

**ON THE LOCAL CAUCHY PROBLEM
FOR HAMILTON JACOBI EQUATIONS
WITH A FUNCTIONAL DEPENDENCE**

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ABSTRACT. The Cauchy problem for a nonlinear functional differential equation is considered. A theorem on the existence of classical solutions defined on the Haar pyramid is proved. The method of differential inequalities is used.

Differential equations with a deviated argument and differential integral equations can be obtained by specializing given operators.

1. Introduction. We will denote by $C(X, Y)$ the class of all continuous functions from X into Y where X and Y are metric spaces. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. For $y = (y_1, \dots, y_n) \in R^n$, we put $\|y\| = |y_1| + \dots + |y_n|$. Let E be the Haar pyramid

$$E = \{(x, y) \in R^{1+n} : x \in [0, a], -b + Mx \leq y \leq b - Mx\},$$

where $b, M \in R_+^n$, $R_+ = [0, +\infty)$, $b = (b_1, \dots, b_n)$, $M = (M_1, \dots, M_n)$ and $b - Ma > 0$. Write

$$E_0 = [-r_0, 0] \times [-b, b] \quad \text{where } r_0 \in R_+, \quad E^* = E_0 \cup E$$

and

$$E_x = E^* \cap ([-r_0, x] \times R^n), \quad \tilde{E}_x = E \cap ([0, x] \times R^n)$$

where $0 \leq x \leq a$. Suppose that $f : E \times C(E^*, R) \times R^n \rightarrow R$ and $\varphi : E_0 \rightarrow R$ are given functions. Consider the Cauchy problem

- (1) $\partial_x z(x, y) = f(x, y, z, \nabla_y z(x, y)),$
- (2) $z(x, y) = \varphi(x, y) \quad \text{on } E_0,$

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where $\nabla_y z = (\partial_{y_1} z, \dots, \partial_{y_n} z)$. In the paper we consider classical solutions of the above problem, i.e., functions $u \in C(E_c, R)$, $0 < c \leq a$, having partial derivatives $\partial_x u, \nabla_y u$ on \tilde{E}_c and satisfying equation (1) on \tilde{E}_c and initial condition (2). We assume that the operator f satisfies the following Volterra condition: for every point $(x, y) \in E$ there is a set $E[x, y]$ such that

$$(i) \quad E[x, y] \subset E_x,$$

(ii) if $z, \bar{z} \in C(E^*, R)$ and $z(t, s) = \bar{z}(t, s)$ for $(t, s) \in E[x, y]$, then $f(x, y, z, q) = f(x, y, \bar{z}, q)$, $q = (q_1, \dots, q_n) \in R^n$.

Note that the Volterra condition means the following property of the operator f : the value of f at the point (x, y, z, q) depends on (x, y, q) and on the restriction of the function z to the set $E[x, y]$ only.

Example 1.1. Consider the equation with a deviated argument

$$(3) \quad \partial_x z(x, y) = F(x, y, z(x, y), z(\alpha(x, y), \beta(x, y)), \nabla_y z(x, y))$$

where $F : E \times R^2 \times R^n \rightarrow R$, $\alpha : E \rightarrow R$, $\beta : E \rightarrow R^n$ and $(\alpha(x, y), \beta(x, y)) \in E_0 \cup E$ for $(x, y) \in E$. We assume that $-r_0 \leq \alpha(x, y) \leq x$ on E . Equation (1.3) can be derived from (1.1) by putting

$$f(x, y, z, q) = F(x, y, z(x, y), z(\alpha(x, y), \beta(x, y)), q).$$

Then $E[x, y] = \{(x, y), (\alpha(x, y), \beta(x, y))\}$.

For the above F consider the differential-integral equation

$$(4) \quad \partial_x z(x, y) = F(x, y, z(x, y), \int_{-b+Mx}^{b-Mx} z(x-r_0, s) ds, \nabla_y z(x, y)).$$

Then

$$E[x, y] = \{(t, s) : t = x - r_0, s \in [-b + Mx, b - Mx]\} \cup \{(x, y)\}.$$

Recently numerous papers were published concerning initial problems for nonlinear functional differential equations or systems. The first group of results ([21], [22]) is connected with global initial problems for equations

$$(5) \quad \partial_x z(x, y) = G(x, y, z, \nabla_y z(x, y))$$

(or adequate hyperbolic systems) where the variable z represents the functional argument. Existence results for (5) can be characterized as follows: theorems have simple assumptions and their proofs are very natural ([21], [22]). Unfortunately a small class of functional differential equations is covered by this theory. The results given in [21], [22] are not applicable to differential-integral equations of the Volterra type and to equations with a deviated argument.

There are a lot of papers concerning initial value problems for equations

$$(6) \quad \partial_x z(x, y) = H(x, y, (Vz)(x, y), \nabla_y z(x, y))$$

where V is an operator of the Volterra type and H is defined on finite-dimensional Euclidean space. The main assumptions in existence theorems for (6) concern the operator V . They are formulated in the form of norm inequalities in appropriate function spaces ([3], [10]). These inequalities are linear, which is the main shortcoming of the theory.

A new model of a functional dependence in partial differential equations is proposed in [8], [11].

The formulation is as follows.

Let $B = [-r_0, 0] \times [-\tau, \tau]$ where $r_0 \in R_+$ and $\tau = (\tau_1, \dots, \tau_n) \in R_+^n$. For a function $z : [-r_0, a] \times R^n \rightarrow R$ and for a point $(x, y) \in [0, a] \times R^n$ we define a function $z_{(x,y)} : B \rightarrow R$ by

$$(7) \quad z_{(x,y)}(t, s) = z(x + t, y + s), \quad (t, s) \in B.$$

Suppose that $F : [0, a] \times R^n \times C(B, R) \times R^n \rightarrow R$ and $\psi : [-r_0, 0] \times R^n \rightarrow R$ are given functions. Consider the initial problem global with respect to y

$$(8) \quad \partial_x z(x, y) = F(x, y, z_{(x,y)}, \nabla_y z(x, y))$$

$$(9) \quad z(x, y) = \psi(x, y) \quad \text{on } [-r_0, 0] \times R^n.$$

Several differential equations, differential equations with a deviated argument, differential-integral equations and functional differential equations with operators of Volterra type can be derived from (8) by specializing the operator F . The paper [11] contains a survey of results for problems (8), (9).

More detailed comparisons between different models of functional dependence are presented in [11].

Generalized solutions of nonlinear systems in the Cinquini-Cibrario sense ([5]) were considered in [14], [15]. Existence results for the Cauchy problem were obtained in these papers by using iterative methods. All these results are global with respect to y .

Initial-boundary value problems for differential-integral equations were considered in [16]. The method of semi-groups of linear operators is used. The functional dependence in equations considered in [16] concerns the first variable only. The spatial variable in the unknown function appears in a classical sense.

For further bibliography on hyperbolic functional differential problems, see the survey paper [11].

Now we present relations between local and global (with respect to y) initial problems for differential and functional differential equations.

Let E be the Haar pyramid, and suppose that $\tilde{F} : E \times R^{1+n} \rightarrow R$ and $\tilde{\varphi} : [-b, b] \rightarrow R$ are given functions. Consider the Cauchy problem without the functional dependence

$$(10) \quad \begin{aligned} \partial_x z(x, y) &= \tilde{F}(x, y, z(x, y), \nabla_y z(x, y)), \\ z(0, y) &= \tilde{\varphi}(y) \quad \text{for } y \in [-b, b]. \end{aligned}$$

We now formulate the following assumptions on \tilde{F} and $\tilde{\varphi}$.

Assumption \tilde{H} . *Suppose that*

1) *the function \tilde{F} of the variables (x, y, p, q) and its partial derivatives*

$$\nabla_y \tilde{F} = (\partial_{y_1} \tilde{F}, \dots, \partial_{y_n} \tilde{F}), \quad \partial_p \tilde{F}, \quad \nabla_q \tilde{F} = (\partial_{q_1} \tilde{F}, \dots, \partial_{q_n} \tilde{F})$$

are continuous on $E \times R^{1+n}$ and

$$|\partial_{q_i} \tilde{F}(x, y, p, q)| \leq M_i \quad \text{on } E \times R^{1+n} \quad \text{for } 1 \leq i \leq n,$$

where the constants (M_1, \dots, M_n) appear in the definition of E ,

2) *the functions \tilde{F} , $\nabla_y \tilde{F}$, $\partial_p \tilde{F}$ are bounded and $\nabla_y \tilde{F}$, $\partial_p \tilde{F}$, $\nabla_q \tilde{F}$ satisfy the Lipschitz condition with respect to (y, p, q) on $E \times R^{1+n}$,*

3) the function $\tilde{\varphi}$ together with its derivatives $(\partial_{y_1}\tilde{\varphi}, \dots, \partial_{y_n}\tilde{\varphi}) = \nabla_y\tilde{\varphi}$ are continuous on $[-b, b]$ and $\nabla_y\tilde{\varphi}$ satisfies the Lipschitz condition.

Lemma 1.2. *If Assumption \tilde{H} is satisfied, then exactly one solution \tilde{u} exists of problem (10) on \tilde{E}_c for sufficiently small $c \in (0, a]$. The solution \tilde{u} is of class C^1 and $\nabla_y\tilde{u}$ satisfies the Lipschitz condition with respect to y on \tilde{E}_c . Moreover, the solution \tilde{u} depends continuously on given functions.*

We only give the main ideas of the proof.

The proof of the existence of a solution of (10) is divided into two steps.

(i) Assume additionally that \tilde{F} and $\tilde{\varphi}$ are of class C^2 on $E \times R^{1+n}$ and $[-b, b]$, respectively. Consider the characteristic system corresponding to (10)

$$y'(x) = -\nabla_q\tilde{F}(x, y(x), p(x), q(x)),$$

$$p'(x) = \tilde{F}(x, y(x), p(x), q(x)) - \sum_{i=1}^n q_i(x)\partial_{q_i}\tilde{F}(x, y(x), p(x), q(x)),$$

$$q'(x) = \nabla_y\tilde{F}(x, y(x), p(x), q(x)) + q(x)\partial_p\tilde{F}(x, y(x), p(x), q(x))$$

and its solutions $\tilde{y}(\cdot, \eta), \tilde{p}(\cdot, \eta), \tilde{q}(\cdot, \eta)$, $\eta \in [-b, b]$ satisfying the initial conditions

$$y(0) = \eta, \quad p(0) = \tilde{\varphi}(\eta), \quad q(0) = \nabla_y\tilde{\varphi}(\eta), \quad \eta \in [-b, b].$$

Let us denote by

$$\tilde{t}(\cdot, \eta) = [\tilde{t}_{ij}(\cdot, \eta)]_{i,j=1,\dots,n}$$

the characteristics of the second order corresponding to (10). They are solutions of a system of ordinary differential equations and satisfy the initial conditions

$$t_{ij}(0) = \partial_{y_i}\partial_{y_j}\tilde{\varphi}(\eta), \quad 1 \leq i, j \leq n, \quad \eta \in [-b, b].$$

The righthand sides of the system are polynomials of the second order with respect to t_{ij} . Details can be found in [17], [19].

We next claim that the equation $y = \tilde{y}(x, \eta)$ can be solved with respect to η . Let $\eta = \tilde{\eta}(x, y)$ be this solution. Define $\tilde{u}(x, y) = \tilde{p}(x, \tilde{\eta}(x, y))$. Then there is a $c \in (0, a]$ such that

- (a) \tilde{u} is of class C^2 on \tilde{E}_c ,
- (b) $\nabla_y \tilde{u}(x, y) = \tilde{q}(x, \tilde{\eta}(x, y))$, $\partial_{y_i} \partial_{y_j} \tilde{u}(x, y) = \tilde{t}_{ij}(x, \tilde{\eta}(x, y))$, $1 \leq i, j \leq n$ on \tilde{E}_c ,
- (c) \tilde{u} is the solution of (10) on \tilde{E}_c ,
- (d) the functions $\tilde{q}(\cdot, \eta)$, $\tilde{t}(\cdot, \eta)$ are bounded.

(ii) Now we consider the original assumptions on \tilde{F} and $\tilde{\varphi}$. Let $\{\tilde{F}^{(k)}\}$ and $\{\tilde{\varphi}^{(k)}\}$ be sequences of functions uniformly convergent to \tilde{F} and $\tilde{\varphi}$ and satisfying (i). Let us denote by $\{\tilde{u}^{(k)}\}$ the sequence of solutions of corresponding initial problems. All these solutions are given on \tilde{E}_c with $c \in (0, a]$ sufficiently small and independent on k . There exists a subsequence $\{\tilde{u}^{(k_i)}\}$ and a function \tilde{u} such that

$$\tilde{u} = \lim_{i \rightarrow \infty} \tilde{u}^{(k_i)}, \quad \partial_x \tilde{u} = \lim_{i \rightarrow \infty} \partial_x \tilde{u}^{(k_i)}, \quad \nabla_y \tilde{u} = \lim_{i \rightarrow \infty} \nabla_y \tilde{u}^{(k_i)},$$

uniformly on \tilde{E}_c . This \tilde{u} satisfies all the conditions of the lemma.

Uniqueness and continuous dependence of the solution \tilde{u} on given functions can be proved using classical theorems on differential inequalities.

Thus we see that initial problems for nonlinear equations have the following property: the proof of the existence of solutions of problem (10) (or adequate hyperbolic systems) and the existence results for the Cauchy problem which is global with respect to y ([17]) are based on the same ideas.

The situation is completely different for equations (or systems) with a functional dependence. Let us see why. We will consider equations involving a generalized Hale operator.

It follows from (7) that $z_{(x,y)} : B \rightarrow R$ is the restriction of the function z to the set $[x - r_0, x] \times [y - \tau, y + \tau]$ and this restriction is shifted to the set B . If $\tau \neq 0$, then there is an $(x, y) \in E$ such that $([x - r_0, x] \times [y - \tau, y + \tau]) \not\subset E_0 \cup E$. Therefore, the formulation (8), (9) is not suitable for the local Cauchy problem considered in the Haar pyramid and, consequently, the results of papers [8], [14], [15] are not applicable to problems (1), (2).

Until now there have not been any results on the existence of classical or generalized solutions to problems (1), (2). The aim of this paper is to prove a theorem on the existence of classical solutions of the problem. Uniqueness theorems with nonlinear estimates with respect to the functional variable can be found in [1], [4], [9], [12].

The proof of the existence of a solution to problem (1), (2), is based on the following idea. We construct the set X_c which is the closed subset of the Banach space consisting of all functions $z \in C(\tilde{E}_c, R)$, $0 < c \leq a$. For $u \in X_c$ consider the initial problem

$$(11) \quad \begin{aligned} \partial_x z(x, y) &= f(x, y, \tilde{u}, \nabla_y z(x, y)), \\ z(0, y) &= \varphi(0, y) \quad \text{for } y \in [-b, b], \end{aligned}$$

where $\tilde{u}(x, y) = u(x, y)$ on \tilde{E}_c and $\tilde{u}(x, y) = \varphi(x, y)$ on E_0 . Let $\tilde{v}(\cdot; u)$ denote the solution of (11). We formulate sufficient conditions for the existence and uniqueness of the solution $\tilde{v}(\cdot; u)$ of the above problem. We consider the operator U defined on X_c as follows: $Uu = \tilde{v}(\cdot; u)$. We prove that, under suitable assumptions of f and φ , there is a $0 < c \leq a$ such that $U : X_c \rightarrow X_c$ and U has exactly one fixed point \bar{u} on X_c . Let $\bar{v} : E_c \rightarrow R$ be a function given by $\bar{v} = \bar{u}$ on \tilde{E}_c and $\bar{v} = \varphi$ on E_0 . This \bar{v} is the classical solution of (1), (2). We use the ideas introduced in [19] for hyperbolic systems and generalized Cauchy problems.

Write $\Omega_0 = E \times R^n$ and suppose that $G : \Omega_0 \rightarrow R$ and $\omega : [-b, b] \rightarrow R$ are given functions. Consider the nonlinear equation

$$(12) \quad \partial_x z(x, y) = G(x, y, \nabla_y z(x, y))$$

with the initial condition

$$(13) \quad z(0, y) = \omega(y) \quad \text{for } y \in [-b, b].$$

The above method of the proof of the existence result for (1), (2) is based on the theorem on the existence of the solution of problem (12), (13) and on the estimates of the partial derivatives of the solution and the Lipschitz coefficients for the derivatives.

Now we state the auxiliary theorem.

Theorem 1.3. *Suppose that*

(i) *the function G of the variables (x, y, q) is continuous and bounded on Ω_0 ,*

(ii) *the derivatives $\nabla_y G$, $\nabla_q G$ exist on Ω_0 and $\nabla_y G$, $\nabla_q G \in C(\Omega_0, R^n)$,*

(iii) *there are $\bar{P}, P_0 \in R_+$ such that*

$$\begin{aligned} \|\nabla_y G(x, y, q)\| &\leq \bar{P}, \\ \|\nabla_y G(x, y, q) - \nabla_y G(x, \bar{y}, \bar{q})\| &\leq P_0[\|y - \bar{y}\| + \|q - \bar{q}\|], \\ \|\nabla_q G(x, y, q) - \nabla_q G(x, \bar{y}, \bar{q})\| &\leq P_0[\|y - \bar{y}\| + \|q - \bar{q}\|], \end{aligned}$$

and

$$|\partial_{q_i} G(x, y, q)| \leq M_i, \quad i = 1, \dots, n,$$

on Ω_0 ,

(iv) *the function $\omega : [-b, b] \rightarrow R$ is of class C^1 on R^n and*

$$\|\nabla_y \omega(y)\| \leq \bar{J}, \quad \|\nabla_y \omega(y) - \nabla_y \omega(\bar{y})\| \leq J\|y - \bar{y}\| \text{ on } [-b, b].$$

Under these assumptions the unique solution $v : \tilde{E}_\delta \rightarrow R$ of problems (12), (13) exists, where

$$\delta = \min \left\{ a, \frac{1}{P_0(1+J)} \right\}.$$

Moreover, the solution v satisfies the conditions

$$\begin{aligned} \|\nabla_y v(x, y) - \nabla_y v(x, \bar{y})\| &\leq \Gamma(x)\|y - \bar{y}\|, \\ \|\nabla_y v(x, y) - \nabla_y v(\bar{x}, y)\| &\leq [\bar{P} + P^*\Gamma(x)]|x - \bar{x}| \end{aligned}$$

and

$$\|\nabla_y v(x, y)\| \leq \bar{J} + \bar{P}x$$

on \tilde{E}_δ where

$$\Gamma(x) = \frac{P_0(1+J)x + J}{1 - P_0(1+J)x}, \quad P^* = \sum_{j=1}^n M_j.$$

We only give the main ideas of the proof of the theorem.

(i) First we assume additionally that G and ω are of class C^2 on Ω_0 and $[-b, b]$ respectively. Then the assumptions on the Lipschitz condition for $\nabla_y G, \nabla_q G$ and $\nabla_y \omega$ mean that adequate derivatives of the second order are bounded. Let us denote by

$$\tilde{y}(\cdot, \eta), \tilde{p}(\cdot, \eta), \tilde{q}(\cdot, \eta), \tilde{t}(\cdot, \eta) = [\tilde{t}_{ij}(\cdot, \eta)]_{i,j=1, \dots, n}$$

the solution of the system of ordinary differential equations consisting of the characteristic system

$$\begin{aligned} y'(x) &= -\nabla_q G(Q(x)), \\ p'(x) &= G(Q(x)) - \sum_{i=1}^n q_i(x) \partial_{q_i} G(Q(x)), \\ q'(x) &= \nabla_y G(Q(x)) \end{aligned}$$

where $Q(x) = (x, y(x), p(x), q(x))$ and the system

$$\begin{aligned} (14) \quad t'_{ij}(x) &= \sum_{k,r=1}^n \partial_{q_k} \partial_{q_r} G(Q(x)) t_{ki}(x) t_{rj}(x) + \partial_{y_i} \partial_{y_j} G(Q(x)) \\ &+ \sum_{k=1}^n [\partial_{q_k} \partial_{y_i} G(Q(x)) t_{kj}(x) + \partial_{q_k} \partial_{y_j} G(Q(x)) t_{ki}(x)], \\ &1 \leq i, j \leq n, \end{aligned}$$

satisfying the initial conditions

$$\begin{aligned} y(0) &= \eta, \quad p(0) = \omega(\eta), \quad q(0) = \nabla_y \omega(\eta), \\ t(0) &= [\partial_{y_i} \partial_{y_j} \omega(\eta)]_{i,j=1, \dots, n}, \end{aligned}$$

where $\eta \in [-b, b]$. It follows that the equation $y = \tilde{y}(x, \eta)$ has a unique solution with respect to $\eta : \eta = \tilde{\eta}(x, y)$ and $\tilde{\eta}$ is of class C^1 on \tilde{E}_δ . Write $v(x, y) = \tilde{p}(x, \tilde{\eta}(x, y))$, $(x, y) \in \tilde{E}_\delta$. Then

$$\begin{aligned} \nabla_y v(x, y) &= \tilde{q}(x, \tilde{\eta}(x, y)), \\ [\partial_{y_i} \partial_{y_j} v(x, y)]_{i,j=1, \dots, n} &= \tilde{t}(x, \tilde{\eta}(x, y)) \quad \text{on } \tilde{E}_\delta \end{aligned}$$

and v is the solution of problem (12), (13). Moreover, v is of class C^2 on \tilde{E}_δ . Write

$$\beta(x, \eta) = \max_{1 \leq j \leq n} \sum_{k=1}^n |\tilde{t}_{kj}(x, \eta)|.$$

It follows from (1.14) that β satisfies the differential inequality

$$D_- \beta(x, \eta) \leq P_0[\beta(x, \eta) + 1]^2 \quad \text{and} \quad \beta(0, \eta) \leq J,$$

where D_- is the lefthand lower Dini derivative with respect to x . Let $\tilde{\omega}$ denote the solution of the problem

$$\zeta'(x) = P_0[\zeta(x) + 1]^2, \quad \zeta(0) = J.$$

Then $\beta(x, \eta) \leq \tilde{\omega}(x)$. Since $\tilde{\omega}(x) = \Gamma(x)$ it follows that $\beta(x, \eta) \leq \Gamma(x)$ and consequently

$$(15) \quad \|\nabla_y \partial_{y_j} v(x, y)\| \leq \Gamma(x) \quad \text{on} \quad \tilde{E}_\delta.$$

In a similar way we prove that

$$(16) \quad \|\nabla_y v(x, y)\| \leq \bar{J} + \bar{P}x \quad \text{on} \quad \tilde{E}_\delta.$$

It is easily seen that

$$\begin{aligned} \|\nabla_y \partial_x v(x, y)\| &\leq \|\nabla_y G(x, \nabla_y v(x, y))\| \\ &\quad + \sum_{i,k=1}^n |\partial_{q_k} G(x, y, \nabla_y v(x, y)) \partial_{y_i} \partial_{y_k} v(x, y)|. \end{aligned}$$

According to (15), we have

$$(17) \quad \|\nabla_y \partial_x v(x, y)\| \leq \bar{P} + P^* \Gamma(x) \quad \text{on} \quad \tilde{E}_\delta.$$

(ii) Now consider the original assumptions on G and ω . There are sequences $\{G^{(k)}\}$ and $\{\omega^{(k)}\}$ such that

- (a) $F^{(k)}$ and $\omega^{(k)}$ satisfy all the assumptions of Theorem 1.3.
- (b) $F^{(k)}$ and $\omega^{(k)}$ are of class C^2 on Ω_0 and $[-b, b]$, respectively.

(c) $\lim_{k \rightarrow \infty} G^{(k)} = G$ uniformly on Ω_0 and $\lim_{k \rightarrow \infty} \omega^{(k)} = \omega$ uniformly on $[-b, b]$.

Let $\{v^{(k)}\}$ be the sequence of solutions of adequate differential problems. The functions $v^{(k)}$ are defined on \tilde{E}_δ and satisfy (15)–(17). There is a subsequence $\{v^{(k_i)}\}$ and a function v such that

$$\lim_{k \rightarrow \infty} v^{(k_i)} = v, \quad \lim_{k \rightarrow \infty} \partial_x v^{(k_i)} = \partial_x v, \quad \lim_{k \rightarrow \infty} \nabla_y v^{(k_i)} = \nabla_y v$$

uniformly on \tilde{E}_δ . This v satisfies all the conditions of Theorem 1.3.

Lemma 1.4. *Suppose that all the assumptions of Theorem 1.3 are satisfied and*

- 1) $|G(0, y, q)| \leq \bar{J}$ on $[-b, b] \times R^n$,
- 2) the derivative $\partial_x G$ exists on Ω_0 and

$$|\partial_x G(x, y, q)| \leq P_0 \quad \text{on } \Omega_0.$$

Then the solution v of problems (12), (13), satisfies the conditions

$$|\partial_x v(x, y)| \leq \bar{J} + \bar{P}x$$

and

$$\begin{aligned} |\partial_x v(x, y) - \partial_x v(x, \bar{y})| &\leq [\bar{P} + P^* \Gamma(x)] \|y - \bar{y}\|, \\ |\partial_x v(x, y) - \partial_x v(\bar{x}, y)| &\leq [\bar{P} + P^*(\bar{P} + P^* \Gamma(x))] |x - \bar{x}| \end{aligned}$$

on \tilde{E}_δ .

We give comments on the proof of the lemma. Let us first consider a reduced form of the lemma. Suppose that G and ω are of class C^2 . Write $u = \partial_x v$. Then

$$|\partial_x u(x, y)| \leq P_0 + \sum_{i=1}^n M_i |\partial_{y_i} u(x, y)| \quad \text{on } \tilde{E}_\delta$$

and $|u(0, y)| \leq \bar{J}$ for $y \in [-b, b]$. It follows from a theorem on partial differential inequalities on the Haar pyramid ([23, Theorem 73.1], see also [13, Theorem 9.2.1]) that

$$(18) \quad |\partial_x v(x, y)| \leq \bar{J} + \bar{P}x \quad \text{on } \tilde{E}_\delta.$$

It is easily seen from (1.12) that

$$(19) \quad \|\nabla_y \partial_x v(x, y)\| \leq \bar{P} + P^* \Gamma(x)$$

and

$$(20) \quad |\partial_x \partial_x v(x, y)| \leq \bar{P} + P^*(\bar{P} + P^* \Gamma(x))$$

on \tilde{E}_δ .

We can now proceed analogously to step (ii) of the proof of Theorem 1.3. Details are omitted.

Remark 1.5. First order partial functional differential equations find applications in different fields of knowledge. Differential-integral systems have been proposed ([2]) as simple mathematical models for the nonlinear phenomenon of harmonic generation of laser radiation through piezoelectric crystals for nondispersive materials and of Maxwell-Hopkinson type. Systems of differential equations containing operators acting on an unknown density of populations in dependence on their age, size, and DNA content, are considered in [18]. An equation with a deviated argument ([6]) describes a density of households at time t , depending on their estates, in the theory of the distribution of wealth. Another system of integral-differential equations appears in mathematical biology in order to investigate an age-dependent epidemic of a disease with vertical transmissions [7]. The paper [20] deals with integral differential equations motivated by applications in the theory of screening of granular bodies.

2. Function spaces. We will denote by $\|\cdot\|_{(x;0)}$ the supremum norm in the space $C(E_x, R)$.

For any $x \in [0, a]$ we consider the following subspaces of the space $C(E_x, R)$. Let $C^{0,L}(E_x, R)$ be the class of all $z \in C(E_x, R)$ such that

$$[|z|]_{(x;L)} = \sup \left\{ \frac{|z(t, s) - z(\bar{t}, \bar{s})|}{|t - \bar{t}| + \|s - \bar{s}\|} : (t, s), (t, \bar{s}) \in E_x \right\}.$$

We will use the symbol $\|\cdot\|_{(x;0,L)}$ to denote the norm in the space $C^{0,L}(E_x, R)$, and we put

$$\|z\|_{(x;0,L)} = \|z\|_{(x;0)} + [|z|]_{(x;L)}.$$

Let us denote by $C^1(E_x, R)$ the class of all continuous functions $z : E_x \rightarrow R$ such that the derivatives $\partial_x z, \nabla_y z$ exists on E_x and $\partial_x z \in C(E_x, R), \nabla_y z \in C(E_x, R^n)$. For $z \in C^1(E_x, R)$, we put

$$\|z\|_{(x;1)} = \|z\|_{(x;0)} + \|\partial_x z\|_{(x;0)} + \|\nabla_y z\|_{(x;0)},$$

where

$$\|\nabla_y z\|_{(x;0)} = \max\{\|\nabla_y z(t, s)\| : (t, s) \in E_x\}.$$

Let $C^{1,L}(E_x, R)$ denote the class of all functions $z \in C^1(E_x, R)$ such that $\|z\|_{(x;1,L)} < +\infty$ where

$$\|z\|_{(x;1,L)} = \|z\|_{(x;1)} + [|\partial_x z|]_{(x;L)} + [|\nabla_y z|]_{(x;L)}$$

and

$$[|\nabla_y z|]_{(x;L)} = \sup \left\{ \frac{\|\nabla_y z(t, s) - \nabla_y z(\bar{t}, \bar{s})\|}{|t - \bar{t}| + \|s - \bar{s}\|} : (t, s), (\bar{t}, \bar{s}) \in E_x \right\}.$$

If $x = 0$, then $C(E_x, R)$ is the space of all initial functions for problems (1), (2). For simplicity of notation we use the symbols $\|\cdot\|_0, \|\cdot\|_{0,L}, \|\cdot\|_1, \|\cdot\|_{1,L}$ to denote the norms in the spaces

$$C(E_0, R), \quad C^{0,L}(E_0, R), \quad C^1(E_0, R), \quad C^{1,L}(E_0, R),$$

respectively. The symbol $\|\cdot\|_0$ will also denote the supremum norm in the space $C(E_0, R^n)$.

Let us denote by $CL(E_x, R)$ the set of all linear and continuous operators defined on $C(E_x, R)$ and taking values in R . The norm in the space $CL(E_x, R)$ will be denoted by $\|\cdot\|_{CL;x}$.

We will prove that, under suitable assumptions on f and φ and for sufficiently small $c \in (0, a]$ there exists a solution \bar{z} of problems (1), (2) such that $\bar{z} \in C^{1,L}(E_c, R)$.

3. Existence of classical solutions. We start with the formulation of assumptions on f and φ . Write $\Omega = E \times C(E^*, R) \times R^n$.

Assumption $H[f]$. Suppose that

(i) the function $f : \Omega \rightarrow R$ is continuous and there exists a nondecreasing function $\alpha : R_+ \rightarrow R_+$ such that

$$\|f(x, y, z, q)\| \leq \alpha(\|z\|_{(x;0)}) \quad \text{on } \Omega,$$

(ii) for each $P = (x, y, w, q) \in E \times C^1(E_x, R) \times R^n$ the following derivatives exist

$$\begin{aligned} \partial_x f(P), \quad \nabla_y f(P) &= (\partial_{y_1} f(P), \dots, \partial_{y_n} f(P)), \\ \nabla_q f(P) &= (\partial_{q_1} f(P), \dots, \partial_{q_n} f(P)) \end{aligned}$$

and the functions $\partial_x f, \nabla_y f, \nabla_q f$ are continuous on $E \times C^1(E^*, R) \times R^n$,

(iii) for each $P \in E \times C^1(E_x, R) \times R^n$ the Frechet derivative $\partial_z f(P)$ and $\partial_z f(P) \in CL(E_x, R)$ exist,

(iv) there exist positive constants C_0, C_1, C such that for each $P \in E \times C^1(E_x, R) \times R^n$, we have

$$|\partial_x f(P)|, \quad \|\nabla_y f(P)\| \leq C_0 + C_1 \|z\|_{(x;1)}, \quad \|\partial_z f(P)\|_{CL;c} \leq C,$$

and

$$|\partial_{q_i} f(P)| \leq M_i \quad \text{for } i = 1, \dots, n,$$

(v) there exists a nondecreasing function $\beta : R_+ \rightarrow R_+$ such that for

$$(x, y, q), (x, \bar{y}, \bar{q}) \in E \times R^n, \quad z \in C^{1,L}(E_x, R), \quad h \in C^1(E_x, R)$$

we have

$$\begin{aligned} &\|\nabla_y f(x, y, z, q) - \nabla_y f(x, \bar{y}, z + h, \bar{q})\| \\ &\leq \beta(\|z\|_{(x;1,L)})[\|y - \bar{y}\| + \|h\|_{(x;1)} + \|q - \bar{q}\|], \\ &\|\nabla_q f(x, y, z, q) - \nabla_q f(x, \bar{y}, z + h, \bar{q})\| \\ &\leq \beta(\|z\|_{(x;1,L)})[\|y - \bar{y}\| + \|h\|_{(x;1)} + \|q - \bar{q}\|] \end{aligned}$$

and

$$\begin{aligned} &\|\partial_z f(x, y, z, q) - \partial_z f(x, \bar{y}, z + h, \bar{q})\|_{CL;x} \\ &\leq \beta(\|z\|_{(x;1,L)})[\|y - \bar{y}\| + \|h\|_{(x;1)} + \|q - \bar{q}\|]. \end{aligned}$$

Assumption $H[\varphi]$. Suppose that

- (i) the function $\varphi : E_0 \rightarrow R$ is of class C^1 and $\|\partial_x \varphi\|_0 \leq B_1$, $\|\nabla_y \varphi\|_0 \leq B_1$,
- (ii) there is a $B_2 \in R_+$ such that $[\|\partial_x \varphi\|]_{(0;L)} \leq B_2$, $[\|\nabla_y \varphi\|]_{(0;L)} \leq B_2$.

Assumption $H[\varphi, f]$. If $r_0 > 0$, then the consistency condition

$$\partial_x \varphi(0, y) = f(0, y, \varphi, \nabla_y \varphi(0, y))$$

is satisfied for $y \in [-b, b]$.

Remark 3.1. It is important in Assumption $H[f]$ that we have assumed the local Lipschitz condition for the derivatives $\nabla_y f$, $\nabla_q f$, $\partial_z f$ on some special function spaces.

Let us consider the simplest assumptions. Suppose that there is an \bar{L} such that

$$(21) \quad \|\nabla_y f(x, y, z, q) - \nabla_y f(x, \bar{y}, z + h, \bar{q})\| \leq \bar{L}(\|y - \bar{y}\| + \|h\|_{(x;0)} + \|q - \bar{q}\|)$$

and that suitable inequalities for the derivatives $\nabla_q f$, $\partial_z f$ are satisfied.

Of course, our results are true under the above stronger assumptions. Now we show that the formulation (v) of Assumption $H[f]$ is important. More precisely, we show that there is a class of nonlinear equations satisfying (v) of Assumption $H[f]$ and not satisfying (21).

Example 3.2. Suppose that $F : E \times R^{1+n} \rightarrow R$, $\alpha : E \rightarrow R$, $\beta : E \rightarrow R^n$ are given functions, and consider the equation with a deviated argument

$$(22) \quad \partial_x z(x, y) = F(x, y, z(\alpha(x, y)), \beta(x, y)), \nabla_y z(x, y).$$

Assume that the functions F, α, β are continuous and that

- (i) there are the derivatives $\nabla_y F$, $\partial_p F$, $\nabla_q F$, the functions $\nabla_y F$, $\partial_p F$, $\nabla_q F$ are continuous, and there are $L_0, \tilde{L} \in R_+$ such that

$$\|\nabla_y F(P)\|, \quad |\partial_p F(P)|, \quad \|\nabla_q F(P)\| \leq L_0$$

for $P = (x, y, p, q) \in E \times R^{1+n}$ and that these derivatives satisfy the Lipschitz condition with respect to (y, p, q) with the constant \tilde{L} ,

(ii) for $(x, y) \in E$ we have

$$(\alpha(x, y), \beta(x, y)) \in E_0 \cup E \quad \text{and} \quad \alpha(x, y) \leq x,$$

(iii) the derivatives $\nabla_y \alpha$ and $\nabla_y \beta$ exist as continuous functions, and $\tilde{C}, \bar{C} \in R_+$ exist such that

$$\|\nabla_y \alpha(x, y)\| \leq \tilde{C}, \quad \|\nabla_y \beta(x, y)\| \leq \tilde{C}$$

where

$$\nabla_y \beta(x, y) = [\partial_{y_i} \beta_j(x, y)]_{i,j=1,\dots,n},$$

and the derivatives $\nabla_y \alpha, \nabla_y \beta$ satisfy the Lipschitz condition with respect to y with the constant \bar{C} .

Put $f(x, y, z, q) = F(x, y, z(\alpha(x, y), \beta(x, y)), q)$. Then equation (1) is equivalent to (22). We consider the function $\nabla_y f$ only. It follows that, for $z \in C^1(E_x, R)$, we have

$$\begin{aligned} \partial_{y_i} f(x, y, z, q) &= \partial_{y_i} F(Q) + \partial_p F(Q) \\ &\cdot \left[\partial_x z(Q_0) \partial_{y_i} \alpha(x, y) + \sum_{j=1}^n \partial_{y_j} z(Q_0) \partial_{y_j} z(Q_0) \partial_{y_i} \beta_j(x, y) \right] \end{aligned}$$

where $Q = (x, y, z(\alpha(x, y), \beta(x, y)), q)$ and $Q_0 = (\alpha(x, y), \beta(x, y))$.

For $(x, y, q), (x, \bar{y}, \bar{q}) \in E \times R^n$ and $z \in C^{1,L}(E_x, R)$, $h \in C^1(E_x, R)$, we get

$$\begin{aligned} &\|\nabla_y f(x, y, z, q) - \nabla_y f(x, \bar{y}, z + h, \bar{q})\| \\ &\leq \tilde{L}[1 + \tilde{C}(\|\partial_x z\|_{(x;0)} + \|\nabla_y z\|_{(x;0)})^2] \|y - \bar{y}\| \\ &\quad + \tilde{L}[1 + \tilde{C}(\|\partial_x z\|_{(x;0)} + \|\nabla_y z\|_{(x;0)})] (\|h\|_{(x;0)} + \|q - \bar{q}\|) \\ &\quad + L_0[2\tilde{C}^2(\|\partial_x z\|_{(x;L)} + \|\nabla_y z\|_{(x;L)}) \\ &\quad \quad + \bar{C}(\|\partial_x z\|_{(x;0)} + \|\nabla_y z\|_{(x;0)})] \|y - \bar{y}\| \\ &\quad + L_0 \tilde{C}[\|\partial_x h\|_{(x;0)} + \|\nabla_y h\|_{(x;0)}]. \end{aligned}$$

It follows from the above considerations that condition (v) of Assumption $H[f]$ is satisfied. We see at once that the function $\nabla_y f$ does not satisfy the global Lipschitz condition (21).

Theorem 3.3. *If Assumptions $H[f]$, $H[\varphi]$, $H[\varphi, f]$ are satisfied, then $c \in (0, a]$ exists such that problem (1), (2) has exactly one classical solution \bar{u} on E_c .*

Proof. Let us denote by $C(\tilde{E}_c, R; \lambda)$ where $c \in (0, a]$, the Banach space of all continuous functions from \tilde{E}_c into R with the norm

$$\|z\|_{[\lambda]} = \sup\{\|z(x, y)\| \exp(-\lambda x) : (x, y) \in E_c^*\}$$

where $\lambda > C$. Let W be the set of all functions $u : E_c \rightarrow R$ such that

- (i) u is of class C^1 ,
- (ii) $u(x, y) = \varphi(x, y)$ on E_0 .

For $u \in W$ denote by $u|_{\tilde{E}_c}$ the restriction of u to the set \tilde{E}_c . Let W^* denote the set of all functions $u|_{\tilde{E}_c}$ where $u \in W$. Let X be the class of all functions z belonging to W^* and satisfying the conditions

$$(23) \quad \|\nabla_y z(x, y)\| \leq \tilde{M}, \quad |\partial_x z(x, y)| \leq \tilde{M},$$

$$(24) \quad \|\nabla_y z(x, y) - \nabla_y z(\bar{x}, \bar{y})\| \leq S_0|x - \bar{x}| + (2B_2 + 1)\|y - \bar{y}\|,$$

$$(25) \quad \|\partial_x z(x, y) - \partial_x z(\bar{x}, \bar{y})\| \leq S_1|x - \bar{x}| + S_0\|y - \bar{y}\|,$$

on \tilde{E}_c , where

$$\tilde{M} = 2B_1 + \frac{C_0 + C_1 B_0}{C_1 a + C + C_1},$$

$$S_0 = C\tilde{M} + 2P(B_2 + 1), \quad S_1 = (P + C\tilde{M})(P + 1) + P^2(2B_2 + 1)$$

and

$$P = C_0 + C_1(\tilde{M}a + B_0 + 2\tilde{M}).$$

Let c denote the constant defined by

$$c = \min \left\{ a, \frac{1}{2K(1 + B_2)}, \frac{1}{2(C_1 a + C + 2C_1)} \right\},$$

where

$$K = \max\{(\tilde{M} + 1)\beta(S)[\tilde{M} + 1 + S_0 + 2B_2 + 1], C(2B_2 + 1)\},$$

$$S = \tilde{M}a + B_0 + 2\tilde{M} + 2B_2 + 1 + 2S_0 + S_1.$$

The set X is a closed subset of the Banach space $C(\tilde{E}_c, R; \lambda)$.

Let u be an arbitrary element of X . Consider the initial problem (1), (2), where

$$G(x, y, q) = f(x, y, \tilde{u}, q) \quad \text{on } \Omega_0$$

and

$$(26) \quad \omega(y) = \varphi(0, y) \quad \text{on } [-b, b]$$

and $\tilde{u}(x, y) = u(x, y)$ on \tilde{E}_c , $\tilde{u}(x, y) = \varphi(x, y)$ on E_0 . We will prove that a unique solution $z(\cdot; u)$ of problem (1), (2), (26) exists on E_c , and this solution satisfies (23)–(25). We will use Theorem 1.3 and Lemma 1.4.

Since $u \in X$, then we have

$$\|\tilde{u}\|_{(x;1)} \leq Ma + B_0 + 2\tilde{M}, \quad \|\tilde{u}\|_{(x;1.L)} \leq S.$$

For the function G given by (26) we have

$$\partial_{y_j} G(x, y, q) = \partial_{y_j} f(x, y, \tilde{u}, q) + \partial_z f(x, y, \tilde{u}(x, y), q)(\partial_{y_j} \tilde{u}),$$

where $i = 1, \dots, n$, and

$$\begin{aligned} \partial_x G(x, y, q) &= \partial_x f(x, y, \tilde{u}, q) + \partial_z f(x, y, \tilde{u}, q)(\partial_x \tilde{u}), \\ \nabla_q G(x, y, q) &= \nabla_q f(x, y, \tilde{u}, q). \end{aligned}$$

It follows from Assumption $H[f]$ that

$$\begin{aligned} |\partial_x G(x, y, q)|, \quad \|\nabla_y G(x, y, q)\| &\leq P + C\tilde{M}, \\ |\partial_{q_i} G(x, y, q)| &\leq M_i, \quad i = 1, \dots, n, \end{aligned}$$

and

$$\begin{aligned} \|\nabla_y G(x, y, q) - \nabla_y G(x, \bar{y}, \bar{q})\| \\ \leq \beta(S)[(\tilde{M} + S_0 + 2B_2 + 1)\|y - \bar{y}\| + \|q - \bar{q}\|] + C(2B_2 + 1)\|y - \bar{y}\| \\ \leq K[\|y - \bar{y}\| + \|q - \bar{q}\|]. \end{aligned}$$

Let

$$\tilde{\Gamma}(x) = \frac{K(1 + B_2)x + B_2}{1 - K(1 + B_2)x}.$$

It follows from Theorem 1.3 and Lemma 1.4 that

- (i) problem (1), (2) and (26) have a unique solution $z(\cdot; u)$ on \tilde{E}_c ,
- (ii) the solution satisfies the conditions:

$$(27) \quad \begin{aligned} & \|\nabla_y z(x, y; u) - \nabla_y z(\bar{x}, \bar{y}; u)\| \\ & \leq [P + C\tilde{M} + P\tilde{\Gamma}(x)]|x - \bar{x}| + \tilde{\Gamma}(x)\|y - \bar{y}\|. \end{aligned}$$

$$(28) \quad \begin{aligned} |\partial_x z(x, y; u) - \partial_x z(\bar{x}, \bar{y}; u)| & \leq [(P + C\tilde{M})(P + 1) + P^2\tilde{\Gamma}(x)]|x - \bar{x}| \\ & + [P + C\tilde{M} + P\tilde{\Gamma}(x)]\|y - \bar{y}\| \end{aligned}$$

and

$$(29) \quad |\partial_x z(x, y; u)|, \quad \|\nabla_y z(x, y; u)\| \leq B_1 + (P + C\tilde{M})x$$

on \tilde{E}_c . Estimates (27), (28) and the condition $\tilde{\Gamma}(c) = 2B_2 + 1$ imply

$$(30) \quad \|\nabla_y z(x, y; u) - \nabla_y z(\bar{x}, \bar{y}; u)\| \leq S_0|x - \bar{x}| + (2B_2 + 1)\|y - \bar{y}\|$$

and

$$(31) \quad |\partial_x z(x, y; u) - \partial_x z(\bar{x}, \bar{y}; u)| \leq S_1|x - \bar{x}| + S_0\|y - \bar{y}\|.$$

For all $x \in [0, c]$ we have

$$B_1 + (P + C\tilde{M})x \leq B_1 + (C_0 + C_1B_0)c + \tilde{M}[C_1a + 2C_1 + C]c \leq \tilde{M}.$$

Then by (29) we get

$$(32) \quad |\partial_x z(x, y; u)|, \quad \|\nabla_y z(x, y; u)\| \leq \tilde{M} \quad \text{on } \tilde{E}_c.$$

Let U be the operator defined on X in the following way: for $u \in X$ we put

$$(Uu)(x, y) = z(x, y; u) \quad \text{on } \tilde{E}_c.$$

It follows from (30)–(32) that the function Uu satisfies (23)–(25) and therefore $U : X \rightarrow X$.

Now we prove that U is a contraction. Let $u, v \in X$ and $\tilde{v}(x, y) = v(x, y)$ on \tilde{E}_c , $\tilde{v}(x, y) = \varphi(x, y)$ on E_0 . It follows from Assumption $H[f]$ that

$$\begin{aligned} |\partial_x [z(x, y; u) - z(x, y; v)]| &\leq C \|u - v\|_{[\lambda]} \exp(\lambda x) \\ &\quad + \sum_{j=1}^n M_j |\partial_{y_j} [z(x, y; u) - z(x, y; v)]| \end{aligned}$$

on \tilde{E}_c and

$$z(0, y; u) - z(0, y; v) = 0 \quad \text{on } [-b, b].$$

By the comparison theorem for hyperbolic differential inequalities [13], [23], we get

$$|z(x, y; u) - z(x, y; v)| \leq \frac{C}{\lambda} \|u - v\|_{[\lambda]} \exp(\lambda x) \quad \text{on } \tilde{E}_c,$$

and hence

$$\|Uu - Uv\|_{[\lambda]} \leq \frac{C}{\lambda} \|u - v\|_{[\lambda]}.$$

Since $C < \lambda$, then by the Banach fixed point theorem, it follows that there exists $\bar{z} \in X$ such that $\bar{z} = U\bar{z}$. Let $\bar{u}(x, y) = \bar{z}(x, y)$ on \tilde{E}_c , $\bar{u} = \varphi(x, y)$ on E_0 . This \bar{u} is the solution of problem (1) and (2), satisfying all the conditions of our theorem.

Remark 3.4. As particular cases of (1) and (2), we obtain the Cauchy problem for equations with a retarded argument (3) and for differential-integral equations (4). It is easy to see that Theorem 3.3 can be extended on the following systems of functional differential equations

$$\partial_x z_i(x, y) = f_i(x, y, z, \nabla_y z_i(x, y)), \quad i = 1, \dots, k,$$

with the initial condition $z(x, y) = \phi(x, y)$ on E_0 where

$$f = (f_1, \dots, f_k) : E \times C(E^*, R^k) \times R^n \longrightarrow R^k, \quad \phi : E_0 \longrightarrow R^k$$

and $z = (z_1, \dots, z_k)$.

REFERENCES

1. A. Augustynowicz and Z. Kamont, *On Kamke's functions in uniqueness theorems for first order partial differential-functional equations*, *Nonlinear Anal.* **14** (1990), 837–850.
2. P. Bassanini and M.C. Salvatori, *Un problema ai limiti per sistemi integrodifferenziali nonlineari di tipo iperbolico*, *Boll. Un. Mat. Ital. B* (5) **13** (1981), 785–798.
3. P. Brandi and R. Ceppitelli, *Existence, uniqueness and continuous dependence for a first order nonlinear partial differential equations in a hereditary structure*, *Ann. Polon. Math.* **47** (1986), 121–136.
4. P. Brandi and C. Marcelli, *Haar inequality in hereditary setting and applications*, *Rend. Sem. Mat. Univ. Padova* **96** (1996), 177–194.
5. M. Cinquini Cibrario, *Sopra una classe di sistemi di equazioni non lineari a derivate parziali in piu variabili indipendenti*, *Ann. Mat. Pura Appl.* **140** (1985), 223–253.
6. W. Eichhorn and W. Gleissner, *On a functional differential equation arising in the theory of the distribution of wealth*, *Aequationes Math.* **28** (1985), 190–198.
7. M. El Doma, *Analysis of nonlinear integro-differential equations arising in the age-dependent epidemic models*, *Nonlinear Anal.* **11** (1987), 913–937.
8. D. Jaruszewska-Walczak, *Existence of solutions of first order partial differential-functional equations*, *Boll. Un. Mat. Ital. B* (7) **4** (1990), 57–82.
9. Z. Kamont, *On the Cauchy problem for nonlinear partial differential-functional equations of the first order*, *Serdica Bulg. Math. Publ.* **5** (1979), 327–339.
10. ———, *Existence of solutions of first order partial differential-functional equations*, *Ann. Soc. Math. Polon. Ser. I Comment. Math.* **25** (1985), 249–263.
11. ———, *Initial value problems for hyperbolic differential-functional systems*, *Boll. Un. Mat. Ital.* **8** (1994), 965–984.
12. Z. Kamont and A. Salvadori, *Uniqueness of solutions to hyperbolic functional differential problems*, *Nonlinear Anal.* **30** (1997), 4584–4594.
13. V. Lakshmikantham and E. Leela, *Differential and integral inequalities*, Academic Press, New York, 1969.
14. H. Leszczyński, *On the existence and uniqueness of weak solutions of the Cauchy problem for weakly coupled systems of nonlinear partial differential-functional equations of the first order*, *Boll. Un. Mat. Ital. B* (7) **7** (1993), 323–354.
15. ———, *On CC-solutions to the initial-boundary value problem for systems of nonlinear first order partial differential-functional equations*, *Rend. Mat. Appl.* **15** (1995), 173–209.
16. R. Nagel and E. Sinestrari, *Nonlinear hyperbolic Volterra integrodifferential equations*, *Nonlinear Anal.* **27** (1996), 167–186.
17. A. Pliš, *On the estimation of the existence domain for solutions of a nonlinear partial differential equation*, *Bull. Polon. Sci. Cl. III* (3) **4** (1956), 125–129.
18. E. Sinestrari and G.F. Webb, *Nonlinear hyperbolic systems with nonlocal boundary conditions*, *J. Math. Anal. Appl.* **121** (1987), 449–464.

19. J. Szarski, *Characteristics and Cauchy problem for nonlinear partial differential equations of first order*, University of Kansas, Lawrence, Kansas, 1959.

20. ———, *On integro-differential equation*, Ann. Polon. Math. **14** (1964), 321–333.

21. ———, *Generalized Cauchy problem for differential-functional equations with first order partial derivatives*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astr. Phys. **24** (1976), 575–580.

22. ———, *Cauchy problem for an infinite system of differential-functional equations with first order partial derivatives*, Ann. Soc. Math. Polon., Comment. Math. tom spec. **1** (1978), 293–300.

23. ———, *Differential inequalities*, Polish Sci. Publ., Warszawa, 1967.

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