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ON THE EQUATION $y^x \pm x^y = z^2$

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ABSTRACT. In this note, we find all solutions of the diophantine equation $y^x \pm x^y = z^2$ when $\gcd(x, y) = 1$, the product xy is even and $\min(x, y) > 1$. Namely, there is only the obvious solution $3^2 - 2^3 = 1$.

0. Introduction. The very special equation $y^x - x^y = 1$ was considered by Catalan [**2**], as a particular case of the famous Catalan's equation $y^n - x^m = 1$. Catalan claimed that the only solution is $(x, y) = (2, 3)$, but did not give any proof. Indeed, a proof was published by Moret-Blanc in 1876. As indicated in Ribenboim [**5**, p. 95], a short proof of this result can now be obtained using the theorems of V.A. Lebesgue (1850) and Ko Chao (1960). There are also special proofs by Rotkiewicz (1960) and Skandalis (1982), see Ribenboim's book for more details.

Here we consider the much more general equation $y^x - x^y = z^2$. To solve it is not easy. Our method is not completely classical; we use a combination of lower bounds of linear forms in p-adic and Archimedian logarithms. This combination enables us to get a good upper bound for y. This bound is small enough to completely solve the problem, thanks to a computer verification. We notice that without this idea the bounds reachable would be too large to permit this verification.

We assume that xy is even because we have no idea how to solve this equation when x and y are both odd. We assume also that x and y are coprime, but even if this is not so, our work is no less interesting.

Our result is the following.

Theorem. *The only positive solution of the diophantine equation*

$$y^x \pm x^y = z^2$$

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such that $\gcd(x,y)=1$, xy *is even and* $\min(x,y)>1$ *is*

$$3^2-2^3=1^2.$$

We first study the equation with the "+" sign.

1. The equation $x^y+y^x=z^2$.

1.1. Preliminaries. We consider the diophantine equation $x^y + y^x = z^2$ when x is even and positive and y is odd, $y>1$. From the factorization (this is where we use the fact that x is even; of course, this factorization plays a key role in the proof)

$$x^y=(z-y^{x/2})(z+y^{x/2}),$$

we see that, for a suitable $\varepsilon=\pm 1$, we have

$$\begin{aligned} z+\varepsilon y^{x/2} &= 2^{y-1}u^y,\\ z-\varepsilon y^{x/2} &= 2v^y, \end{aligned}$$

where $x=2uv,\ \gcd(2u,v)=1$. This gives

$$y^{x/2}=\varepsilon(2^{y-2}u^y-v^y). \tag{1.1}$$

If $x=2$ or 4, then $y=2^{y-2}-1$ or $y^2=4^{y-1}-1$ which do not lead to any solutions. From now on, we assume that $x\ge 6$.

1.2. An upper bound for x in terms of y. From (1.1) one gets at once

$$\frac{x}{2}\log y<\log\left(\max\left\{\left(\frac{x}{2}\right)^y,\frac{1}{4}x^y\right\}\right)=\log(x^y/4),$$

where the equality above follows from the fact that $y\ge 3$. Since x is an even integer, the above inequality implies that

$$x\le 2\left\lfloor\frac{y\log x-\log 4}{\log y}\right\rfloor. \tag{1.2}$$

This gives an upper bound for x in terms of y which we denote by xmax(y).

Remark . Writing $x = ty$, inequality (2.2) implies that

$$t \le 2 + \frac{2\log t}{\log y} < 2 + \frac{2t}{e\log y},$$

where the rightmost inequality comes from the inequality $\log t \le t/e$; this shows that $t \le 2 + o(1)$ when y tends to infinity. A direct computation leads to $x < 3y$ for $y \ge 9$ and also to $1 + \sqrt{x} \le 2\sqrt{y}$. On the other hand, inequality (1.2) together with the fact that x is even imply that $x \le 20$ for $y \le 7$. A quick check proves that there are no solutions such that $y \le 7$ and $6 \le x \le 20$. From now on, we assume that $y \ge 9$.

1.3. Lower bounds for x in terms of y. We distinguish the two cases, namely $2u < v$ and $2u > v$. Indeed, the case $2u = v$ cannot occur because $\gcd(2u, v) = 1$.

(i) If $2u < v$, then

$$\begin{aligned} y^{x/2} &= v^y - 2^{y-2}u^y = v^y\left(1 - \frac{1}{4}\left(\frac{2u}{v}\right)^y\right) \\ &> (\sqrt{x})^y\left(\frac{1}{4} - \left(\frac{2u}{2u+1}\right)^y\right). \end{aligned} \tag{1.3}$$

We now show that

$$\frac{1}{4} - \left(\frac{2u}{2u+1}\right)^y > \frac{1}{4} - e^{-y/(1+\sqrt{x})}.$$

Indeed, we have

$$\log\left(\frac{2u}{2u+1}\right) < -\frac{1}{2u+1} \le -\frac{1}{1+\sqrt{x}},$$

which in turn is smaller than $-1/(2\sqrt{y})$ by the Remark following inequality (1.2).

(ii) If $2u > v$, then

$$y^{x/2} = 2^{y-2}u^y - v^y = (2u)^y \left(\frac{1}{4} - \left(\frac{v}{2u}\right)^y\right)$$
$$\geq (2u)^y \left(\frac{1}{4} - \left(\frac{2u-1}{2u}\right)^y\right) \geq \sqrt{x}^y \left(\frac{1}{4} - e^{-y/\sqrt{x}}\right), \tag{1.4}$$

where the last inequality above follows from arguments similar to the ones employed in case $2u < v$.

The above arguments show that we always have

$$y^{x/2} \geq (\sqrt{x})^y \left(\frac{1}{4} - e^{-1/2\sqrt{y}}\right).$$

This is equivalent to

$$x \geq \frac{y \log x}{\log y} + \frac{2 \log(1/4 - e^{-(1/2)\sqrt{y}})}{\log y}. \tag{1.5}$$

This leads to a lower bound for x in terms of y which we denote by xmin (y).

1.4. Absolute bounds for x and y. From equation (1.1), it follows that if $2^w \mid x$, then

$$y^{x/2} + \varepsilon v^y \equiv 0 \pmod{2^{wy-2}}.$$

We now apply Theorem 1 of Bugeaud and Laurent [**1**]: with their notations, here we have

$$\Lambda = y^{x/2} \pm v^y, \qquad p = 2,$$
$$D = e = g = t = 1, \qquad v_2(\Lambda) \geq wy - 2$$

and

$$\alpha_1 = y, \qquad \alpha_2 = v, \qquad b_1 = x/2, \qquad b_2 = y.$$

We take

$$A_1 = y, \qquad A_2 = \max\{2, v\}.$$

With these notations it follows, by Theorem 1 in [**1**], that

$$v_2(\Lambda) \leq 2(KL - 1/2) = 2KL - 1 \tag{1.6}$$

whenever $K \geq 3$ and $L \geq 2$ are integers such that

$$\begin{aligned} 2K(L-1)\log 2 > {} & 3\log(KL) + (K-1)\log b \\ & + 2L\left(\frac{1}{2} - \frac{KL}{6RS}\right)(R\log y + S\log x) \end{aligned} \tag{1.7}$$

where

$$b = \frac{(R-1)y + (S-1)x}{2}\left(\prod_{k=1}^{K-1} k!\right)^{-2/(K^2-K)}$$

and R and S are two positive integers such that K, L, R and S satisfy inequalities (1) of [**1**].

We now employ the method described in Section 5.1. of [**1**]. Let

$$a_1 = \frac{\log y}{\log 2}, \qquad a_2 = \frac{\log x}{\log 2}.$$

We choose $K = \lfloor kLa_1a_2 \rfloor + 1$ where k is a positive parameter. From Lemma 13 in [**1**], we know that

$$\begin{aligned} \log b \leq {} & \log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) - \frac{1}{2}\log k - \log 2 \\ & + \frac{3}{2} + \log\frac{(1+\sqrt{k-1})\sqrt{k}}{k-1}. \end{aligned} \tag{1.8}$$

Moreover, by Lemma 12 in [**1**], we have

$$\left(\frac{1}{2} - \frac{KL}{6RS}\right)(Ra_1 + Sa_2) \leq \frac{2}{3}\sqrt{(K-1)La_1a_2} + \frac{4}{3}\sqrt{La_1a_2} + \frac{2(a_1+a_2)}{3}.$$

Hence, inequality (1.7) is satisfied if

$$\begin{aligned} kL(L-1)a_1a_2 > {} & 3\frac{\log(kL^2a_1a_2 + L)}{2\log 2} + \frac{kLa_1a_2\log b}{2\log 2} \\ & + \frac{1}{3}\sqrt{k}L^2a_1a_2 + \frac{2L^{3/2}\sqrt{a_1a_2}}{3} + \frac{L(a_1+a_2)}{3}. \end{aligned} \tag{1.9}$$

In conclusion, from inequality (1.6), it follows that

$$wy - 2 \le 2\lfloor kLa_1a_2\rfloor L + 2L - 1 \tag{1.10}$$

whenever k and L are such that inequalities (1.8) and (1.9) are satisfied.

If we suppose that $v \le x^{0.55}$, we then get the bound $y \le 210\,000$. More precisely, we apply the previous method with the choices $k = 0.9282$ and $L = \lfloor 0.92B\rfloor + 2$ where $B = \log b/\log 2$. This gives $L = 24$ for an initial value of $y = 220\,000$. From inequality (1.2), we get $x < 450\,000$.

We now use an archimedian linear form in logarithms to prove that if $v > x^{0.55}$, then $y \le 210\,000$ as well. We consider again equation (1.1). From this equation and the hypothesis $v = x^t$ with $t > 0.55$, it follows that $\varepsilon = -1$. By dividing with v^y, we get

$$y^{x/2}/v^y - 1 = \frac{1}{4}(x/v^2)^y.$$

Thus,

$$0 < (x/2)\log y - y\log v < v^{-((2t-1)/t)\cdot y}.$$

Indeed, to see why the last inequality holds, write the previous relation as $e^\Lambda = 1+\eta$, notice that η is positive and that $0 < \Lambda = \log(1+\eta) < \eta$.

We apply the corollary of the main result of [**3**] obtained in [**4**] to this archimedian linear form in logarithms. This result is:

Proposition. *Consider the linear form*

$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Put

$$D = [\mathbf{Q}(\alpha_1,\alpha_2) : \mathbf{Q}]\,/\,[\mathbf{R}(\alpha_\mathbf{1},\alpha_\mathbf{2}) : \mathbf{R}].$$

Let a_1, a_2, h and k be real positive numbers, and ρ a real number > 1. Put $\lambda = \log\rho$, $\chi = h/\lambda$ and suppose that $\chi \ge \chi_0$ for some number $\chi_0 \ge 0$ and that

$$h \ge D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + f(\lceil K_0\rceil)\right) + 0.023, \tag{1.11}$$

$$a_i \ge \max\{1, \rho\,|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)\}, \quad i = 1,2, \tag{1.12}$$

$$a_1a_2 \ge \lambda^2, \tag{1.13}$$

where

$$f(x) = \log \frac{(1+\sqrt{x-1})\sqrt{x}}{x-1} + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log\frac{3}{4} + \frac{\log(x/(x-1))}{x-1},$$

and

$$K_0 = \frac{1}{\lambda}\left(\frac{\sqrt{2+2\chi_0}}{3} + \sqrt{\frac{2(1+\chi_0)}{9} + \frac{2\lambda}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) + \frac{4\lambda\sqrt{2+\chi_0}}{3\sqrt{a_1a_2}}}\right)^2 a_1a_2.$$

Put

$$v = 4\chi + 4 + 1/\chi, \qquad m = \max\{2^{5/2}(1+\chi)^{3/2}, (1+2\chi)^{5/2}/\chi\}.$$

Then we have the lower bound

$$\log|\Lambda| \geq -\frac{1}{\lambda}\left(\frac{v}{6} + \frac{1}{2}\sqrt{\frac{v^2}{9} + \frac{4\lambda v}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) + \frac{8\lambda m}{3\sqrt{a_1a_2}}}\right)^2 a_1a_2 - \max\{\lambda(1.5+2\chi) + \log(((2+2\chi)^{3/2} + (2+2\chi)^2\sqrt{k^*})A + (2+2\chi)), D\log 2\}$$

where

$$A = \max\{a_1, a_2\},$$

and

$$k^* = \frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2 + \frac{1}{\lambda}\left(\frac{2}{3\chi} + \frac{2}{3}\frac{(1+2\chi)^{1/2}}{\chi}\right).$$

We use the relations $v = x^t$ with $0.55 < t < 1$ and $\mathrm{xmin}\,(y) \leq x \leq \mathrm{xmax}\,(y)$. We choose $\lambda = 4.55$. This gives the upper bound $y \leq 204\,000$.

1.5. The computer verification: Filling the gap. We come back to the initial equation

$$x^y + y^x = z^2$$

where x is even, y is odd, $7 < y < 250000$ (just to be safe!) and $\gcd(x, y) = 1$. Of course, we do not compute the powers x^y and y^x. We use the weaker condition, namely that

$$\left(\frac{x^y + y^x}{p}\right) = -1$$

for some prime number p, $p \leq 281$, where $(\div)$ is the Legendre symbol. We used single precision in C working on a 64 bit computer, namely a DEC station. The program was written by Yves Roy and the CPU time needed was less than one day. We are deeply grateful to Yves Roy for his kind assistance. No solution was found.

2. The equation $y^x - x^y = z^2$. Now we study the equation with the "$-$" sign. Since the analysis is very similar to the previous one, we omit some details.

2.1. Preliminaries. We consider the equation $y^x - x^y = z^2$ when one of the numbers x or y is even, $x > 1$, $y > 1$. We first notice that x is even. Indeed, otherwise, x is odd and y is even. Since $x \geq 3$, it follows that $y^x \equiv 0 \pmod 4$. Moreover, since x^y is an odd square, it follows that $x^y \equiv 1 \pmod 4$. Thus,

$$z^2 = y^x - x^y \equiv -1 \pmod 4,$$

which is a contradiction.

Since x is even, we can rewrite our equation as

$$x^y = y^x - z^2 = (y^{x/2} - z)(y^{x/2} + z).$$

Since x and y are coprime, it follows that x and z are coprime as well. From the above equation, we see that for a suitable $\varepsilon \in \{\pm 1\}$ we have

$$y^{x/2} + \varepsilon z = 2^{y-1} u^y,$$
$$y^{x/2} - \varepsilon z = 2 v^y,$$

for some u and v with $x = 2uv$ and $\gcd(2u, v) = 1$. This gives

$$y^{x/2} = 2^{y-2} u^y + v^y. \tag{2.1}$$

If $x = 2$, then $y^2 = 2^{y-2} + 1$. Hence, $y = 3$. This leads to the solution

$$3^2 - 2^3 = 1^2.$$

If $x = 4$, then $y = 4^{y-1} + 1$ which does not lead to any solution. Thus, from now on, we assume that $x \geq 6$.

2.2. An upper bound for x in terms of y. By equation (2.1),

$$y^{x/2} = \frac{x^y}{4v^y} + v^y,$$

and $1 \leq v = x/(2u) \leq x/2$. It is then easily seen that the righthand side is maximal when $v = 1$ or $v = x/2$. Thus

$$y^{x/2} < \frac{1}{2}x^y.$$

From the last two inequalities and the fact that x is even, it follows that

$$x \leq 2\left\lfloor \frac{y \log x - \log 2}{\log y} \right\rfloor. \tag{2.2}$$

This leads to an upper bound for x in terms of y which we call xmax (y). Notice that, in fact, this xmax (y) is only slightly larger than the corresponding one for the equation $x^y + y^x = z^2$ (the only difference is the appearance of $\log(2)$ instead of $\log(4)$ in the numerator of the righthand side of (2.2). One can easily show that the difference between our present xmax (y) and the previous one is at most 2).

Remark . Inequality (2.2) implies $x \leq 3y$ for $y \geq 9$. On the other hand, inequality (2.2) together with the fact that x is even imply that $x \leq 20$ for $y \leq 7$. A quick check proves that there are no solutions such that $y \leq 7$ and $6 \leq x \leq 20$. From now on, we assume that $y \geq 9$.

2.3. A lower bound for x in terms of y. Since $y^x - x^y = z^2 > 0$, it follows that $y^x > x^y$. This is equivalent to

$$\frac{x}{\log x} > \frac{y}{\log y}.$$

Since $y \geq 9$, $x \geq 6$ and the function $(x/\log x)$ is increasing for $x > e$, it follows that $x > y$. Thus $x \geq y + 1$. Denote $y + 1$ by $\text{xmin}\,(y)$.

2.4. Absolute bounds for x and y. From equation (2.1), we see that if $2^w \mid x$, then

$$y^{x/2} - v^y \equiv 0 \pmod{2^{wy-2}}.$$

This is the same congruence as the one appearing in Section 1. By the analysis done in Section 1, it follows that $y < 210\,000$ if $v \leq x^{0.55}$.

We now use an archimedian linear form in logarithms to bound y when $v > x^{0.55}$. Let $v = x^t$ for some $t > 0.55$. Since $v > \sqrt{x}$, it follows that $v > 2u$. By dividing equation (2.1) by $y^{x/2}$, we obtain

$$1 - v^y/y^{x/2} = \tfrac{1}{4}(2u)^y/y^{x/2}.$$

Notice that, since $y^x > x^y$, it follows that $y^{x/2} > x^{y/2}$. From the above inequality it follows that

$$1 - v^y/y^{x/2} < \frac{1}{4}\left(\frac{2u}{\sqrt{x}}\right)^y.$$

Let $\lambda = (2u/\sqrt{x})^y$. Clearly, $0 < \lambda < 1$. Let $\Lambda = (x/2)\log y - y\log v$. Then

$$0 < \Lambda = -\log\left(1 - \frac{\lambda}{4}\right) = \frac{\lambda}{4} + \frac{\lambda^2}{32} + \cdots < \frac{\lambda}{3}.$$

Since $\lambda = (4u^2/x)^{y/2} = (x/v^2)^{y/2} = v^{-((2t-1)/2t)y}$, we see that if $v = x^t$ and $t \geq 0.55$, then

$$0 < |(x/2)\log y - y\log v| < v^{(-(2t-1)/2t)y}. \tag{2.3}$$

If one compares inequality (2.3) above with the corresponding inequality of Section 1, one notices that the only difference is the appearance of a factor of 2 in the denominator of the exponent of v in the right-hand side. If one follows through the above arguments one again gets $y \leq 210\,000$ and $x < 450\,000$.

In conclusion, it suffices to search for the solutions of our equation in the range $9 \leq y < 210\,000$ and $\text{xmin}\,(y) \leq x \leq \text{xmax}\,(y)$. A similar study as in case "+" leads again to the conclusion that there is no other solution. This ends the proof of our theorem.

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