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THE GENERAL STABLE RANK IN NONSTABLE K-THEORY

YIFENG XUE

ABSTRACT. In this paper we show that for every C^*-algebra $\mathcal{A}$ the natural homomorphism $i_{\mathcal{A}} : U(\mathcal{A}) \to K_1(\mathcal{A})$ is injective if and only if $S\mathcal{A}$ has 1-cancellation and $i_{M_n(\mathcal{A})}$ is injective for any $n \geq 1$ if and only if $\operatorname{gsr}(S\mathcal{A}) = 1$. These results improve [**12**]. As applications, we figure out the value of $\operatorname{gsr}(S\mathcal{A})$ or $\operatorname{gsr}(\Omega(\mathcal{A}))$ when the unital C^*-algebra $\mathcal{A}$ is of real rank zero or purely infinite simple; we also investigate the manner of $i_{\mathcal{A}\otimes\mathcal{B}}$ for certain infinite C^*-algebra $\mathcal{A}$ and any nuclear C^*-algebra $\mathcal{B}$. We have proven that if $\mathcal{B}$ is a nonunital purely infinite simple C^*-algebra or a certain stable corona algebra, then $i_{\mathcal{A}\otimes\mathcal{B}}$ is always an isomorphism.

0. Introduction. For the unital C^*-algebra $\mathcal{A}$, we write $\mathcal{U}(\mathcal{A})$, respectively $\mathcal{U}_0(\mathcal{A})$, to denote the unitary group of $\mathcal{A}$, respectively the connected component of the unit in $\mathcal{U}(\mathcal{A})$. The quotient group $U(\mathcal{A}) = \mathcal{U}(\mathcal{A})/\mathcal{U}_0(\mathcal{A})$ whose multiplication is given by $[u][v] = [uv]$ is called the U-group of $\mathcal{A}$, where $[u]$ stands for the equivalence class of u in $\mathcal{U}(\mathcal{A})$. If $\mathcal{A}$ has no unit, we put $U(\mathcal{A}) = U(\mathcal{A}^+)$, where $\mathcal{A}^+$ is $\mathcal{A}$ obtained by unit adjointed. For any unital C^*-algebra $\mathcal{A}$, we denote by $M_n(\mathcal{A})$ the matrix algebra of $n \times n$ over $\mathcal{A}$. Set $\mathcal{U}_1(\mathcal{A}) = \mathcal{U}(\mathcal{A})$, respectively $\mathcal{U}_1^0(\mathcal{A}) = \mathcal{U}_0(\mathcal{A})$ and

$$\mathcal{U}_n(\mathcal{A}) = \mathcal{U}(M_n(\mathcal{A})), \mathcal{U}_n^0(\mathcal{A}) = \mathcal{U}_0(M_n(\mathcal{A})), U_n(\mathcal{A}) = \mathcal{U}_n(\mathcal{A})/\mathcal{U}_n^0(\mathcal{A}).$$

For the C^*-algebra $\mathcal{A}$, we set $\Omega(\mathcal{A}) = C(S^1, \mathcal{A})$, $S\mathcal{A} = C_0(0,1) \otimes \mathcal{A}$, the suspension of $\mathcal{A}$. We notice that $(S\mathcal{A})^+$ can be expressed as

$$(S\mathcal{A})^+ \cong \{f \in C_0([0,1], \mathcal{A}) f(0) = f(1) = \lambda 1, \\ f(t) = \lambda 1 + x_t, \lambda \in \mathbf{C}, x_t \in \mathcal{A}\}.$$

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Let $\mathcal{A}$ be a C^*-algebra with unit 1. We view $\mathcal{A}^n$ as the set of all $n \times 1$ matrices over $\mathcal{A}$. According to [**11**] and [**12**], the topological stable rank, the connected stable rank and the general stable rank of $\mathcal{A}$ are defined respectively as follows:

$$\begin{aligned}\operatorname{tsr}(\mathcal{A}) &= \min\{n \in \mathbf{N} \mid \mathcal{A}^m \text{ is dense in } Lg_m(\mathcal{A}), \forall\, m \geq n\}\\ \operatorname{csr}(\mathcal{A}) &= \min\{n \in \mathbf{N} \mid \mathcal{U}_m^0(\mathcal{A}) \text{ acts transitively on}\\ &\qquad S_m(\mathcal{A}), \forall\, m \geq n\}\\ \operatorname{gsr}(\mathcal{A}) &= \min\{n \in \mathbf{N} \mid \mathcal{U}_m(\mathcal{A}) \text{ acts transitively on}\\ &\qquad S_m(\mathcal{A}), \forall\, m \geq n\},\end{aligned}$$

where $S_n(\mathcal{A}) = \{(a_1, \dots, a_n)^T \in \mathcal{A}^n \mid \sum_{i=1}^n a_i^* a_i = 1\}$ and

$$Lg_n(\mathcal{A}) = \left\{(a_1, \dots, a_n)^T \in \mathcal{A}^n \mid \exists (b_1, \dots, b_n)^T \in \mathcal{A}^n \ni \sum_{i=1}^n b_i a_i = 1\right\}.$$

If no such integer exists, we set $\operatorname{tsr}(\mathcal{A}) = \infty$, $\operatorname{csr}(\mathcal{A}) = \infty$, or $\operatorname{gsr}(\mathcal{A}) = \infty$, respectively.

According to [**1**, Section 9], there is a natural homomorphism $i_{\mathcal{A}} : U(\mathcal{A}) \to K_1(\mathcal{A})$ for any C^*-algebra $\mathcal{A}$ where $K_0(\mathcal{A})$, $K_1(\mathcal{A})$ are the K-groups defined in [**1**]. Under what conditions is $i_{\mathcal{A}}$ injective? When is $i_{\mathcal{A}}$ surjective? These two problems are very important in computing K-groups in terms of $U(\mathcal{A})$ or $U(S\mathcal{A})$.

Rieffel has found that these problems are closely connected to the $\operatorname{csr}(\cdot)$ and $\operatorname{gsr}(\cdot)$. He showed that, for any $n \geq \max(\operatorname{gsr}(\Omega(\mathcal{A})), \operatorname{csr}(\mathcal{A}))$, $i_{M_{n-1}(\mathcal{A})} : U_{n-1}(\mathcal{A}) \to K_1(\mathcal{A})$ is an isomorphism [**12**, Theorem 2.9]. Using different approaches, Cuntz showed that if $\mathcal{A}$ is a unital purely infinite simple C^*-algebra, $i_{\mathcal{A}}$ is an isomorphism [**3**, Theorem 1.9] and Lin proved that if $\mathcal{A}$ is a unital C^*-algebra with real rank zero, then $i_{\mathcal{A}}$ is injective [**5**, Lemma 2.2]. Besides, Thomsen showed that there is a natural isomorphism between quasi-unitary group of $\mathcal{A}$ and $K_1(\mathcal{A})$ for certain C^*-algebra $\mathcal{A}$, cf. [**14**, Theorems 4.3, 4.5]. Although many results have been obtained up to now, the problems seem far from being solved.

In this paper we will be concerned with the problem when $i_{\mathcal{A}}$ is a monomorphism. We give an equivalent description of the problem, that is, $i_{\mathcal{A}}$ is injective if and only if $S\mathcal{A}$ has 1-cancellation. Using this result, we prove that $i_{M_n}(\mathcal{A})$ is injective for all $n \geq 1$ if and only if $\operatorname{gsr}(S\mathcal{A}) = 1$. As a result, we get that $\operatorname{gsr}(\Omega(\mathcal{A})) = \operatorname{gsr}(\mathcal{A})$ if $\mathcal{A}$ is a unital C^*-algebra with $RR(\mathcal{A}) = 0$ and, moreover, if $\mathcal{A}$ is a purely infinite simple C^*-algebra with unit $1_{\mathcal{A}}$ such that $[1_{\mathcal{A}}]$ is torsion-free, respectively has torsion, in $K_0(\mathcal{A})$, then $\operatorname{gsr}(\Omega(\mathcal{A})) = 2$, respectively $\operatorname{gsr}(\Omega(\mathcal{A})) = \infty$. We also prove that if $\mathcal{A}$ is a nonunital purely infinite simple C^*-algebra or a certain stable corona algebra and $\mathcal{B}$ is a nuclear C^*-algebra, then $i_{\mathcal{A}\otimes\mathcal{B}}$ is an isomorphism.

1. The 1-cancellation of C^*-algebras. Let p, q be two projections in the C^*-algebra $\mathcal{A}$. We say that p is equivalent to q, denoted $p \sim q$, if there is a $u \in \mathcal{A}$ such that $p = u^*u$, $q = uu^*$. We write $[p]$ to denote the equivalence class of p with respect to "$\sim$."

Borrowing ideas from [**4**, Theorem 1.5] and [**11**, Corollary 10.7], we establish the following notation.

Definition 1.1. Let $\mathcal{A}$ be a C^*-algebra with unit 1. $\mathcal{A}$ is said to have 1-cancellation if for any projection p in $M_2(\mathcal{A})$ with $\operatorname{diag}(p, 1_k) \sim \operatorname{diag}(p_1, 1_k)$ in $M_{k+2}(\mathcal{A})$ for some $k \geq 1$, we have $p \sim p_1$ where $p_1 = \operatorname{diag}(1, 0) \in M_2(\mathcal{A})$ and 1_k is the unit of $M_k(\mathcal{A})$. If $\mathcal{A}$ has no unit, we work with $\mathcal{A}^+$.

Obviously if the unital C^*-algebra $\mathcal{A}^+$ has cancellation or if $\operatorname{gsr}(\mathcal{A}) \leq 2$, then $\mathcal{A}$ has 1-cancellation, cf. [**1**] and [**11**, Proposition 10.5].

Now let $\phi : \mathcal{A} \to \mathcal{B}$ be a $*$ homomorphism between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. We denote by ϕ_n the induced $*$ homomorphism of ϕ on $M_n(\mathcal{A})$ and let ϕ_* denote the induced homomorphism of ϕ on $U(\mathcal{A})$ or $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$. We also let $\rho : \mathcal{A}^+ \to \mathbf{C}$ denote the canonical homomorphism.

Definition 1.2. Let $\mathcal{A}$ be a C^*-algebra. A projection e in $M_2((S\mathcal{A})^+)$ is called to a 1-projective loop if $e(0) = e(1) = p_1$ and $\rho_2(e(t)) = p_1$ for all $t \in [0, 1]$.

Let $\operatorname{PL}(\mathcal{A})$ denote the set of all 1-projective loops in $M_2((S\mathcal{A})^+)$.

Lemma 1.1. *Let $\mathcal{A}$ be a C^*-algebra.*

(1) *If $\mathcal{A}$ has 1-cancellation, then $\mathcal{U}_2(\mathcal{A}^+)$ acts transitively on $S_2(\mathcal{A}^+)$;*

(2) *Assume that for any projection e in* $\mathrm{PL}(\mathcal{A})$ *with* $\mathrm{diag}\,(p,1_k) \sim \mathrm{diag}\,(p_1,1_k)$ *in $M_{k+2}((S\mathcal{A})^+)$ for some $k \geq 1$, we have $p \sim p_1$ in $M_2((S\mathcal{A})^+)$. Then $S\mathcal{A}$ has 1-cancellation.*

Proof. (1) Let $(a_1,a_2)^T \in S_2(\mathcal{A}^+)$. Then the projection $p = \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \end{bmatrix}\begin{bmatrix} a_1^* & a_2^* \\ 0 & 0 \end{bmatrix}$ is equivalent to the projection $p_1 = \begin{bmatrix} a_1^* & a_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ a_2 & 0 \end{bmatrix}$ in $M_2(\mathcal{A}^+)$. Thus there is a $z \in \mathcal{U}_4(\mathcal{A}^+)$ such that $\mathrm{diag}\,(p,0) = z^*\mathrm{diag}\,(p_1,0)z$ in $M_4(\mathcal{A}^+)$, and consequently,

$$\mathrm{diag}\,(1_2-p,1_2) = z^*\mathrm{diag}\,(1_2-p_1,1_2)z \sim \mathrm{diag}\,(p_1,1_2).$$

Since $\mathcal{A}$ has 1-cancellation, we have $1_2-p \sim p_1 \sim 1_2-p_1$ in $M_2(\mathcal{A}^+)$. Therefore there is a $v \in \mathcal{U}_2(\mathcal{A}^+)$ such that $p = vp_1v^*$ by [**1**]. Set

$$c = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_1^* & a_2^* \\ 0 & 0 \end{bmatrix} v \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.1}$$

Then by simple computation we obtain that

$$c_{12} = c_{21} = c_{22} = 0 \quad \text{and} \quad c_{11}^*c_{11} = c_{11}c_{11}^* = 1. \tag{1.2}$$

So, combining (1.1) with (1.2), we get that

$$p_1 = v^*pvp_1 = v^*\begin{bmatrix} a_1 & 0 \\ a_2 & 0 \end{bmatrix}\begin{bmatrix} a_1^* & a_2^* \\ 0 & 0 \end{bmatrix} v \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = v^*\begin{bmatrix} a_1 & 0 \\ a_2 & 0 \end{bmatrix}\begin{bmatrix} c_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

and hence $(a_1,a_2)^T = (v\mathrm{diag}\,(c_{11}^*,c_{11}))(1,0)^T$.

(2) Let p be a projection in $M_2((S\mathcal{A})^+)$ such that

$$\mathrm{diag}\,(p,1_k) \sim \mathrm{diag}\,(p_1,1_k) \quad \text{in } M_{k+2}((S\mathcal{A})^+) \tag{1.3}$$

for some k. Put $q(t) = \rho_2(p(t))$, $t \in [0,1]$. Then by (1.3) we have $q \sim p_1$ in $M_2((C_0(0,1))^+) \cong M_2(C(S^1))$ for $\mathrm{tsr}\,(M_2(C(S^1))) = 1$. Thus there exists $u \in \mathcal{U}_2((C_0(0,1))^+)$ by [**1**] such that $q = u^*p_1u$. Put $e(t) = u(t)pu^*(t)$, $t \in [0,1]$. Then $\rho_2(e(t)) = p_1$, $t \in [0,1]$, and

$$e(0) = e(1) = u(0)p(0)u^*(0) = u(0)q(0)u^*(0) = p_1,$$

for we always identify $q(0) = q(1) = p(0) = p(1)$, and $\mathrm{diag}\,(e, 1_k) \sim \mathrm{diag}\,(p_1, 1_k)$ in $M_{n+k}((S\mathcal{A})^+)$ by (1.3). Therefore, by assumption, $e \sim p_1$ in $M_2((S\mathcal{A})^+)$, i.e., $p \sim p_1$ in $M_2((S\mathcal{A})^+)$. □

Inspired by Lemma 1.1 (1) and [**12**], we define the integer $\mathrm{Gsr}\,(\mathcal{A})$ for each unital C^*-algebra $\mathcal{A}$ by

$$\mathrm{Gsr}\,(\mathcal{A}) = \min\{n \in \mathbf{N} \mid \mathcal{U}_2(M_m(\mathcal{A})) \text{ acts transitively on } S_2((M_m(\mathcal{A})))\forall\, m \geq n\}.$$

If no such integer exists we set $\mathrm{Gsr}\,(\mathcal{A}) = \infty$. If $\mathcal{A}$ has no unit, we set $\mathrm{Gsr}\,(\mathcal{A}) = \mathrm{Gsr}\,(\mathcal{A}^+)$. The following proposition characterizes the $\mathrm{Gsr}\,(\cdot)$.

Proposition 1.1. *Let $\mathcal{A}$ be a unital C^*-algebra. Then we have*

(1) $\mathrm{gsr}\,(\mathcal{A}) - 1 \leq \mathrm{Gsr}\,(\mathcal{A}) \leq \max\{\mathrm{gsr}\,(\mathcal{A}) - 1, 1\}$;

(2) $\mathrm{Gsr}\,(M_n(\mathcal{A})) \leq \{\mathrm{Gsr}\,(\mathcal{A})/n\}$, *where $\{x\}$ stands for the least integer which is greater than or equal to x.*

Proof. (1) For each $k \geq \max\{1, \mathrm{gsr}\,(\mathcal{A}) - 1\}$, and hence $2k \geq \mathrm{gsr}\,(\mathcal{A})$, let $A = (a_{ij})_{k\times k}$, $B = (b_{ij})_{k\times k} \in M_k(\mathcal{A})$ with $A^*A + B^*B = 1_k$. So

$$(a_{1j}, \dots, a_{kj}, b_{1j}, \dots, b_{kj})^T \in S_{2k}(\mathcal{A}),$$

$1 \leq j \leq k$ and $\sum_{t=1}^k (a_{ti}^* a_{tj} + b_{ti}^* b_{tj}) = 0$, $i \neq j$. Therefore there is a $u^{(1)} \in \mathcal{U}_{2k}(\mathcal{A})$ such that

$$\begin{aligned} u^{(1)}&(a_{11}, \dots, a_{k1}, b_{11}, \dots, b_{k1})^T \\ &= (1, 0, \dots, 0)^T u^{(1)}(a_{1j}, \dots, a_{kj}, b_{1j}, \dots, b_{kj})^T \\ &= (0, a_{2j}^{(1)}, \dots, a_{kj}^{(1)}, b_{2j}^{(1)}, \dots, b_{kj}^{(1)})^T, \quad 2 \leq j \leq k. \end{aligned}$$

By the same argument as above, we can find $u^{(2)}, \dots, u^{(k)}$ in $\mathcal{U}_{2k}(\mathcal{A})$ such that

$$u^{(k)} \cdots u^{(1)}(A, B)^T = (1_k, 0)^T \quad \text{in } S_2(M_k(\mathcal{A})).$$

On the other hand, suppose that $k \geq \mathrm{Gsr}\,(\mathcal{A})$ and $(a_1, \dots, a_{k+1})^T \in S_{k+1}(\mathcal{A})$. Set $A = [\,a\ O_{k\times(k-1)}\,]$ and $B = \mathrm{diag}\,(a_{k+1}, 1_{k-1}) \in M_k(\mathcal{A})$, where $a = (a_1, \dots, a_k)^T$. Then $(A, B)^T \in S_2(M_k(\mathcal{A}))$. Since $k \geq \mathrm{Gsr}\,(\mathcal{A})$, it follows that there is a $u \in \mathcal{U}_{2k}(\mathcal{A})$ such that $(A, B)^T = u(1_k, 0)^T$. We write u as the form $u = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$, where $C, D \in M_k(\mathcal{A})$. Thus we deduce from $u \in \mathcal{U}_{2k}(\mathcal{A})$ that D has the form $D = \begin{bmatrix} d \\ O_{(k-1)\times k} \end{bmatrix}$ and $W = \begin{bmatrix} a & C \\ a_{k+1} & d \end{bmatrix} \in \mathcal{U}_{k+1}(\mathcal{A})$, where $d = (d_1, \dots, d_k)$. Therefore $(a_1, \dots, a_{k+1})^T = W(1, 0, \dots, 0)^T$.

(2) Suppose that $\mathrm{Gsr}\,(\mathcal{A}) < \infty$ and $k \geq \{\mathrm{Gsr}\,(\mathcal{A})/n\}$. Then $\mathrm{Gsr}\,(\mathcal{A}) \leq kn$. Noting that $M_k(M_n(\mathcal{A})) \cong M_{kn}(\mathcal{A})$, we obtain that $\mathcal{U}_2(M_k(M_n(\mathcal{A})))$ acts transitively on $S_2(M_k(M_n(\mathcal{A})))$. The assertion follows. □

Corollary 1.1. *Let $\mathcal{A}$ be a C^*-algebra with unit* 1. *Then $M_n(\mathcal{A})$ has* 1*-cancellation for all $n \geq 1$ if and only if* $\mathrm{gsr}\,(\mathcal{A}) \leq 2$.

Proof. That $M_n(\mathcal{A})$ has 1-cancellation for each $n \geq 1$ shows that $\mathcal{U}_2(M_n(\mathcal{A}))$ acts transitively on $S_2(M_n(\mathcal{A}))$ by Lemma 1.1 (1). Thus we have $\mathrm{Gsr}\,(\mathcal{A}) = 1$ and hence $\mathrm{gsr}\,(\mathcal{A}) \leq 2$ by Proposition 1.1.

Conversely, since every projection in $M_n(\mathcal{A})$ corresponds uniquely to a finitely generated projective $\mathcal{A}$-module, it follows from [**11**] that $M_n(\mathcal{A})$ has 1-cancellation for each $n \geq 1$. □

2. The proof of the main result.

Lemma 2.1. *Let $\mathcal{A}$ be a C^*-algebra and $e \in PL\,(\mathcal{A})$. Then there is a continuous map $v_t : [0,1] \to \mathcal{U}_2(\mathcal{A}^+)$ such that $v_0 = 1_2$, $\rho_2(v_t) = 1_2$ and $e_t = v_t^* p_1 v_t$.*

Proof. Put $g_s(t) = e(16s(1-s)t(1-t))$, $0 \leq s,\ t \leq 1$. Then $g_s \in \mathrm{PL}\,(\mathcal{A})$ is a path from p_1 to p_1. Using the same method as in the proof of [**1**], we can find a continuous map $v_s : [0,1] \to \mathcal{U}_2((S\mathcal{A})^+)$ with $v_0(t) = 1_2$, $v_s(0) = v_s(1) = 1_2$ and $\rho_2(v_s(t)) = 1_2$ such that $g_s = v_s^* p_1 v_2$ for all $s, t \in [0,1]$.

Now take $s = (1 - \sqrt{1-t})/2$, $u_t = v_s(1/2)$, $0 \leq t \leq 1$. Then $u_0 = \rho_2(u_t) = 1_2$ and $e_t = u_t^* p_1 u_t$ for all $t \in [0,1]$. □

For a C^*-algebra $\mathcal{A}$ and a closed two-side ideal $\mathcal{J}$ of $\mathcal{A}$, we have the following exact sequence:

$$0 \longrightarrow \mathcal{J} \xrightarrow{j} \mathcal{A} \xrightarrow{\pi} \mathcal{B} \longrightarrow 0, \tag{2.1}$$

where $j : \mathcal{J} \to \mathcal{A}$ is the inclusive map and $\pi : \mathcal{A} \to \mathcal{B} = \mathcal{A}/\mathcal{J}$ is a quotient map.

Let $\partial : K_1(\mathcal{B}) \to K_0(\mathcal{J})$ denote the index map of (2.1) which is defined in [**1**, Definition 8.3.1] and put $\eta = \partial \circ i_{\mathcal{B}} : U(\mathcal{B}) \to K_0(\mathcal{J})$. Then η has the form

$$\eta([u]) = [wp_1w^*] - [p_1], \quad \forall\, u \in \mathcal{U}(\mathcal{B}^+), \tag{2.2}$$

where $w \in \mathcal{U}_2(\mathcal{A}^+)$ with $\pi_2(w) = \operatorname{diag}(u, u^*)$.

Borrowing some techniques from [**1**, Proposition 8.3.3], we can prove the following useful lemma.

Lemma 2.2. *Let $\mathcal{J}, \mathcal{A}, \mathcal{B}$ be as above, and suppose that $\mathcal{J}$ has* 1-*cancellation. Then we have the following exact sequence of groups.*

$$U(\mathcal{J}) \xrightarrow{j_*} U(\mathcal{A}) \xrightarrow{\pi_*} U(\mathcal{B}) \xrightarrow{\eta} K_0(\mathcal{J}). \tag{2.3}$$

Proof. Since $\pi(\mathcal{U}_0(\mathcal{A}^+)) = \mathcal{U}_0(\mathcal{B}^+)$ and $\pi \circ i = 0$, it follows that $\operatorname{Im} i_* = \operatorname{Ker} \pi_*$. We will prove $\operatorname{Im} \pi_* = \operatorname{Ker} \eta$ in the following.

It is easy to check that $\operatorname{Im} \pi_* \subset \operatorname{Ker} \eta$. Now let v be in $\mathcal{U}(\mathcal{B}^+)$ with $\eta([v]) = 0$. Then there is a $w \in \mathcal{U}_2(\mathcal{A}^+)$ with $\pi_2(w) = \operatorname{diag}(v, v^*)$ such that $[\,w\ p_1\ w^*\,] = [p_1]$ in $K_0(\mathcal{J})$ by (2.2). Since $\mathcal{J}$ has 1-cancellation, it follows from the definition of $K_0(\mathcal{J})$ that $u \in \mathcal{U}_4(\mathcal{J}^+)$ exists such that

$$u \operatorname{diag}\,(\,w\ p_1\ w^*, 0\,)\, u^* = \operatorname{diag}(p_1.0). \tag{2.4}$$

Assume that $a = \pi_4(u) \in \mathcal{U}_4(\mathcal{B}^+)$ and set $w_0 = a^*u \operatorname{diag}(w, 1_2)$. Then $\pi_4(w_0) = \operatorname{diag}(v, v^*, 1_2)$ and w_0 commutes with $\operatorname{diag}(p_1, 0)$ by (2.4). Therefore w_0 has the form $\operatorname{diag}(w_1, w_2)$ where $w_1 \in \mathcal{U}(\mathcal{A}^+)$, $w_2 \in \mathcal{U}_3(\mathcal{A}^+)$. This indicates that $[v] = \pi_*([w_1])$. $\square$

Remark 2.1. Lemma 2.2 somewhat generalizes Theorem 2 of [**8**]. We should notice that, if (2.1) is split exact, we can deduce that j_*

is injective and π_* is surjective in (2.3) by means of [**1**], without the hypothesis that $\mathcal{J}$ has 1-cancellation.

We now present our main result of the paper as follows.

Theorem 2.1. *For the C^*-algebra $\mathcal{A}$, $i_{\mathcal{A}}$ is injective if and only if $S\mathcal{A}$ has* 1*-cancellation.*

Proof. $\Leftarrow$. Since $C\mathcal{A} = C_0([0,1),\mathcal{A})$ is contractible and $S\mathcal{A}$ has 1-cancellation, applying Lemma 2.2 to the exact sequence of C^*-algebras

$$0 \longrightarrow S\mathcal{A} \longrightarrow C\mathcal{A} \xrightarrow{\pi} \mathcal{A} \longrightarrow 0, \tag{2.5}$$

we obtain that $\eta : U(\mathcal{A}) \to K_0(S\mathcal{A})$ is injective, where $\pi(f) = f(1)$ for all $f \in C\mathcal{A}$. Noting that $\eta = \partial \circ i_{\mathcal{A}}$ and $\partial : K_1(\mathcal{A}) \to K_0(S\mathcal{A})$ is the natural isomorphism given in [**1**, Theorem 8.2.2], we obtain that $i_{\mathcal{A}}$ is injective.

$\Rightarrow$. By Lemma 1.1 (2), we only need to prove that if $e \in PL(\mathcal{A})$ with $\operatorname{diag}(e, 1_k) \sim \operatorname{diag}(p_1, 1_k)$ in $M_{k+2}((S\mathcal{A})^+)$ for some k, then $e \sim p_1$ in $M_k((S\mathcal{A})^+)$.

Applying Lemma 2.1 to the above 1-projective loop e, we obtain that there is a continuous map $u_t : [0,1] \to \mathcal{U}_2(\mathcal{A}^+)$ such that $u_0 = 1_2 = \rho_2(u_t)$ and $e_t = u_t^* p_1 u_t$. Therefore, u_1 has the form $\operatorname{diag}(a,b)$ where $a, b \in \mathcal{U}(\mathcal{A}^+)$ with $\rho(a) = \rho(b) = 1$. Since $i_{\mathcal{A}}$ is injective and

$$[\operatorname{diag}(a^*, b^*, 1)] = [\operatorname{diag}(a^*, b^*)\operatorname{diag}(b^*, b)] = [u_1^*] = [1_2]$$

in $K_1(\mathcal{A})$, we get that there is a path s_t from 1 to $a^* b^*$ in $\mathcal{U}(\mathcal{A}^+)$ with $\rho(s_t) = 1$, $0 \le t \le 1$. Put $w_t = \operatorname{diag}(1, s_t) u_t$. Then $w_0 = 1_2 = \rho_2(w_t)$, $0 \le t \le 1$ and w_t is a path from 1_2 to $\operatorname{diag}(a, a^*)$ in $\mathcal{U}_2(\mathcal{A}^+)$ such that

$$e_t = u_t^* p_1 u_t = w_t^* p_1 w_t \quad \forall\, t \in [0,1]. \tag{2.6}$$

Now applying (2.2) to (2.5) and (2.6), we get that

$$\eta([\alpha]) = \partial \circ i_{\mathcal{A}}([a]) = [e] - [p_1] = 0 \quad \text{in } K_0(S\mathcal{A})$$

for $\operatorname{diag}(e, 1_k) \sim \operatorname{diag}(p_1, 1_k)$ in $M_{2+k}((S\mathcal{A})^+)$ and $\pi_2(w) = w_1 = \operatorname{diag}(a, a^*)$. Thus $i_{\mathcal{A}}([a]) = 0$ in $K_1(\mathcal{A})$ and $a \in \mathcal{U}_0(\mathcal{A}^+)$ because

$i_{\mathcal{A}}$ is injective by assumption. Let a_t be a path in $\mathcal{U}_0(\mathcal{A}^+)$ with $\rho(a_t) = 1$ from 1 to a, and put $c_t = w_t^* \mathrm{diag}\,(a_t, a_t^*)$, $0 \le t \le 1$. Then $c \in \mathcal{U}_2((S\mathcal{A})^+)$ with $\rho(c_t) = 1_2$ such that $c_t^* e(t) c_t = p_1$, $0 \le t \le 1$, i.e., $e \sim p_1$ in $M_2((S\mathcal{A})^+)$. ◻

Theorem 2.4 yields the following important results.

Corollary 2.1. *Let $\mathcal{A}$ be a C^*-algebra. Then $i_{M_n(\mathcal{A})}$ is injective for all $n \ge 1$ if and only if* $\mathrm{gsr}\,(S\mathcal{A}) = 1$.

Proof. $\Rightarrow$. By Theorem 2.4, $(S(M_n(\mathcal{A})))^+$ has 1-cancellation for all $n \ge 1$. Thus $\mathcal{U}_2((S(M_n(\mathcal{A})))^+)$ acts transitively on $S_2((S(M_n(\mathcal{A})))^+)$ by Lemma 1.1 (1). Now, from the split exact sequence of C^*-algebras,

$$0 \longrightarrow S(M_n(\mathcal{A})) \longrightarrow M_n((S\mathcal{A})^+) \xrightarrow{\rho_n} M_n(\mathbf{C}) \longrightarrow 0 \tag{2.7}$$

we get that $\mathcal{U}_2(M_n((S\mathcal{A})^+))$ acts transitively on $S_2(M_n((S\mathcal{A})^+))$. Thus $\mathrm{gsr}\,(S\mathcal{A}) \le 2$ by Proposition 1.1 (1). Since $(S\mathcal{A})^+$ is a finite C^*-algebra, we have $\mathrm{gsr}\,(S\mathcal{A}) = 1$.

$\Leftarrow$. By Corollary 1.5, $M_n((S\mathcal{A})^+)$ has 1-cancellation for all $n \ge 1$. So, by (2.7), $(S(M_n(\mathcal{A})))^+$ has 1-cancellation for all $n \ge 1$. Consequently, we have $i_{M_n(\mathcal{A})}$ is injective by Theorem 2.1. ◻

Corollary 2.2 [**12**, Theorem 2.9]. *Let $\mathcal{A}$ be a unital C^*-algebra, and*

$$r = \max\{\mathrm{gsr}\,(\Omega(\mathcal{A})), \mathrm{csr}\,(\mathcal{A})\}.$$

Then for all $n \ge \max(2, r)$, $i_{M_{n-1}(\mathcal{A})}$ is an isomorphism.

Proof. By [**11**, Theorem 10.10], $i_{M_{n-1}(\mathcal{A})}$ is surjective for any $n \ge \mathrm{csr}\,(\mathcal{A})$. Since it is a routine to check that $\mathrm{gsr}\,(\Omega(\mathcal{A})) = \max\{\mathrm{gsr}\,(S\mathcal{A}), \mathrm{gsr}\,(\mathcal{A})\}$ from the split exact sequence of C^*-algebras

$$0 \longrightarrow S\mathcal{A} \longrightarrow \Omega(\mathcal{A}) \longrightarrow \mathcal{A} \longrightarrow 0,$$

we have $\mathrm{gsr}\,(S\mathcal{A}) \le n$. Thus $\mathrm{Gsr}\,(M_{n-1}((S\mathcal{A})^+)) = 1$ by Proposition 1.1. It follows from (2.7) and Theorem 2.1 that $i_{M_{n-1}(\mathcal{A})}$ is injective. The proof is completed. ◻

3. Some applications. A projection p in the C^*-algebra $\mathcal{A}$ is called to be infinite if there is a $v \in \mathcal{A}$ such that $vv^* < v^*v = p$. A simple C^*-algebra $\mathcal{A}$ is said to be purely infinite if $\overline{x\mathcal{A}x}$, the closure of $x\mathcal{A}x$ in $\mathcal{A}$, contains an infinite projection for any positive element $x \in \mathcal{A}$, cf. [**3**]. Recall that a C^*-algebra $\mathcal{A}$ has the property $RR(\mathcal{A}) = 0$ if every self-adjoint element in $\mathcal{A}$ can be approximated by a self-adjoint element with finite spectra in $\mathcal{A}$, cf. [**2**].

Proposition 3.1. *Let $\mathcal{A}$ be a C^*-algebra with unit* 1*.*

(1) *If $RR(\mathcal{A}) = 0$, then* $\operatorname{gsr}(\Omega(\mathcal{A})) = \operatorname{gsr}(\mathcal{A})$;

(2) *Assume that $\mathcal{A}$ is purely infinite simple. If* [1] *is torsion-free in $K_0(\mathcal{A})$, then* $\operatorname{gsr}(\Omega(\mathcal{A})) = 2$; *if* [1] *has torsion, then* $\operatorname{gsr}(\Omega(\mathcal{A})) = \infty$*.*

Proof. (1) Since $RR(\mathcal{A}) = 0$ indicates that $RR(M_n(\mathcal{A})) = 0$ for all n by [**2**, Theorem 2.10], it follows from [**5**, Lemma 2.2] that $i_{M_n(\mathcal{A})}$ is injective. Therefore, by Corollary 2.1, $\operatorname{gsr}(S\mathcal{A}) = 1$ and hence $\operatorname{gsr}(\Omega(\mathcal{A})) = \operatorname{gsr}(\mathcal{A})$.

(2) By [**16**, Theorem 1.3] and assertion (1), $\operatorname{gsr}(S\mathcal{A}) = 1$. If [1] is torsion-free in $K_0(\mathcal{A})$ we have $\operatorname{csr}(\mathcal{A}) = 2$ by [**15**, Theorem 1]. Thus $\operatorname{gsr}(\mathcal{A}) \leq \operatorname{csr}(\mathcal{A}) = 2$. Since $\mathcal{A}$ contains an isometry, we have $\operatorname{gsr}(\mathcal{A}) = 2$. Therefore $\operatorname{gsr}(\Omega(\mathcal{A})) = 2$.

If $k \geq 1$ is the order of [1] in $K_0(\mathcal{A})$, then for any integer n with $n \equiv 1 \bmod k$, we can find n isometries $S_1, \dots, S_n$ in $\mathcal{A}$ such that $\sum_{i=1}^n S_i S_i^* = 1$ by the proof of [**3**, Lemma 1.8]. Clearly, $(S_1^*, \dots, S_n^*)^T \in S_n(\mathcal{A})$ and $(S_1^*, \dots, S_n^*)^T \neq u(1, 0, \dots, 0)^T$ for any $u \in \mathcal{U}_n(\mathcal{A})$. Thus we have $\operatorname{gsr}(\mathcal{A}) = \infty$ and hence $\operatorname{gsr}(\Omega(\mathcal{A})) = \infty$. □

In the following we will consider the $i_{\mathcal{A}\otimes\mathcal{B}}$ when $\mathcal{A}$ is purely infinite simple C^*-algebra or is a stable corona algebra and $\mathcal{B}$ is a nuclear C^*-algebra.

Proposition 3.2. *Suppose that $\mathcal{A}$ is a nonunital purely infinite simple C^*-algebra and $\mathcal{B}$ is a nuclear C^*-algebra. Then $i_{\mathcal{A}\otimes\mathcal{B}}$ is an isomorphism.*

Proof. We first prove that, for any $(a_1, \dots, a_n)^T \in \mathcal{A}^n$ and any $\varepsilon > 0$ there are a nonunital hereditary C^*-subalgebra $\mathcal{D}$ of $\mathcal{A}$ and $(b_1, \dots, b_n)^T \in \mathcal{D}^n$ such that $\|a_i - b_i\| \le (4\varepsilon/5)$, $1 \le i \le n$.

In fact, since $\mathcal{A}$ is nonunital, purely infinite and simple, it follows from [**6**, Condition (ii)] and [**2**, Theorem 2.6] that there is a projection r in $\mathcal{A}$ such that

$$\left\| \sum_{i=1}^{n} (a_i^* a_i + a_i a_i^*)(1-r) \right\| < \frac{\varepsilon^2}{25}. \tag{3.1}$$

Since $(1-r)\mathcal{A}(1-r)$ is purely infinite simple by [**16**, Theorem 1.3], we can find a sequence of pairwise orthogonal projections $\{R_i\}$ in $\mathcal{A}$ with $R_i < 1-r$. Set $x = \sum_{i=1}^{\infty} 2^{-i} R_i$. Then $0 \le x < 1-r$ and $\overline{x\mathcal{A}x}$ has no unit and, furthermore, $\mathcal{D} = \overline{(r+x)\mathcal{A}(r+x)} \subset \mathcal{A}$ has no unit as well, cf. the proof of [**15**, Theorem 2]. Set $b_i = (r+x)a_i(r+x) \in \mathcal{D}$. Then, from (3.1), we get that for $i = 1, \dots, n$,

$$\begin{aligned} \|a_i - b_i\| &= \|a_i(1-r-x) + (1-r-x)a_i(r+x)\| \\ &\le \|a_i(1-r)\| + \|a_i(1-r)\|\|x\| + \|(1-r)a_i\| \\ &\quad + \|x\|\|(1-r)a_i\|\|r+x\| \\ &\le \frac{4\varepsilon}{5}. \end{aligned}$$

Now for any $(a_1 + \lambda_1, \dots, a_n + \lambda_n)^T \in \mathrm{Lg}_n((\mathcal{A} \otimes \mathcal{B})^+)^n$, there are a nonunital hereditary C^*-subalgebra $\mathcal{D} \subset \mathcal{A}$ and $(b_1 + \lambda_1, \dots, b_n + \lambda_n)^T \in \mathrm{Lg}_n((\mathcal{A} \otimes \mathcal{B})^+)^n$ such that $\|a_i - b_i\| \le (4\varepsilon/5)$, $i = 1, \dots, n$, by the above argument. Noting that $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{K}$ by [**16**, Theorem 1.2], where $\mathcal{K}$ is the algebra of all compact operators on the separable Hilbert space H over the field $\mathbf{C}$, we obtain that there is a $(c_1 + \mu_1, \dots, c_n + \mu_n)^T \in \mathrm{Lg}_n((\mathcal{D} \otimes B)^+)$ such that $\|(b_i + \lambda_i) - (c_i + \mu_i)\| \le (\varepsilon/5)$, $1 \le i \le n$, by [**11**, Theorem 6.4]. Consequently, $\|(a_i + \lambda_i) - (c_i + \mu_i)\| \le \varepsilon$, $1 \le i \le n$. This means that $\mathrm{tsr}\,(\mathcal{A} \otimes \mathcal{B}) \le 2$. So $\mathrm{gsr}\,(\mathcal{A} \otimes \mathcal{B}) \le \mathrm{csr}\,(\mathcal{A} \otimes \mathcal{B}) \le \mathrm{tsr}\,(C([0,1]) \otimes \mathcal{A} \otimes \mathcal{B}) \le 2$ by [**9**, Lemma 2.4]. Finally we have that $i_{\mathcal{A} \otimes \mathcal{B}}$ is an isomorphism by Corollary 2.2. $\square$

Let $\mathcal{A}$ be a nonunital C^*-algebra. We denote by $M(\mathcal{A})$ the multiplier algebra of $\mathcal{A}$, cf. [**10**] and $SM(\mathcal{A}) = M(\mathcal{A} \otimes \mathcal{K})$ the stable multiplier algebra of $\mathcal{A}$. Set $SQ(\mathcal{A}) = M(\mathcal{A}\otimes\mathcal{K})/\mathcal{A}\otimes\mathcal{K}$ (the stable corona algebra of $\mathcal{A}$).

The following proposition gives a simple proof of $U(SQ(\mathcal{A}) \otimes \mathcal{B}) \cong K_1(SQ(\mathcal{A}) \otimes \mathcal{B})$ for certain $\mathcal{A}$ and $\mathcal{B}$ obtained by Thomsen for the quasi-unitary group, cf. [**14**, Theorem 4.9].

Proposition 3.3. *Let $\mathcal{A}$ be a C^*-algebra with unit* 1 *or a countable approximate identity consisting of projections and $\mathcal{B}$ a nuclear C^*-algebra. Then $i_{SQ(\mathcal{A})\otimes B}$ is an isomorphism.*

In order to prove this proposition, we need a lemma.

Lemma 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra which contains a pair of orthogonal isometries. Then $i_{\mathcal{A}}$ is surjective.*

Proof. Let 1 be a unit of $\mathcal{A}$ and S_1, S_2 two isometries in $\mathcal{A}$ such that $S_1S_1^* + S_2S_2^* = p$ is a projection in $\mathcal{A}$. For $n \geq 2$, set $T_1 = S_1^{n-1}$, $T_2 = S_1^{n-2}S_2, \dots, T_{n-1} = S_1S_2$, $T_n = S_2$. Then it is easy to verify that $T_i^*T_i = 1$ and $T_j^*T_i = 0$, $i \neq j$, $i, j = 1, \dots, n$. So $q_n = \sum_{i=1}^n T_iT_i^*$ is a projection in $\mathcal{A}$. Set

$$X = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_n(\mathcal{A}),$$

$$Y = \begin{bmatrix} X & 1_N - XX^* \\ 0 & X^* \end{bmatrix} \in \mathcal{U}_{2n}(\mathcal{A}).$$

Then $X^*X = 1_n$ and

$$Y \operatorname{diag}(u, 1_n) Y^* = \operatorname{diag}(b, 1_{2n-1}), \tag{3.2}$$

where $u = (u_{ij})n \times n \in \mathcal{U}_n(\mathcal{A})$ and $b = \sum_{i,j=1}^n T_iu_{ij}T_j^* + 1 - q_n \in \mathcal{U}(\mathcal{A})$. Since

$$Y = \begin{bmatrix} 1_n & X \\ 0 & 1_n \end{bmatrix} \begin{bmatrix} 0 & 1_n - 2XX^* \\ 1_n & 0 \end{bmatrix} \begin{bmatrix} 1_n & X^* \\ 0 & 1_n \end{bmatrix} \begin{bmatrix} 1_n & 0 \\ -X & 1_n \end{bmatrix} \in \mathcal{U}_{2n}^0(\mathcal{A}),$$

it follows from (3.2) that $[u] = i_{\mathcal{A}}([b])$ in $K_1(\mathcal{A})$. $\square$

Corollary 3.1. *Let $\mathcal{A}$ be a unital purely infinite simple C^*-algebra and $\mathcal{B}$ a nuclear C^*-algebra. Then $i_{\mathcal{A}\otimes\mathcal{B}}$ is surjective.*

Proof. Obviously, $\mathcal{A}\otimes\mathcal{B}$ contains a pair of orthogonal isometries if $\mathcal{B}$ is unital since $\mathcal{A}$ has this property by the definition of the purely infinite simple C^*-algebra. Thus the assertion follows.

If $\mathcal{B}$ is nonunital, then from the following split exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{A}\otimes\mathcal{B} \longrightarrow \mathcal{A}\otimes\mathcal{B}^+ \longrightarrow \mathcal{A} \longrightarrow 0 \tag{3.3}$$

and Remark 2.1, we get that the diagram of exact sequences

(3.4)
$$\begin{array}{ccccccccc}
[1] & \longrightarrow & U(\mathcal{A}\otimes\mathcal{B}) & \longrightarrow & U(\mathcal{A}\otimes\mathcal{B}^+) & \longrightarrow & U(\mathcal{A}) & \longrightarrow & [1] \\
 & & \downarrow{\scriptstyle i_{\mathcal{A}\otimes\mathcal{B}}} & & \downarrow{\scriptstyle i_{\mathcal{A}\otimes\mathcal{B}^+}} & & \downarrow{\scriptstyle i_{\mathcal{A}}} & & \\
0 & \longrightarrow & K_1(\mathcal{A}\otimes\mathcal{B}) & \longrightarrow & K_1(\mathcal{A}\otimes\mathcal{B}^+) & \longrightarrow & K_1(\mathcal{A}) & \longrightarrow & 0
\end{array}$$

is commutative. Since $i_{\mathcal{A}\otimes\mathcal{B}^+}$ is surjective and $i_{\mathcal{A}}$ is injective by [**3**, Lemma 1.8], we can deduce from (3.4) that $i_{\mathcal{A}\otimes\mathcal{B}}$ is surjective. ◻

Proof of Proposition 3.3. We first assume that $\mathcal{B}$ has unit $1_{\mathcal{B}}$. Since $\operatorname{gsr}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{B}) \le \operatorname{csr}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{B}) \le 2$ by [**13**, Theorem 3.10], we have that $\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{B}$ has 1-cancellation. Applying Lemma 2.2 to the exact sequence of C^*-algebras,

$$0 \longrightarrow \mathcal{A}\otimes\mathcal{K}\otimes\mathcal{B} \overset{j\otimes id_{\mathcal{B}}}{\longrightarrow} M(\mathcal{A}\otimes\mathcal{K})\otimes\mathcal{B} \overset{\pi\otimes id_{\mathcal{B}}}{\longrightarrow} SQ(\mathcal{A})\otimes\mathcal{B} \longrightarrow 0,$$

we obtain the following commutative diagram of exact sequences of groups

(3.5)
$$\begin{array}{ccccccc}
\longrightarrow & U(SM(\mathcal{A})\otimes\mathcal{B}) & \longrightarrow & U(SQ(\mathcal{A})\otimes\mathcal{B}) & \overset{\eta}{\longrightarrow} & K_0(\mathcal{A}\otimes\mathcal{B}) \\
 & \downarrow & & \downarrow{\scriptstyle i_{SQ(\mathcal{A})\otimes\mathcal{B}}} & & \| \\
\longrightarrow & K_1SM(\mathcal{A})\otimes\mathcal{B} & \longrightarrow & K_1(SQ(\mathcal{A})\otimes\mathcal{B}) & \overset{\partial}{\longrightarrow} & K_0(\mathcal{A}\otimes\mathcal{B}).
\end{array}$$

Now the hypotheses on $\mathcal{A}$ and $\mathcal{B}$ indicate that $U(SM(\mathcal{A})\otimes\mathcal{B}))=0$ by [**7**, Theorem 2.5]. Thus $i_{SQ(\mathcal{A})\otimes\mathcal{B}}$ is injective by (3.5).

Since $\mathbf{C}1\otimes L(H)\subset SM(\mathcal{A})$ where $L(H)$ is the algebra of all linear bounded operators on H, we can pick two isometries S_1, S_2 in $L(H)$ such that $S_1S_1^*+S_2S_2^*=I_H$. Thus $SQ(\mathcal{A})\otimes\mathcal{B}$ contains isometric $T_i=(\pi\otimes id_B)(1\otimes S_i\otimes 1_{\mathcal{B}})$, $i=1,2$, with $T_1T_1^*+T_2T_2^*=1\otimes 1_{\mathcal{B}}$. So $i_{SQ(\mathcal{A})\otimes\mathcal{B}}$ is surjective by Lemma 3.3.

If $\mathcal{B}$ has no unit, then replacing $\mathcal{A}$ by $SM(\mathcal{A})$ in (3.3), we also get that $i_{SQ(\mathcal{A})\otimes B}$ is an isomorphism. □

Remark 3.1. We have known from [**15**, Corollary 2.5] that if $\mathcal{A}$ is a σ-unital purely infinite simple C^*-algebra, then $SQ(\mathcal{A})$ is a unital purely infinite simple C^*-algebra. In this case $i_{SQ(\mathcal{A})\otimes\mathcal{B}}$ is an isomorphism if $\mathcal{B}$ is a nuclear C^*-algebra. We also notice that, using the same method as that in the proof of [**14**, Theorem 4.3], we can prove that $i_{\mathcal{O}_n\otimes\mathcal{B}}$ is an isomorphism where $\mathcal{O}_n$, $2\le n\le\infty$, is the Cuntz algebra and $\mathcal{B}$ is any C^*-algebra. Combining these facts with Corollary 3.1 and Proposition 3.2, we could raise a question: Is $i_{\mathcal{A}\otimes\mathcal{B}}$ always injective for any unital purely infinite simple C^*-algebra and any nuclear C^*-algebra $\mathcal{B}$?

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Department of Mathematics, East China University of Science and Technology, Shanghai 200237, P.R. China