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GENERAL RADICAL THEORY IN RINGS

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1. **The general radical theory.** A radical property, speaking roughly, is one which can be "divided out". Thus, for example, any abelian group G has a largest torsion subgroup $H = \{x \in G \mid nx = 0$ for some positive integer $n\}$ such that G/H is torsion-free. Another perfectly typical example is the sum I of all nil ideals (every element nilpotent) of a ring R. It is easy to see that I is itself a nil ideal and that R/I has no nonzero nil ideals. It will be clear as we go along that most of what we do could just as well be done in a much more general category (yielding, as special cases, the parallel theories in groups, rings, modules, algebras, and so on). However, to avoid too much generality we will stick to rings and agree that (unless otherwise stated) all rings considered will belong to some arbitrary (but fixed) universal class W of not necessarily associative rings. Mostly we can as well take W to be the class of all such rings, but the class could be more restricted, provided that it has the properties:

(1) Hereditary; that is, $I \lhd R \in W$ implies $I \in W$ (where $I \lhd R$ means I is an ideal of R).

(2) Homomorphically closed; that is $R \in W$ implies all $R/I \in W$.

One of the motivations for studying radicals is that often the "dividing out" process yields a ring which in some sense is simpler than the original one and hence possibly more amenable. A typical example is the Wedderburn-Artin theorem: If R is a (associative) ring with descending chain condition on left (or right) ideals and I is the sum of all nilpotent ideals of R, then R/I is the direct sum of a finite number of simple rings each a complete matrix ring over some division ring. Even more, it is always hoped that one may be able to extract information about the original ring. For example, if the Jacobson radical J of a ring R is nil then any idempotent element u (that is $u^2 = u$) of R/J has an inverse image in R which is idempotent. Much of what can be done will of course depend on the kind of ring and the particular radical. However quite a bit can be said about radicality in general, and in any case it is worthwhile looking at the general theory, if only to bring a certain amount of order into the bewildering tangle of results in the area.

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The axiomatization we give for radical is due to Kuroš [1] and, independently, Amitsur [2]. A subclass $P \subseteq W$ will be called *radical* if every $R \in W$ contains an ideal $I = P(R)$ which is the "largest" P-ideal of R (in the sense that $I \in P$ and $J \lhd R$ with $J \in P$ implies $J \subseteq I$), and such that R/I contains no nonzero P-ideals.

We quote some theorems without proof: (*Note.* The proofs are easy, and most can be found in [3].)

THEOREM 1. *P is a radical class if and only if* (R1) *P is homomorphically closed, and* (R2) *$R \notin P$ implies some $0 \neq R/I \in SP$, where $SP = \{K \in W \mid if\ 0 \neq I \lhd K\ then\ I \notin P\}$ (note that SP is called the semisimple class of the radical P).*

A criterion for radicality generally easier to check than the original definition (or Theorem 1) is

THEOREM 2. *P is a radical class if and only if* (R1), (R2a) *if I, $R/I \in P$ then $R \in P$, and* (R2b) *if $\{I_i\}$ is a chain of ideals of R with all $I_i \in P$ then $\bigcup I_i \in P$.*

There is a far-reaching (though not quite perfect) duality between semisimplicity as a property and radicality. We could, in fact, axiomatize semisimplicity as: A class Q is called *semisimple* if it has the properties:

(S1) If $R \in Q$ and $0 \neq I \lhd R$ then some $0 \neq I/J \in Q$, and

(S2) if $R \notin Q$ then there is some $0 \neq I \lhd R$ with $I \in UQ$, where $UQ = \{K \in W \mid \text{all } 0 \neq K/I \notin Q\}$.

The duality is in interchanging ideal with image, when interchanging radical and semisimple. Thus (R2) and (S2) are dual but (R1) and (S1) are not quite.

If the class W in which we are working consists of associative rings, or even alternative rings (namely rings in which all $x^2y = x(xy)$ and $xy^2 = (xy)y$), there is a theorem which says that all semisimple classes are hereditary. Thus (in this case) we could replace (S1) by (S1′): Q is hereditary; which is indeed dual to (R1).

Typical dual theorems:

THEOREM 3. *If P is a radical class then SP is semisimple and $USP = P$.*

THEOREM 4. *If Q is a semisimple class then UQ is radical and $SUQ = Q$.*

Note that if Q satisfies only (S1) then UQ is still radical (it is called the *upper radical* defined by the class Q) and $Q \subseteq SUQ$. In fact

THEOREM 5. *If Q satisfies* (S1) *then UQ is radical and SUQ is the smallest semisimple class containing Q.*

This last part is easy to show, for suppose $Q \subseteq T$ where T is a semisimple class. Now clearly both S and U are order-reversing, so SU is an order-preserving class function. Thus $SUQ \subseteq SUT = T$. This theorem is illustrative of a large class of "smallest" theorems, about which more will be said later.

We now turn to radical and related constructions. These usually involve induction, often transfinite. It is a curious fact that one of the so-called paradoxes of set theory, namely the Burali-Forti paradox, plays an important role here. This takes various forms, the simplest of which is that the class of all ordinals is too big to be a set. That is, the paradox arises (as has happened) when one tries to treat the collection of all ordinals as a set. One of the neatest examples (and also one of the oldest) is the Levitzki Construction [4] of the lower Baer radical B of a ring R, which arises when one tries to divide out the property of nilpotence (of ideals). The construction proceeds. (*Note*: We are now operating in the class of associative rings.): Let

$$I_1 = \sum \text{all nilpotent } J \lhd R,$$

and for β an arbitrary ordinal define

$$I_\beta = \begin{cases} \bigcup_{\alpha<\beta} I_\alpha & \text{when } \beta \text{ is a limit ordinal, otherwise} \\ \sum J \lhd R & \text{such that } J/I_{\beta-1} \text{ is nilpotent.} \end{cases}$$

Clearly the set $\{I_\beta\}$ so defined forms a chain (that is, $\alpha < \beta$ implies $I_\alpha \subseteq I_\beta$) and if it were true that always $I_\alpha \neq I_{\alpha+1}$ then there would be a 1-1 correspondence between a set and the class of all ordinals. But this produces the paradox, so we conclude that for some ordinal γ we have $I_\gamma = I_{\gamma+1}$, and we call $I_\gamma = B(R)$ the lower Baer radical of R. The class $B = \{K \mid B(K) = K\}$ is radical and for an arbitrary ring R it is true by definition that $R/B(R)$ has no nonzero nilpotent ideals. Remark that it is also true that $B(R) = \bigcap$ all prime ideals of R (so B is sometimes called the prime radical). Furthermore $B = \bigcup$ { all prime rings} $= \bigcup$ { all semiprime rings (no nonzero nilpotent ideals)}.

2. **Lower radical constructions.** Returning to our universal class W of not necessarily associative rings, we will give the Kuroš Construction of the lower radical LM for an arbitrary class $M \subseteq W$ (as modified in [5]). We start by letting M_1 be the homomorphic closure of M, that is

$$M_1 = HM = \{K \in W \mid K = R/I \text{ for some } R \in M\},$$

and for an ordinal β, assuming classes M_α known for all $\alpha < \beta$,

$$M_\beta = \{K \in W \mid \text{every } 0 \neq K/I \text{ has a nonzero ideal in } M_\alpha \text{ for some } \alpha < \beta\}.$$

Then $LM = \bigcup M_\beta$ taken over all ordinals β.

THEOREM 6. *LM is radical and is the smallest radical class containing M (another smallest theorem).*

Note that while the B-F paradox is used in proving LM is radical (see [3]), it does not apply to the M_β since they are classes. For example, even if W is restricted to be all associative rings and one takes the simplest case possible, namely $M = \{R\}$ where $R = \{0, x\}$ with $2x = x^2 = 0$, it develops that $LM = M_3$ is already too big to be a set. Thus the Kuroš steps while appearing simple, may in fact be enormous.

Incidentally, transfinite induction, which is used freely in the lower radical construction, makes some people a bit uneasy. Thus it was distinctly pleasing to learn [6] that for associative rings $LM = M_\omega$ where ω is the first transfinite ordinal, so that only ordinary induction need be used. (This result is sharp, by a construction of Heinicke [7].) In [6] it was also shown that for alternative rings $LM = M_\omega^2$ but is is not known if this is sharp. On the other hand, Rjabuhin [8] constructed a sequence of rings showing that when W = class of all not necessarily associative rings and M = all simple rings, $LM \neq M_\alpha$ for any ordinal α.

The lower radical is useful in many ways. For one thing it shows how to divide out an arbitrary property (whether radical or not) and in a minimal way. Also the lower radical can point the way to correct generalizations. For example, if in associative rings $N = \{\text{all nilpotent rings}\}$ then the lower Baer radical $B = LN$. However, for a nonassociative ring nilpotence is ambiguous (Do we require all $x_1(x_2(\cdots)) = 0$ or all products of n things associated arbitrarily, or what?). However, if $Z = \{R \mid xy = 0 \text{ for all } x, y \in R\}$, namely all zero rings, then also $B = LZ$ and Z can be generalized unambiguously.

There are a surprising number of apparently quite different constructions which turn out to give the lower radical, sometimes in general, sometimes only for more restricted W.

One example [9] is

$$M_1 = HM,$$

$$M_{n+1} = \{R \in W \mid \text{some } 0 \neq I \lhd R \text{ with } I \in M_n\},$$

and define $M^* = \{R \in W \mid \text{every } 0 \neq R/I$ has a nonzero ideal in some $M_n\}$. It turns out that M^* is always radical (and in general bigger than LM as examples show), but when $W =$ all associative, or even alternative, rings $M^* = LM$. (Note that M^* is also equal to a radical constructed by Watters [10].)

Another new one [11] also begins with $M_1 = HM$, and when $\beta - 1$ exists define

$$M_\beta = \{R \mid I, R/I \in M_{\beta-1} \text{ for some } I \lhd R\},$$

while when β is a limit ordinal, define

$$M_\beta = \{R \mid R = \bigcup I_i \text{ for some chain } \{I_i\} \text{ of ideals of } R \text{ such that each } I_i \in M_\alpha \text{ for some } \alpha < \beta\}.$$

Then $LM = \bigcup M_\beta$, taken over all ordinals β.

This construction is in many ways better adapted than earlier ones to proving properties of LM. For example, an old result we proved with some pain in [12] has now a simple proof:

THEOREM 7. *If M is hereditary then so is LM.*

PROOF. First M hereditary implies $M_1 = HM$ hereditary. Suppose $M_{\beta-1}$ hereditary and $0 \neq J \lhd R \in M_\beta$. Now we have some I, $R/I \in M_{\beta-1}$ so from $(I + J)/I \lhd R/I$ we have $J/(I \cap J) \cong (I + J)/I \in M_{\beta-1}$. But also $I \cap J \in M_{\beta-1}$, since $I \cap J \lhd I$, and therefore $J \in M_\beta$.

If β is a limit ordinal and for $\alpha < \beta$ all M_α are hereditary, let $R \in M_\beta$ so $R = \bigcup I_i$ for some chain of ideals $I_i \in M_\alpha$. But if $J \lhd R$ then $J = \bigcup (J \cap I_i)$ where each $J \cap I_i \lhd I_i$, so is in M_α. Therefore $J \in M_\beta$ and we are done.

The hereditary property is a very useful one for a radical to have, thus this theorem is often helpful. It is true, in fact, that many radicals are hereditary, such as the nil radical and the Jacobson radical, but some (such as the idempotent radical $= \{$all idempotent rings$\}$) are not.

More generally, defining a class M to be left (right, subring) hereditary if $R \in M$ implies $I \in M$ for all left ideals (right ideals, subrings) of R, then, as pointed out by Robert Rossa [13], virtually the same proof as that just given establishes

THEOREM 7′. *If M is left (right, subring) hereditary then so is LM.*

We digress for a moment to consider the following: A radical class

P is called (left) *strong* [14] if $I \in P$, where I is a left ideal of a ring R, implies $I \subseteq P(R)$. Clearly [14, Proposition 2] P is strong if and only if no $R \in SP$ has a nonzero left ideal in P. It is known that the Jacobson and Levitzki radicals are strong and it has been conjectured (the Koethe Conjecture) that this is also true of the nil radical. It was shown [14, p. 377] that the Brown-McCoy radical is not strong whereas the lower Baer radical B is strong [14, p. 374]. We include as an illustration an alternative proof of this last fact (somewhat simpler than that of [14]). Let $R \in SB$ with $0 \neq I \in B$ a left ideal. By the Levitzki Construction there must be some nilpotent $0 \neq A \lhd I$. Now any nilpotent left ideal of R is contained in a nilpotent ideal so since $IA \subseteq A$, we have $IA \subseteq B(R) = 0$. Thus if N is the sum of all the nilpotent ideals of I we have $IN = 0$ and so $N^2 = 0$. But then $N = B(I) = I$ and hence the contradiction $I^2 = 0$.

The question has been raised as to whether a left hereditary radical must be left strong. We will use Theorem 7′ to construct an example to show that this is not the case. Let $F = Z/(2)$, the integers mod 2, and let I be generated over F by x, y where $x^2 = x$, $xy = 0$, $yx = y$, and $y^2 = 0$. Then I is associative and has an ideal $J = \{0, y\}$. Letting M be the class of all rings isomorphic to any of $\{0, J, I, F\}$ then $M = M_1$ and is left hereditary. Thus by Theorem 7′ we have LM left hereditary. However, it is clear that $I \cong \{[\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}]\}$ where $a, b \in F$, which is a left ideal of F_2. But F_2 is simple and since a simple ring is a member of LM only if it is in M_1 it follows that F_2 has zero LM-radical. Thus LM is left hereditary but not left strong. On the other hand, the idempotent radical is left strong but not left hereditary, so these two properties are independent.

3. **Largest and smallest theorems.** We will conclude with some remarks about smallest theorems and their dual largest theorems. (Most of this material is from [15].) A class function F is called *admissible* if

(1) $M \subseteq FM$ for all $M \subseteq W$,

(2) $M \subseteq N$ implies $FM \subseteq FN$,

(3) *If* $\{M_\alpha\}$ is a chain defined for all ordinals then $F \bigcup M_\alpha = \bigcup FM_\alpha$.

THEOREM 8. *If F is an admissible function then every class M is contained in a smallest radical class P such that $FP = P$.*

CONSTRUCTION. $M_1 = M$ and

$$M_\beta = \begin{cases} LFM_{\beta-1} & \text{when } \beta - 1 \text{ exists, or} \\ \bigcup_{\alpha<\beta} M_\alpha & \text{when } \beta \text{ is a limit ordinal.} \end{cases}$$

Then $P = \bigcup_\beta M_\beta$.

The idea here is to have some property of radicals in mind, then try to find an admissible F such that a radical P has the desired property if and only if $FP = P$. When this can be done, Theorem 8 says that the property admits a smallest theorem.

A typical example is as follows: Define

$$I_1 M = \{K \mid K \lhd R \text{ for some } R \in M\},$$
$$I_{n+1} M = I_1 I_n M \quad \text{and} \quad IM = \bigcup_n I_n M.$$

Clearly the class IM is hereditary and a radical P is hereditary if and only if $IP = P$. It is also easy to see that I is an admissible function, and so by Theorem 8 every class M is contained in a smallest hereditary radical. (Of course one can get this result more easily by noticing that IM is the smallest hereditary class containing M. Hence the smallest hereditary radical containing M is just LIM.)

For the dual theorem we need an F with not only properties (1), (2), and (3), but also (4) FM satisfies (S1). We call such a function *S-admissible.*

THEOREM 9. *If F is S-admissible then every class M is contained in a smallest semisimple class Q such that $FQ = Q$.*

CONSTRUCTION. Let $M_1 = M$ and

$$M_\beta = \begin{cases} SUFM_{\beta-1} & \text{when } \beta - 1 \text{ exists,} \\ \bigcup_{\alpha<\beta} M_\alpha & \text{when } \beta \text{ is a limit ordinal, and} \end{cases}$$
$$Q = \bigcup_\beta M_\beta.$$

Notice that the function I is also S-admissible, so as a corollary: Every class M is contained in a smallest hereditary semisimple class. From this it follows that in associative (or even alternative) classes any M is contained in a smallest semisimple class. However this is not true in general (see [16]).

Recently J. F. Watters [17] has shown that smallest theorems exist for F which may not satisfy property (3). The constructions of Theorems 8 and 9 still lead to radical (or semisimple) classes, but in general not the smallest ones with the desired property. In fact the process in [17] is quite the opposite, namely it consists of intersection from above. That is, in place of the construction of Theorem 8, for example, one simply takes the intersection of all radicals P such that $M \subseteq P$ and $FP = P$.

Another problem which might be considered is as follows: Suppose one is given a particular class M. Does M contain a radical class, or even more, does it contain a radical with some specified property? If so, does it contain a largest radical with the desired property?

This is rather too large a project, but we have a partial answer; in fact, there is sort of a four-way duality between largest and smallest; radical, and semisimple, as follows:

THEOREM 10. *If a class M contains a largest class M_1 with property* (R2) *then M contains a largest radical class and a largest hereditary radical class. Moreover, if a given semisimple property admits a smallest theorem, then M contains a largest radical whose semisimple class has the property, and conversely.*

That is, there is a smallest theorem for a semisimple class if and only if there is a largest theorem for the corresponding radical.

The dual theorem is

THEOREM 11. *If a class M contains a largest class M_1 with property* (S2) *then M contains a largest semisimple class and, for a given radical property, a smallest theorem exists if and only if M contains a largest semisimple class whose radical has the property.*

The constructions are as follows: For Theorem 10 define inductively

$$M_{n+1} = \{R \in M_n \mid \text{every } 0 \neq R/I \text{ has an ideal in } M_n\}.$$

Then $P = \bigcap_n M_n$ is the largest radical in M and $P' = \{R \in P \mid I\{R\} \subseteq P\}$ is the largest hereditary radical contained in M.

For Theorem 11 the construction must in general go transfinite, namely

$$M_\beta = \begin{cases} \bigcap\limits_{\alpha<\beta} M_\alpha & \text{when } \beta \text{ is a limit ordinal, otherwise} \\ R \in M_{\beta-1} \mid \text{if } 0 \neq I \lhd R \text{ then some } 0 \neq I/J \in M_{\beta-1}. \end{cases}$$

Then $Q = \bigcap_\beta M_\beta$ is the largest semisimple class contained in M.

There are a number of open questions in this area. For example, one would like a better way of characterizing those radical or semisimple properties admitting smallest theorems, other than to simply search for an admissible function to express the property. Also one would like to know more about the relationship of properties of the initial class M to those of the smallest radical class, particularly those already possessed by the lower radical. Another problem is to clarify the role played by the class M (of Theorems 10 or 11) relative to which dualization occurs. The conditions stated for M are sufficient but probably not necessary. On the other hand, some sort of conditions are going to be needed, since there are classes containing radicals (or semisimple classes) not containing even a maximal one.

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