \quad i = 1, 2, 3, 4, \end{aligned} \tag{8}$$

in each of the equations (5), (6) and (7). We will choose a_i, b_i, $i = 1, 2, 3, 4$, such that the equations (5) and (6) hold identically for all values of u and v. In the equation obtained from (7) by substituting the values of x_i, y_i, $i = 1, 2, 3, 4$, given above, the coefficient of v^4 cancels out on both sides and we will choose a_i, b_i, $i = 1, 2, 3, 4$, such that the coefficients of uv^3 and u^2v^2 also cancel out. Thus, equation (7) would reduce to

$$\left[\sum_{i=1}^{4}\{(-a_i + b_i)^4 - (a_i + b_i)^4\}\right]u^4 + 4\left[\sum_{i=1}^{4}\{(-a_i + b_i)^3 - (a_i + b_i)^3\}\alpha_i\right]u^3v = 0,$$

which, on simplifying and ignoring the factor u^3, leads to the following solution for u and v:

$$u = \sum_{i=1}^{4}(a_i^2 + 3b_i^2)a_i\alpha_i, \qquad v = -\sum_{i=1}^{4}(a_i^2 + b_i^2)a_ib_i. \tag{9}$$

We must now choose a_i, b_i, $i = 1, 2, 3, 4$, so as to satisfy the conditions mentioned above. Equations (5) and (6) will hold identically if the a_i, b_i, $i = 1, 2, 3, 4$, satisfy the following conditions:

$$a_1 + a_2 + a_3 + a_4 = 0, \tag{10}$$

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0, \tag{11}$$

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = 0. \tag{12}$$

Further, equating the coefficients of uv^3 and u^2v^2 on both sides in the equation obtained from (7), we get the conditions:

$$a_1\alpha_1^3 + a_2\alpha_2^3 + a_3\alpha_3^3 + a_4\alpha_4^3 = 0, \tag{13}$$

and

$$a_1b_1\alpha_1^2 + a_2b_2\alpha_2^2 + a_3b_3\alpha_3^2 + a_4b_4\alpha_4^2 = 0. \tag{14}$$

We now have to solve equations (10), (11), (12), (13) and (14) for $a_i, b_i,\ i = 1, 2, 3, 4$. The equations (10), (11) and (13) are linear equations in the variables a_i, $i = 1, 2, 3, 4$, and we readily obtain the following solution:

$$\begin{aligned} a_1 &= -(\alpha_2-\alpha_3)\,(\alpha_3-\alpha_4)\,(\alpha_4-\alpha_2)\,(\alpha_2+\alpha_3+\alpha_4)\beta_0, \\ a_2 &= (\alpha_3-\alpha_4)\,(\alpha_4-\alpha_1)\,(\alpha_1-\alpha_3)\,(\alpha_3+\alpha_4+\alpha_1)\beta_0, \\ a_3 &= -(\alpha_4-\alpha_1)\,(\alpha_1-\alpha_2)\,(\alpha_2-\alpha_4)\,(\alpha_4+\alpha_2+\alpha_1)\beta_0, \\ a_4 &= (\alpha_1-\alpha_2)\,(\alpha_2-\alpha_3)\,(\alpha_3-\alpha_1)\,(\alpha_1+\alpha_2+\alpha_3)\beta_0, \end{aligned} \tag{15}$$

where β_0 is an arbitrary parameter. With the values of a_i, $i = 1, 2, 3, 4$, already known, equations (12) and (14) are just two linear equations in the four variables b_i, $i = 1, 2, 3, 4$. We impose the following auxiliary condition on b_i, $i = 1, 2, 3, 4$, given by

$$b_1\beta_1 + b_2\beta_2 + b_3\beta_3 + b_4\beta_4 = 0, \tag{16}$$

where $\beta_1, \beta_2, \beta_3$ and β_4 are arbitrary. Solving the linear equations (12), (14) and (16) for b_i, $i = 1, 2, 3, 4$, we get

$$\begin{aligned} b_1 &= (\alpha_3^2-\alpha_4^2)\beta_2 a_3 a_4 + (\alpha_4^2-\alpha_2^2)\beta_3 a_4 a_2 + (\alpha_2^2-\alpha_3^2)\beta_4 a_2 a_3, \\ b_2 &= -[(\alpha_4^2-\alpha_1^2)\beta_3 a_4 a_1 + (\alpha_1^2-\alpha_3^2)\beta_4 a_1 a_3 + (\alpha_3^2-\alpha_4^2)\beta_1 a_3 a_4], \\ b_3 &= (\alpha_1^2-\alpha_2^2)\beta_4 a_1 a_2 + (\alpha_2^2-\alpha_4^2)\beta_1 a_2 a_4 + (\alpha_4^2-\alpha_1^2)\beta_2 a_4 a_1, \\ b_4 &= -[(\alpha_2^2-\alpha_3^2)\beta_1 a_2 a_3 + (\alpha_3^2-\alpha_1^2)\beta_2 a_3 a_1 + (\alpha_1^2-\alpha_2^2)\beta_3 a_1 a_2]. \end{aligned} \tag{17}$$

Thus, when $a_i, b_i, i = 1, 2, 3, 4$, are given by (15) and (17), the equations (10), (11), (12), (13) and (14) are satisfied. It follows that a solution of the diophantine system I consisting of equations (5), (6) and (7) is given by (8) where the $a_i, b_i,\ i = 1, 2, 3, 4$, are defined by (15) and (17) in terms of the arbitrary parameters $\alpha_i,\ i = 1, 2, 3, 4$, and β_j, $j = 0, 1, 2, 3, 4$, while u and v are defined by (9).

As a numerical example, taking $\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3,\ \alpha_4 = 4$, $\beta_0 = 1,\ \beta_1 = 1,\ \beta_2 = 0,\ \beta_3 = 0,\ \beta_4 = 0$, we get, after removal of common factors and suitable rearrangement, the following solution of the diophantine system I:

$$2370^k + 2447^k + 2515^k + 2563^k = 2375^k + 2433^k + 2542^k + 2545^k, \quad k = 1, 2, 4.$$

3. We will now obtain a second parametric solution of diophantine system I. We write

$$\begin{aligned}
x_1 &= \alpha_1 m - \{(\alpha_2+\alpha_3)p - q\}n,\\
x_2 &= \alpha_2 m + \{(\alpha_1-\alpha_3)p + q\}n,\\
x_3 &= \alpha_3 m + \{(\alpha_1+\alpha_2)p + q\}n,\\
x_4 &= -(\alpha_1+\alpha_2+\alpha_3)m + qn,\\
y_1 &= -\alpha_1 m - \{(\alpha_2+\alpha_3)p - q\}n,\\
y_2 &= -\alpha_2 m + \{(\alpha_1-\alpha_3)p + q\}n,\\
y_3 &= -\alpha_3 m + \{(\alpha_1+\alpha_2)p + q\}n,\\
y_4 &= (\alpha_1+\alpha_2+\alpha_3)m + qn.
\end{aligned} \tag{18}$$

With these values of x_i, y_i, $i = 1, 2, 3, 4$, it is readily seen that equations (5) and (6) are identically satisfied while

$$\sum_{i=1}^{4}(x_i^4 - y_i^4) = 8mn(-m+np)(m+np)(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)\cdot\{(\alpha_1-\alpha_3)(\alpha_1-\alpha_2+\alpha_3)p + 3q(\alpha_1+\alpha_3)\}.$$

Thus, equation (7) will also be satisfied if we take

$$p = -3(\alpha_1+\alpha_3), \qquad q = (\alpha_1-\alpha_3)(\alpha_1-\alpha_2+\alpha_3). \tag{19}$$

This gives us, on substituting these values of p and q in (18), the following solution of diophantine system I:

$$\begin{aligned}
x_1 &= \alpha_1 m + (\alpha_1+2\alpha_3)(\alpha_1+2\alpha_2+\alpha_3)n,\\
x_2 &= \alpha_2 m - (\alpha_1-\alpha_3)(2\alpha_1+\alpha_2+2\alpha_3)n,\\
x_3 &= \alpha_3 m - (2\alpha_1+\alpha_3)(\alpha_1+2\alpha_2+\alpha_3)n,\\
x_4 &= -(\alpha_1+\alpha_2+\alpha_3)m + (\alpha_1-\alpha_3)(\alpha_1-\alpha_2+\alpha_3)n,\\
y_1 &= -\alpha_1 m + (\alpha_1+2\alpha_3)(\alpha_1+2\alpha_2+\alpha_3)n,\\
y_2 &= -\alpha_2 m - (\alpha_1-\alpha_3)(2\alpha_1+\alpha_2+2\alpha_3)n,\\
y_3 &= -\alpha_3 m - (2\alpha_1+\alpha_3)(\alpha_1+2\alpha_2+\alpha_3)n,\\
y_4 &= (\alpha_1+\alpha_2+\alpha_3)m + (\alpha_1-\alpha_3)(\alpha_1-\alpha_2+\alpha_3)n,
\end{aligned} \tag{20}$$

where $\alpha_1, \alpha_2, \alpha_3, m$ and n are arbitrary parameters.

4. Next we shall solve diophantine system II given by the following three equations:

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4 \tag{21}$$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3 \tag{22}$$

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = y_1^4 + y_2^4 + y_3^4 + y_4^4. \tag{23}$$

We write

$$\begin{aligned} x_i &= a_i u + \alpha_i v, \quad i = 1, 2, 3, 4, \\ y_i &= a_{i+1} u + \alpha_i v, \quad i = 1, 2, 3, 4, \end{aligned} \tag{24}$$

where $a_5 = a_1$. With these values of x_i, y_i, $i = 1, 2, 3, 4$, equation (21) is identically satisfied. Substituting the values of x_i, y_i, $i = 1, 2, 3, 4$, in equation (22), we get

$$\left\{ \sum_{i=1}^{4} (a_i^2 - a_{i+1}^2)\alpha_i \right\} u^2 v + \left\{ \sum_{i=1}^{4} (a_i - a_{i+1})\alpha_i^2 \right\} u v^2 = 0.$$

Thus equation (22) will be identically satisfied for all values of u and v if the a_i, $i = 1, 2, 3, 4$, are chosen so as to satisfy the following conditions:

$$(a_1^2 - a_2^2)\alpha_1 + (a_2^2 - a_3^2)\alpha_2 + (a_3^2 - a_4^2)\alpha_3 + (a_4^2 - a_1^2)\alpha_4 = 0, \tag{25}$$

$$(a_1 - a_2)\alpha_1^2 + (a_2 - a_3)\alpha_2^2 + (a_3 - a_4)\alpha_3^2 + (a_4 - a_1)\alpha_4^2 = 0. \tag{26}$$

Substituting the values of x_i, y_i, $i = 1, 2, 3, 4$, in equation (23), we get

$$\begin{aligned} \left\{ \sum_{i=1}^{4} (a_i^3 - a_{i+1}^3)\alpha_i \right\} u^3 v &+ \left\{ \sum_{i=1}^{4} (a_i^2 - a_{i+1}^2)\alpha_i^2 \right\} u^2 v^2 \\ &+ \left\{ \sum_{i=1}^{4} (a_i - a_{i+1})\alpha_i^3 \right\} u v^3 = 0. \end{aligned} \tag{27}$$

We will choose the a_i, $i = 1, 2, 3, 4$, so that the coefficient of uv^3 in equation (27) becomes zero. This gives the condition

$$(a_1 - a_2)\alpha_1^3 + (a_2 - a_3)\alpha_2^3 + (a_3 - a_4)\alpha_3^3 + (a_4 - a_1)\alpha_4^3 = 0. \tag{28}$$

We now have to solve equations (25), (26) and (28) for a_1, a_2, a_3 and a_4. We note that

$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_1) = 0 \tag{29}$$

holds identically, so that it follows from (26), (28) and (29) that

$$\frac{a_1 - a_2}{\lambda_1} = \frac{a_2 - a_3}{\lambda_2} = \frac{a_3 - a_4}{\lambda_3} = \frac{a_4 - a_1}{\lambda_4} \tag{30}$$

where

$$\begin{aligned} \lambda_1 &= (\alpha_2 - \alpha_3)(\alpha_3 - \alpha_4)(\alpha_4 - \alpha_2)(\alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_4\alpha_2), \\ \lambda_2 &= -(\alpha_3 - \alpha_4)(\alpha_4 - \alpha_1)(\alpha_1 - \alpha_3)(\alpha_3\alpha_4 + \alpha_4\alpha_1 + \alpha_1\alpha_3), \\ \lambda_3 &= (\alpha_4 - \alpha_1)(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_4)(\alpha_4\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_4), \\ \lambda_4 &= -(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1). \end{aligned} \tag{31}$$

Using (30), equation (25) reduces to the linear equation

$$(a_1+a_2)\lambda_1\alpha_1 + (a_2+a_3)\lambda_2\alpha_2 + (a_3+a_4)\lambda_3\alpha_3 + (a_4+a_1)\lambda_4\alpha_4 = 0. \tag{32}$$

Thus, the problem is effectively reduced to choosing a_i, $i = 1, 2, 3, 4$, so as to satisfy the linear equations (30) and (32), and we accordingly get the following solution for a_i, $i = 1, 2, 3, 4$:

$$\begin{aligned} a_1 &= \lambda_1^2(\alpha_1 - \alpha_4) + \lambda_2^2(\alpha_2 - \alpha_4) + \lambda_3^2(\alpha_3 - \alpha_4) \\ &\quad + 2\lambda_1\lambda_2(\alpha_2 - \alpha_4) + 2(\lambda_1 + \lambda_2)\lambda_3(\alpha_3 - \alpha_4), \\ a_2 &= -\lambda_1^2(\alpha_1 - \alpha_4) + \lambda_2^2(\alpha_2 - \alpha_4) + \lambda_3^2(\alpha_3 - \alpha_4) \\ &\quad + 2\lambda_2\lambda_3(\alpha_3 - \alpha_4), \\ a_3 &= -\lambda_1^2(\alpha_1 - \alpha_4) - \lambda_2^2(\alpha_2 - \alpha_4) + \lambda_3^2(\alpha_3 - \alpha_4) \\ &\quad - 2\lambda_1\lambda_2(\alpha_1 - \alpha_4), \\ a_4 &= -\lambda_1^2(\alpha_1 - \alpha_4) - \lambda_2^2(\alpha_2 - \alpha_4) - \lambda_3^2(\alpha_3 - \alpha_4) \\ &\quad - 2\lambda_2\lambda_3(\alpha_2 - \alpha_4) - 2\lambda_1(\lambda_2 + \lambda_3)(\alpha_1 - \alpha_4). \end{aligned} \tag{33}$$

When the a_i, $i = 1, 2, 3, 4$, are defined by (33), the values of x_i, y_i, $i = 1, 2, 3, 4$, given by (24) satisfy equations (21) and (22) whereas equation (23), which led to equation (27), reduces to

$$\left\{ \sum_{i=1}^{4} (a_i^3 - a_{i+1}^3)\alpha_i \right\} u^3 v + \left\{ \sum_{i=1}^{4} (a_i^2 - a_{i+1}^2)\alpha_i^2 \right\} u^2 v^2 = 0.$$

This equation is satisfied if we take

$$u = \sum_{i=1}^{4}(a_i^2 - a_{i+1}^2)\alpha_i^2, \qquad v = -\sum_{i=1}^{4}(a_i^3 - a_{i+1}^3)\alpha_i. \tag{34}$$

Thus a solution of the diophantine system II is given in terms of the parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ by (24), with the a_i, $i = 1, 2, 3, 4$, being defined by (31) and (33), and u and v being defined by (34).

As a numerical example we take $\alpha_1 = 5$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, when we get, after removal of common factors and suitable rearrangement, the following solution in positive integers of diophantine system II:

$$\begin{aligned} &5318357^k + 13600563^k + 14484592^k + 20533406^k \\ &\quad = 6580709^k + 9969256^k + 18251462^k + 19135491^k, \quad k = 1, 3, 4. \end{aligned}$$

5. We will now consider diophantine system III given by the following three equations:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 \tag{35}$$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3 \tag{36}$$

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = y_1^4 + y_2^4 + y_3^4 + y_4^4. \tag{37}$$

We observe that, if x_i, $i = 1, 2, 3, 4$, are chosen so that

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \tag{38}$$

then a solution of equations (35), (36) and (37) is given by $y_i = -x_i$, $i = 1, 2, 3, 4$. As integer solutions of equation (38) are well known [**3**, pp. 290–291], we get integer solutions of the system of equations (35), (36) and (37). However, this must be regarded in a way as a trivial solution of this system of equations.

To obtain a nontrivial parametric solution of equations (35), (36) and (37), we will start with a parametric solution of the diophantine system

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= y_1^2 + y_2^2 + y_3^2 \\ x_1^4 + x_2^4 + x_3^4 &= y_1^4 + y_2^4 + y_3^4 \end{aligned} \tag{39}$$

and write $y_4 = -x_4$ so that equations (35) and (37) will be automatically satisfied, and we will finally choose suitably the parameters as well as x_4 so that equation (36) is also satisfied.

A solution of diophantine system (39) is given by

$$\begin{aligned} x_1 &= \alpha_1 m - (\alpha_1 + 2\alpha_2)n, \\ x_2 &= \alpha_2 m + (2\alpha_1 + \alpha_2)n, \\ x_3 &= (\alpha_1 + \alpha_2)m + (\alpha_1 - \alpha_2)n, \\ y_1 &= -\alpha_1 m - (\alpha_1 + 2\alpha_2)n, \\ y_2 &= -\alpha_2 m + (2\alpha_1 + \alpha_2)n, \\ y_3 &= -(\alpha_1 + \alpha_2)m + (\alpha_1 - \alpha_2)n. \end{aligned} \tag{40}$$

It is readily verified that this is indeed a solution of diophantine system (39). Substituting these values of x_i, y_i, $i = 1, 2, 3$, and $y_4 = -x_4$ in equation (36), we get the equation

$$\begin{aligned} &(2\alpha_1^3 + 3\alpha_1^2\alpha_2 + 3\alpha_1\alpha_2^2 + 2\alpha_2^3)m^3 \\ &\qquad + (6\alpha_1^3 + 21\alpha_1^2\alpha_2 + 21\alpha_1\alpha_2^2 + 6\alpha_2^3)mn^2 + x_4^3 = 0. \end{aligned} \tag{41}$$

Considered as a cubic equation in the variables m, n and x_4, equation (41) represents a cubic curve in the projective plane. Moreover, it can be readily verified that the point given by $m = -1$, $n = 1$, $x_4 = 2(\alpha_1 + \alpha_2)$ lies on this cubic curve. By the well-known tangent method, we now get a new solution of equation (41) which is given by

$$\begin{aligned} m &= (-4\alpha_1^4 - 20\alpha_1^3\alpha_2 - 33\alpha_1^2 a_2^2 - 20\alpha_1\alpha_2^3 - 4\alpha_2^4)\xi, \\ n &= (4\alpha_1^4 + 8\alpha_1^3\alpha_2 + 9\alpha_1^2\alpha_2^2 + 8\alpha_1\alpha_2^3 + 4\alpha_2^4)\xi, \\ x_4 &= (8\alpha_1^5 + 36\alpha_1^4\alpha_2 + 64\alpha_1^3\alpha_2^2 + 64\alpha_1^2\alpha_2^3 + 36\alpha_1\alpha_2^4 + 8\alpha_2^5)\xi. \end{aligned} \tag{42}$$

With these values of m and n we get, using (40) and taking $\xi = 1/2$, the following parametric solution of diophantine system III consisting of equations (35), (36) and (37):

$$
(43)\qquad
\begin{aligned}
x_1 &= -4\alpha_1^5 - 18\alpha_1^4\alpha_2 - 29\alpha_1^3\alpha_2^2 - 23\alpha_1^2\alpha_2^3 - 12\alpha_1\alpha_2^4 - 4\alpha_2^5,\\
x_2 &= 4\alpha_1^5 + 8\alpha_1^4\alpha_2 + 3\alpha_1^3\alpha_2^2 - 4\alpha_1^2\alpha_2^3 - 2\alpha_1\alpha_2^4,\\
x_3 &= -10\alpha_1^4\alpha_2 - 26\alpha_1^3\alpha_2^2 - 27\alpha_1^2\alpha_2^3 - 14\alpha_1\alpha_2^4 - 4\alpha_2^5,\\
x_4 &= 4\alpha_1^5 + 18\alpha_1^4\alpha_2 + 32\alpha_1^3\alpha_2^2 + 32\alpha_1^2\alpha_2^3 + 18\alpha_1\alpha_2^4 + 4\alpha_2^5,\\
y_1 &= 2\alpha_1^4\alpha_2 + 4\alpha_1^3\alpha_2^2 - 3\alpha_1^2\alpha_2^3 - 8\alpha_1\alpha_2^4 - 4\alpha_2^5,\\
y_2 &= 4\alpha_1^5 + 12\alpha_1^4\alpha_2 + 23\alpha_1^3\alpha_2^2 + 29\alpha_1^2\alpha_2^3 + 18\alpha_1\alpha_2^4 + 4\alpha_2^5,\\
y_3 &= 4\alpha_1^5 + 14\alpha_1^4\alpha_2 + 27\alpha_1^3\alpha_2^2 + 26\alpha_1^2\alpha_2^3 + 10\alpha_1\alpha_2^4,\\
y_4 &= -4\alpha_1^5 - 18\alpha_1^4\alpha_2 - 32\alpha_1^3\alpha_2^2 - 32\alpha_1^2\alpha_2^3 - 18\alpha_1\alpha_2^4 - 4\alpha_2^5.
\end{aligned}
$$

As a numerical example we take $\alpha_1 = 1$ and $\alpha_2 = 2$ when we get, after removal of common factors and suitable rearrangement, the following solution of diophantine system III:

$$
65^k + (-127)^k + (-192)^k + 210^k = 8^k + 165^k + 173^k + (-210)^k,\quad k = 2, 3, 4.
$$

Since $x_4 = -y_4$, the parametric solution (43) cannot yield a solution of the diophantine system III in positive integers.

6. We will now describe a method of generating solutions of diophantine system III which will yield solutions in positive integers. As all three equations (35), (36) and (37) of the diophantine system III are homogeneous, any rational solution of this system of equations can be multiplied throughout by a suitable constant to yield a solution of this system in integers. We will now use solution (20) of the system of equations (5), (6) and (7). For brevity, we write

$$
(44)\qquad
\begin{aligned}
\alpha_4 &= -(\alpha_1 + \alpha_2 + \alpha_3),\\
\beta_1 &= (\alpha_1 + 2\alpha_3)(\alpha_1 + 2\alpha_2 + \alpha_3),\\
\beta_2 &= -(\alpha_1 - \alpha_3)(2\alpha_1 + \alpha_2 + 2\alpha_3),\\
\beta_3 &= -(2\alpha_1 + \alpha_3)(\alpha_1 + 2\alpha_2 + \alpha_3),\\
\beta_4 &= (\alpha_1 - \alpha_3)(\alpha_1 - \alpha_2 + \alpha_3),
\end{aligned}
$$

and it now follows from the aforementioned solution (20) that if we take

$$
\begin{aligned}
x_1 &= \varepsilon_1(\alpha_1 m + \beta_1 n), & y_1 &= \varepsilon_1(\alpha_1 m - \beta_1 n),\\
x_2 &= \varepsilon_2(\alpha_2 m + \beta_2 n), & y_2 &= \varepsilon_2(\alpha_2 m - \beta_2 n),\\
x_3 &= \varepsilon_3(\alpha_3 m + \beta_3 n), & y_3 &= \varepsilon_3(\alpha_3 m - \beta_3 n),\\
x_4 &= \varepsilon_4(\alpha_4 m + \beta_4 n), & y_4 &= \varepsilon_4(\alpha_4 m - \beta_4 n),
\end{aligned}
\tag{45}
$$

where $\varepsilon_i = \pm 1$, $i = 1, 2, 3, 4$, then equations (35) and (37) are identically satisfied. On substituting these values of x_i, y_i, $i = 1, 2, 3, 4$, in equation (36) and observing that $\varepsilon_i^3 = \varepsilon_i$ for each $i = 1, 2, 3, 4$, we get

$$2n\left[3\left\{\sum_{i=1}^{4}\varepsilon_i\alpha_i^2\beta_i\right\}m^2 + \left\{\sum_{i=1}^{4}\varepsilon_i\beta_i^3\right\}n^2\right] = 0. \tag{46}$$

Thus equation (36) will also be satisfied by the x_i, y_i, $i = 1, 2, 3, 4$, given by (45) if we can find suitable m and n satisfying equation (46). Now equation (46) will have a rational solution for m and n if we can find suitable values of ε_i, $i = 1, 2, 3, 4$, as well as rational α_i, $i = 1, 2, 3$, such that the equation

$$z^2 = -3\left\{\sum_{i=1}^{4}\varepsilon_i\alpha_i^2\beta_i\right\}\left\{\sum_{i=1}^{4}\varepsilon_i\beta_i^3\right\} \tag{47}$$

is satisfied by a rational value of z. We take $\varepsilon_1 = 1$ as there is no loss of generality in doing so. Further, we note that when $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $\varepsilon_3 = 1$ and $\varepsilon_4 = -1$, the solutions obtained will be trivial and we, therefore, exclude this set of values of ε_i from consideration. We will now find, by trial, suitable α_i, $i = 1, 2, 3$, and ε_i, $i = 2, 3, 4$, such that equation (47) is satisfied by a rational value of z. With these values of ε_i, $i = 1, 2, 3, 4$, and α_i, $i = 1, 2, 3$, equation (46) can be solved to obtain rational values of m and n, and using (44) and (45), we can obtain rational, and hence, integer solutions of equations (35), (36) and (37). As an example, it was found by trial that when we take $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $\varepsilon_3 = -1$, $\varepsilon_4 = 1$, $\alpha_1 = 19$, $\alpha_2 = -26$ and $\alpha_3 = -23$, the righthand side of equation (47) becomes a perfect square, and this leads to the following solution of diophantine system III:

$$43^k + 486^k + 815^k + 1058^k = 242^k + 335^k + 907^k + 1014^k,$$
$$k = 2, 3, 4.$$

A computer search carried out in the range $1 \leq \alpha_1 \leq 100$, $-100 \leq \alpha_i \leq 100$ for $i = 2$ and 3 yielded a number of solutions of equation (47) and these, in turn, generated several distinct solutions of the diophantine system III. The values of α_i, $i = 1, 2, 3$, and ε_i, $i = 2, 3, 4$, and the corresponding solutions (except for one solution which is dealt with subsequently) with the values of x_i, y_i, $i = 1, 2, 3, 4$, suitably rearranged are given in Table 1.

We will now show that there are infinitely many sets of rational values of α_i, $i = 1, 2, 3$, and ε_i, $i = 1, 2, 3, 4$, such that equation (47) is satisfied by a rational value of z and, hence, infinitely many integer solutions of the diophantine system III can be obtained by this method.

In equation (47) we fix $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$, $\varepsilon_4 = -1$, and we write

$$\alpha_2 = t(\alpha_1 + \alpha_3),$$

when the righthand side of equation (47) has the squared factor $[3(\alpha_1 + \alpha_3)^2\{t\alpha_1 + (t+1)\alpha_3\}]^2$ which can be removed so that equation (47) is effectively reduced to the equation

$$\begin{aligned} Z^2 = 3(\alpha_1 + \alpha_3)\left\{(t-1)\alpha_1 - (t+2)\alpha_3\right\}&\left\{(8t^2 + 11t + 5)\alpha_1^2\right. \\ &\left.+(8t^2 + 8t - 1)\alpha_1\alpha_3 + (8t^2 + 5t + 2)\alpha_3^2\right\}. \end{aligned} \tag{48}$$

It would be observed from Table 1 that is has been found by trial that when $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$, $\varepsilon_4 = -1$, $\alpha_1 = 1$, $\alpha_2 = -40$ and $\alpha_3 = 4$, the righthand side of equation (47) becomes a perfect square. Using this fact, we find that $\alpha_1 = 1$, $\alpha_3 = 4$, $t = -8$, $Z = 1485$ is a solution of equation (48). We now fix $\alpha_1 = 1$ and $t = -8$ in equation (48) which reduces to

$$Z^2 = 27(\alpha_3 + 1)(2\alpha_3 - 3)(158\alpha_3^2 + 149\alpha_3 + 143). \tag{49}$$

We note that $\alpha_3 = 4$, $Z = 1485$ is a known solution of equation (49). Next, we apply the birational transformation

$$\begin{aligned} \alpha_3 &= -(\xi + 2280)/\xi, \\ Z &= 6840\eta/\xi^2, \end{aligned} \tag{50}$$

to this equation when we get

$$\eta^2 = \xi^3 + 3417\xi^2 + 7688160\xi + 4928083200. \tag{51}$$

TABLE 1. Solutions of $x_1^k + x_2^k + x_3^k + x_4^k = y_1^k + y_2^k + y_3^k + y_4^k$, $k = 2, 3, 4$.

α_1	α_2	α_3	ε_2	ε_3	ε_4	x_1, x_2, x_3, x_4	y_1, y_2, y_3, y_4
1	-53	9	1	-1	-1	209, 303, 754, 1040	39, -247, 880, 974
1	-40	4	1	-1	-1	32, 47, 185, 225	-3, -43, 200, 215
1	-32	7	1	-1	-1	89, 175, 354, 524	49, -121, 414, 500
1	-22	4	1	-1	-1	14, 35, 89, 123	9, -25, 98, 119
1	-6	-4	-1	-1	1	-5, 6, 16, 17	8, 9, 10, 19
2	-37	-6	-1	-1	1	210, -658, 2237, 2617	522, 902, 1897, 2765
2	-9	14	1	-1	1	-23, 102, 208, 275	-53, 58, 240, 257
3	-98	33	1	-1	-1	299, 967, 1292, 2280	-437, 551, 1560, 2228
3	-5	-9	-1	-1	1	-5, 41, 42, 56	8, 35, 49, 54
3	-4	-9	-1	1	1	12, -31, 45, -58	-19, 33, 38, -60
3	85	27	-1	-1	1	209, 394, 735, 1120	119, -266, 880, 1065
4	-40	-53	-1	-1	1	-1169, 27045, 31327, 55456	6141, 22675, 34522, 55171
5	-64	-45	-1	-1	1	-1495, 1875, 4612, 5882	2075, 2812, 3345, 6182
5	-15	-11	-1	-1	1	-26, 52, 93, 111	39, 58, 76, 117
6	-65	-26	-1	-1	1	1784, -2796, 8545, 8765	3884, 4104, 5405, 9985
8	82	23	-1	-1	1	-151, 855, 2723, 3794	-385, 686, 2771, 3777
9	-14	-12	-1	-1	1	-11, 195, 254, 364	84, 130, 299, 349
10	-64	-29	-1	-1	1	1271, -1395, 4202, 4721	2006, 2511, 2525, 5177
10	-57	-22	-1	-1	1	-2360, 2452, 7149, 9057	2872, 4737, 4780, 9549
15	-23	-33	-1	-1	1	-85, 541, 672, 686	96, 581, 595, 722
17	-53	-40	-1	-1	1	-239, 457, 855, 976	349, 543, 664, 1045
17	-35	-22	-1	-1	1	-19, 155, 171, 290	71, 95, 206, 285
18	-61	-78	-1	-1	1	-2224, 7228, 9545, 13173	3753, 6148, 9776, 13205
19	-26	-23	-1	-1	1	43, 486, 815, 1058	242, 335, 907, 1014
19	-24	-67	-1	-1	1	239, 2324, 3855, 4564	604, 2079, 4220, 4319
24	-47	-39	-1	-1	1	-570, 2589, 2924, 4343	1109, 1935, 3354, 4268
33	-70	-93	-1	-1	1	-4029, 15591, 21070, 23890	5110, 16371, 19191, 24730
38	-63	-95	-1	-1	-1	17614, -25505, 36000, -44121	5439, -32825, 37680, -40946
47	-16	81	1	1	-1	3272, -4393, 16878, 17185	-823, 9528, 12050, 19327
51	-64	-75	-1	-1	1	-267, 9130, 11631, 14788	1852, 7869, 13527, 13750
51	-52	-75	-1	-1	1	813, 14008, 19605, 22096	1096, 13845, 19888, 21933

The solution of equation (51) corresponding to the known solution $\alpha_3 = 4$, $Z = 1485$ of (49) is given by

$$\xi = -456, \qquad \eta = 45144. \tag{52}$$

Equation (51) represents an elliptic curve and a rational point, with integer coordinates, on this elliptic curve, and is given by (52). The discriminant of the cubic polynomial on the righthand side of equation (51) is -239427613818938880000, and it is readily verified that the ordinate of the rational point given by (52) does not divide this discriminant. It therefore follows from the Nagell-Lutz theorem [**5**, p. 56] on elliptic curves that this rational point is not a point of finite order. Thus, it follows that equation (51) has infinitely many rational solutions, and these can be obtained by using the group law. Working backwards, we can obtain infinitely many rational solutions of equation (49), and eventually we obtain infinitely many solutions of equation (47). And, thus, the method leads to infinitely many solutions in integers of the diophantine system III.

7. Finally we will obtain the numerical solution (1) of diophantine system IV. We have already mentioned that one solution of diophantine system III, obtained by trial, has not been listed in the table of solutions given above. This solution, obtained by taking $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$, $\varepsilon_4 = 1$, $\alpha_1 = 4$, $\alpha_2 = 91$, $\alpha_3 = 94$, is given below:

$$358^k + (-407)^k + (-815)^k + 1224^k = (-410)^k + (-776)^k + (-407)^k + 1233^k, \quad k = 2, 3, 4.$$

Cancelling out the common term $(-407)^k$ from both sides, we get the numerical solution (1) of diophantine system IV.

REFERENCES

1. L.E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952 (reprint).

2. L. Gatteschi and L.A. Rosati, *Riposta ad una questione proposta da A. Moessner*, Boll. Un. Mat. Ital. (3) **5** (1950), 43–48.

3. L.K. Hua, *Introduction to number theory*, Springer-Verlag, New York, 1982.

4. G. Palama, *Sistemi indeterminati impossibili*, Boll. Un. Mat. Ital. (3) **6** (1951), 113–117.

5. J.H. Silverman and J. Tate, *Rational points on elliptic curves*, Springer-Verlag, New York, 1992.

Ambassador, Embassy of India, Kantari Street, Sahmarani Building, P.O. Box 113-5240, Beirut, Lebanon
E-mail address: ambindia@inco.com.lb