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POSITIVE SOLUTIONS FOR SEMI-POSITONE SYSTEMS IN AN ANNULUS

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ABSTRACT. We study existence and multiplicity results for positive solutions for semi-linear elliptic systems of the form

$$(p(t)u')' = -\lambda f(u,v)p(t); \quad t \in (a,b)$$
$$(p(t)v')' = -\lambda g(u,v)p(t); \quad t \in (a,b)$$
$$u(a) = 0 = u(b), \qquad v(a) = 0 = v(b)$$

where $\lambda > 0$ is a parameter, $p : [a,b] \to \mathbf{R}$ is continuous with $p > 0$ on $[a,b]$ and $g, f : [0,\infty) \times [0,\infty) \to \mathbf{R}$ are continuous such that $f(u,v) \geq -(M/2)$, $g(u,v) \geq -(M/2)$ for every $(u,v) \in [0,\infty) \times [0,\infty)$ for some $M > 0$. Our proofs are based on fixed point theory in a cone. Our results extend existence results for single semi-positone equations to semi-positone systems. We also establish a multiplicity result which is new even in the case of single equations.

1. Introduction. We consider the existence of positive solutions for the system

$$\begin{aligned} &(p(t)u')' = -\lambda f(u,v)p(t); \quad t \in (a,b) \\ &(p(t)v')' = -\lambda g(u,v)p(t); \quad t \in (a,b) \\ &u(a) = 0 = u(b), \qquad v(a) = 0 = v(b) \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter,

$$p : [a,b] \to \mathbf{R} \quad \text{is continuous with} \quad p > 0 \quad \text{on} \quad [a,b], \tag{A.1}$$

and

$$\begin{aligned} &g, f : [0,\infty) \times [0,\infty) \to \mathbf{R} \quad \text{are continuous and there exists} \\ &M > 0 \quad \text{such that} \quad f(u,v) \geq -(M/2),\ g(u,v) \geq -(M/2) \\ &\text{for every} \quad (u,v) \in [0,\infty) \times [0,\infty). \end{aligned} \tag{A.2}$$

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In particular, our results apply to the case when $f(0,0)$ or $g(0,0)$ (or both) are negative (semi-positone systems). Semi-positone problems naturally occur in the study of steady state reaction-diffusion processes with "harvesting." The fact that the reaction term may be negative at the origin makes it a very challenging problem in showing the positivity of the solution. In the case of systems it is even more challenging since we have to show the positivity of every component. In this paper we restrict our analysis to a system of two equations. Our system (1.1) is a generalization of the study of radial solutions of the steady state reaction-diffusion system in an annulus $\Omega = \{x \in \mathbf{R}^n : a < \|x\| < b\}$ given by

$$\begin{aligned} \Delta u &= -\lambda f(u,v); \quad x \in \Omega \\ \Delta v &= -\lambda g(u,v); \quad x \in \Omega \\ u &= 0 = v; \qquad x \in \partial\Omega, \end{aligned}$$

in which case $p(t) = t^{n-1}$. For existence results for the single semi-positone equation, see [**1**–**5**]. In this paper we first consider the case when f, g further satisfy:

(A.3)

$$\lim_{v\to\infty} f(u,v) = \infty, \ \lim_{u\to\infty} g(u,v) = \infty \quad \text{where each limit is uniform}$$

$$\text{with respect to the other variable, and} \quad \lim_{z\to\infty} (h^*(z)/z) = \infty$$

$$\text{where} \quad h^*(z) := \inf_{u,v\geq z} (\min(f(u,v), g(u,v)))$$

and prove an existence result for λ small. Namely, we prove:

Theorem 1.1. *Let* (A.1)–(A.3) *hold. Then there exists* $\lambda^* > 0$ *such that for* $\lambda < \lambda^*$*, the system* (1.1) *has a positive solution* (u_λ, v_λ) *with* $|(u_\lambda(t), v_\lambda(t))| \to \infty$ *as* $\lambda \to 0$ *uniformly for* t *in compact intervals of* (a,b). *Here* $|(u,v)| = |u| + |v|$.

Next we consider the case when f, g satisfy (A.2) and

(A.4)

$$\lim_{v\to\infty} f(u,v) = \infty, \ \lim_{u\to\infty} g(u,v) = \infty \quad \text{where each limit is uniform}$$

$$\text{with respect to the other variable, and} \quad \lim_{z\to\infty} (\tilde{h}(z)/z) = 0$$

$$\text{where} \quad \tilde{h}(z) := \sup_{0\leq u,v\leq z} (\max(f(u,v), g(u,v))),$$

and prove an existence result for λ large. Namely, we prove:

Theorem 1.2. *Let* (A.1), (A.2) *and* (A.4) *hold. Then there exists* $\tilde{\lambda} > 0$ *such that for* $\lambda > \tilde{\lambda}$, *the system* (1.1) *has a positive solution* (u_λ, v_λ) *with* $\lambda^{-1}\max(u_\lambda(t), v_\lambda(t)) \to \infty$ *as* $\lambda \to \infty$ *uniformly for* t *in compact intervals of* (a, b).

Finally, we consider the case when $f(u,v) = f(v)$, $g(u,v) = g(u)$ satisfy:
(A.5)

$$f, g : [0,\infty) \to \mathbf{R} \quad \text{are continuous, nondecreasing,}$$
$$f(0) < 0,\ g(0) < 0, \quad \text{and} \quad \lim_{x\to\infty}(f(x)/x) = \lim_{x\to\infty}(g(x)/x) = \infty$$

which is a special case of (A.3), and
(A.6)

$$\text{there exists} \quad r > 0 \quad \text{and} \quad 0 < \alpha < 1 \quad \text{such that} \quad h(x) \geq (x/r)^\alpha h(r)$$
$$\text{for} \quad x \in [0, r] \quad \text{where} \quad h(x) = \min(f(x) - f(0), g(x) - g(0)),$$

and discuss the existence of at least two positive solutions for a certain range of λ. Namely, we prove:

Theorem 1.3. *Let* (A.1), (A.5) *and* (A.6) *hold and* $m \geq 1$ *be such that* $\bar{h}(r) \leq mh(r)$ *where* $\bar{h}(x) = \max(f(x) - f(0), g(x) - g(0))$. *Then there exists* $C(m) > 0$ *and an interval* I *such that if* $h(r) > C(m)$, *then* (1.1) *has at least two positive solutions for* $\lambda \in I$.

We note that multiplicity results for the case of superlinear semi-positone single equations was known only in the case when $p(t) \equiv 1$, see [**6**]. Thus, Theorem 1.3 is new even in the case of single equations. Also in the case when $f(0) = 0 = g(0)$ the existence of at least two positive solutions for $0 < \lambda < \lambda^{**}$ for some $\lambda^{**} > 0$, follows by modifying the proof of Theorem 1.3.

We prove Theorem 1.1 in Section 2, Theorem 1.2 in Section 3 and Theorem 1.3 in Section 4. In Section 5 we discuss various examples satisfying the hypotheses of Theorems 1.1–1.3. Our proofs are based on fixed point theorems in a cone.

2. Proof of Theorem 1.1. Let w be the solution of

$$\begin{gathered}(p(t)w')' = -\lambda p(t)M \\ w(a) = 0 = w(b).\end{gathered}$$

Then there exists a $\tilde{K} > 0$ such that

$$w(t) \le \lambda\tilde{K}r(t) \tag{2.1}$$

where $r(t) = 1/(b-a)\min(t-a, b-t)$.

Next we note that (u,v) is a positive solution of (1.1) if and only if $\tilde{u} = u + w$, $\tilde{v} = v + w$ is a solution of

$$\begin{gathered}(p(t)\tilde{u}')' = -\lambda\tilde{f}(\tilde{u}-w, \tilde{v}-w)p(t); \quad t \in (a,b) \\ (p(t)\tilde{v}')' = -\lambda\tilde{g}(\tilde{u}-w, \tilde{v}-w)p(t); \quad t \in (a,b) \\ \tilde{u}(a) = 0 = \tilde{u}(b), \qquad \tilde{v}(a) = 0 = \tilde{v}(b)\end{gathered} \tag{2.2}$$

with $\tilde{u} > w$, $\tilde{v} > w$, where

$$\tilde{f}(x,y) = f(\max(x,0), \max(0,y)) + M,$$

and

$$\tilde{g}(x,y) = g(\max(x,0), \max(0,y)) + M.$$

Now for each $(\bar{u},\bar{v}) \in C[a,b] \times C[a,b]$, let $(u,v) := A(\bar{u},\bar{v})$ be the solution of

$$\begin{gathered}(p(t)u')' = -\lambda\tilde{f}(\bar{u}-w, \bar{v}-w)p(t); \quad t \in (a,b) \\ (p(t)v')' = -\lambda\tilde{g}(\bar{u}-w, \bar{v}-w)p(t); \quad t \in (a,b) \\ u(a) = 0 = u(b), \qquad v(a) = 0 = v(b).\end{gathered} \tag{2.3}$$

Let $\mathbf{K}$ be the cone defined by

$$\begin{aligned}\mathbf{K} = \{(u,v) \in C[a,b] \times C[a,b] : & u(t) \ge |u|_0 r(t), \\ & v(t) \ge |v|_0 r(t), t \in [a,b]\}\end{aligned}$$

where $|\cdot|_0$ denotes the supremum norm. Then $A : \mathbf{K} \to \mathbf{K}$ and is completely continuous, see [**3, 7**].

In order to prove Theorem 1.1, we recall:

Theorem A (see [**8**]). *Let* $\mathbf{K}$ *be a cone in a Banach space* E*, and let* $A : \mathbf{K} \to \mathbf{K}$ *be a completely continuous operator. Let* $0 < r < R$ *be such that*

$$u \le Au \Longrightarrow |u|_0 \ne r$$
$$u \ge Au \Longrightarrow |u|_0 \ne R.$$

Here $u \le v$ *if* $v - u \in \mathbf{K}$*. Then* A *has a fixed point* u *with* $r < |u|_0 < R$*.*

Now, to apply Theorem A, we first prove

Lemma 2.1. *For* $\lambda > 0$ *small enough, there exists* $A_\lambda > 0$ *such that* $(u, v) \le A(u, v)$ *implies* $|(u, v)|_0 \ne A_\lambda$*. Further* $A_\lambda \to \infty$ *as* $\lambda \to 0$*. Here* $|(u, v)|_0 := \max\{|u|_0, |v|_0\}$*.*

Proof. Let $(u, v) \in \mathbf{K}$ satisfy $(u, v) \le A(u, v)$, i.e.,

$$u(t) \le \lambda \int_a^b K(t, s)\tilde{f}(u - w, v - w)\, ds; \quad t \in [a, b]$$
$$v(t) \le \lambda \int_a^b K(t, s)\tilde{g}(u - w, v - w)\, ds; \quad t \in [a, b],$$

where $K(t, s)$ is the Green's function of $(p(t)u')' = -hp(t)$ with Dirichlet boundary conditions. Then

$$|u|_0 \le \lambda C \sup\{\tilde{f}(s, t) : |(s, t)| \le |(u, v)|_0\}$$
$$|v|_0 \le \lambda C \sup\{\tilde{g}(s, t) : |(s, t)| \le |(u, v)|_0\}$$

where $C = |K|_0(b - a)$. This implies

$$|(u, v)|_0 \le \lambda C \sup_{0 \le s, t \le |(u,v)|_0} q(s, t) \equiv \lambda C H(|(u, v)|_0)$$

where $q(s, t) = \max(\tilde{f}(s, t), \tilde{g}(s, t))$, or

$$\frac{H(|(u, v)|_0)}{|(u, v)|_0} \ge \frac{1}{\lambda C}. \tag{2.4}$$

Suppose that $\lambda < (1/(2CH(1))$. Then $(H(1)/1) < (1/(2\lambda C))$ and since $\lim_{x\to\infty}(H(x)/x) = \infty$, see (A.3), there exists $A_\lambda > 1$ such that

$$\frac{H(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda C}. \tag{2.5}$$

From (2.4) and (2.5), we deduce that $|(u,v)|_0 \neq A_\lambda$. Since $H(A_\lambda) = (A_\lambda/(2\lambda C)) \geq (1/(2\lambda C)) \to \infty$ as $\lambda \to 0$, $A_\lambda \to \infty$ as $\lambda \to 0$, and therefore $(A_\lambda/\lambda) \to \infty$ as $\lambda \to 0$. We next prove:

Lemma 2.2. *There exists $R_\lambda > A_\lambda$ such that $(u,v) \geq A(u,v) \Rightarrow |(u,v)|_0 \neq R_\lambda$.*

Proof. Let $(u,v) \in \mathbf{K}$ satisfy $(u,v) \geq A(u,v)$, i.e.,

$$u(x) \geq \lambda \int_a^b K(x,y)\tilde{f}(u-w, v-w)\,dy; \quad x \in [a,b]$$

and

$$v(x) \geq \lambda \int_a^b K(x,y)\tilde{g}(u-w, v-w)\,dy; \quad x \in [a,b].$$

Suppose that $|(u,v)|_0 = |u|_0$ and let $[c,d] \subset (a,b)$. Then we have

$$(u-w)(x) \geq |u|_0 r(x) - \lambda\tilde{K}r(x) \geq (|u|_0 - \lambda\tilde{K})\delta; \quad x \in [c,d],$$

where $\delta = \min_{[c,d]} r(x)$. Here, without loss of generality, we assume that $|u|_0 > \lambda\tilde{K}$. Hence,

$$\begin{aligned} |v|_0 &\geq \lambda\bar{c}\min\begin{cases} \inf\{\tilde{g}(s,t) : s \geq (|u|_0 - \lambda\tilde{K})\delta, t \geq 0\} \\ \inf\{\tilde{f}(s,t) : t \geq (|u|_0 - \lambda\tilde{K})\delta, s \geq 0\} \end{cases} \\ &\equiv \overline{B}(|u|_0) \end{aligned}$$

where $\bar{c} = \{\min_{[c,d]\times[c,d]} K(x,y)\}(c-d)$. Further,

$$\begin{aligned} |u|_0 &\geq \lambda\bar{c}\min\begin{cases} \inf\{\tilde{f}(s,t) : s \geq (|u|_0 - \lambda\tilde{K})\delta, t \geq B(|u|_0)\} \\ \inf\{\tilde{g}(s,t) : s \geq B(|u|_0), t \geq (|u|_0 - \lambda\tilde{K})\delta\} \end{cases} \\ &\equiv \lambda\bar{c}\bar{A}(|u|_0) \end{aligned} \tag{2.6}$$

where $B(|u|_0) = (\overline{B}(|u|_0) - \lambda\tilde{K})\delta$ and hence

$$\frac{\bar{A}(|u|_0)}{|u|_0} \leq \frac{1}{\lambda\bar{c}}.$$

Once again, without loss of generality, we assume $|u|_0$ is large enough so that $\overline{B}(|u|_0) > \lambda\tilde{K}$. Since $\lim_{x\to\infty}(\bar{A}(x)/x) = \infty$, see (A.3), there exists $R_\lambda > A_\lambda$ such that $(\bar{A}(R_\lambda)/R_\lambda) > (2/(\lambda\bar{c}))$. Consequently, $|u|_0 \neq R_\lambda$. Similarly, $|v|_0 \neq R_\lambda$ if $|(u,v)|_0 = |v|_0$.

Now, using Lemmas 1 and 2 and Theorem A, we establish Theorem 1.1.

Proof of Theorem 1.1. From Lemmas 2.1, 2.2 and Theorem A, it follows that there exists

$$(\tilde{u}_\lambda, \tilde{v}_\lambda) \in \mathbf{K} \text{ with } |(\tilde{u}_\lambda, \tilde{v}_\lambda)|_0 \geq A_\lambda \text{ such that } (\tilde{u}_\lambda, \tilde{v}_\lambda) = A(\tilde{u}_\lambda, \tilde{v}_\lambda).$$

Without loss of generality, we assume that $|(\tilde{u}_\lambda, \tilde{v}_\lambda)| = |\tilde{u}_\lambda|_0$. Then

$$\begin{aligned} u_\lambda(x) &= \tilde{u}_\lambda(x) - w(x) \\ &\geq (|\tilde{u}_\lambda|_0 - \lambda\tilde{K})r(x) \\ &\geq (A_\lambda - \lambda\tilde{K})r(x) > 0 \quad \text{for } \lambda \text{ small} \end{aligned}$$

since $A_\lambda \to \infty$ as $\lambda \to 0$. Also, we obtain

$$|\tilde{v}_\lambda|_0 \geq \lambda\bar{c}\inf\{\tilde{g}(s,t) : s \geq (A_\lambda - \lambda\tilde{K})\delta, t \geq 0\} \equiv \lambda\bar{c}\bar{A}_\lambda,$$

where $\bar{c} = \{\min_{[c,d]\times[c,d]} K(x,y)\}(c-d)$. Note that $A_\lambda \to \infty$ as $\lambda \to 0$ and so $\bar{A}_\lambda \to \infty$ as $\lambda \to 0$. Consequently, $v_\lambda(x) \geq |\tilde{v}_\lambda|_0 r(x) - \lambda\tilde{K}r(x) \geq \lambda(\bar{c}\bar{A}_\lambda - \tilde{K})r(x) > 0$ for λ small. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2. First we prove:

Lemma 3.1. *For $\lambda > 0$ large enough, there exists $B_\lambda > 0$ with $(B_\lambda/\lambda) \to \infty$ as $\lambda \to \infty$ such that $(u,v) \geq A(u,v) \Rightarrow |(u,v)|_0 \neq B_\lambda$.*

Proof. Let $(u,v) \in \mathbf{K}$ satisfy $(u,v) \geq A(u,v)$. Suppose that $|(u,v)|_0 = |u|_0$. Then, as in Lemma 2.2, see (2.6), we obtain

$$|u|_0 \geq \lambda \bar{c} \bar{A}(|u|_0). \tag{3.1}$$

Now suppose that $\lambda > (2/(\bar{c}M_1))$ where $M_1 = M/2 (> 0)$. Then

$$\frac{\bar{A}(1)}{1} > M_1 > \frac{2}{\lambda \bar{c}}$$

and since $\lim_{z\to\infty}(\bar{A}(z)/z) = 0$, there exists $B_\lambda > 0$ such that

$$\frac{\bar{A}(B_\lambda)}{B_\lambda} = \frac{2}{\lambda \bar{c}}. \tag{3.2}$$

From (3.1), we have $(\bar{A}(|u|_0)/|u_0|) \leq (1/(\lambda\bar{c})) < (2/(\lambda\bar{c}))$ and hence by (3.2) we deduce that $|u|_0 \neq B_\lambda$. In a similar way, we obtain $|v|_0 \neq B_\lambda$ if $|(u,v)|_0 = |v|_0$. From (3.2), it follows that $B_\lambda \to \infty$ as $\lambda \to \infty$, and since $\lim_{z\to\infty} \bar{A}(z) = \infty$, we obtain $B_\lambda/\lambda \to \infty$ as $\lambda \to \infty$.

Next we prove:

Lemma 3.2. *Let λ be as in Lemma* 3.1. *Then there exists $R_\lambda > B_\lambda$ such that*

$$(u,v) \leq A(u,v) \Longrightarrow |(u,v)|_0 \neq R_\lambda.$$

Proof. Let $(u,v) \in \mathbf{K}$ satisfy $(u,v) \leq A(u,v)$. Then $|(u,v)|_0 \leq \lambda C \sup\{h_1(s,t) : 0 \leq s,t \leq |(u,v)|_0\} \equiv \lambda C \tilde{h}(|(u,v)|_0)$, where $h_1(s,t) = \max(\tilde{f}(s,t), \tilde{g}(s,t))$ and $\tilde{h}(z) = \sup\{h_1(s,t) : 0 \leq s,t \leq z\}$. Thus,

$$\frac{\tilde{h}(|(u,v)|_0)}{|(u,v)|_0} \geq \frac{1}{\lambda C} \quad \text{while} \quad \lim_{z\to\infty} \frac{\tilde{h}(z)}{z} = 0.$$

Hence it follows that there exists $R_\lambda > B_\lambda$ such that $|(u,v)|_0 \neq R_\lambda$.

We now establish Theorem 1.2 by using Lemmas 3.1, 3.2 and Theorem A.

Proof of Theorem 1.2. It follows from Lemmas 3.1, 3.2 and Theorem A that there exists $(\tilde{u}_\lambda, \tilde{v}_\lambda) \in \mathbf{K}$ with $|(\tilde{u}_\lambda, \tilde{v}_\lambda)|_0 \geq B_\lambda$ if λ is large enough. Without loss of generality we can assume that $|(\tilde{u}_\lambda, \tilde{v}_\lambda)|_0 = |\tilde{u}|_0$. Then

$$\begin{aligned} u_\lambda(t) &= \tilde{u}_\lambda(t) - w(t) \\ &\geq |\tilde{u}_\lambda|_0 r(t) - \lambda \tilde{K} r(t) \\ &\geq \lambda \left[\frac{B_\lambda}{\lambda} - \tilde{K} \right] r(t) > 0 \end{aligned}$$

for λ large. Consequently, from $\tilde{v}_\lambda(t) = \lambda \int_a^b K(t,s) \tilde{g}(\tilde{u}_\lambda - w, \tilde{v}_\lambda - w)\, ds$, we obtain

$$|\tilde{v}_\lambda|_0 \geq \lambda \bar{c} \inf\{\tilde{g}(s,t) : s \geq \lambda[(B_\lambda/\lambda) - \tilde{K}]\delta, t \geq 0\} = \lambda \bar{c} D_\lambda,$$

and

$$\begin{aligned} v_\lambda(t) &= \tilde{v}_\lambda(t) - w(t) \\ &\geq |\tilde{v}_\lambda|_0 r(t) - \lambda \tilde{K} r(t) \\ &\geq \lambda[\bar{c} D_\lambda - \tilde{K}] r(t), \quad t \in (a,b). \end{aligned}$$

Since $D_\lambda \to \infty$ as $\lambda \to \infty$, this completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.3. Here we assume $f(u,v) = f(v)$, $g(u,v) = g(u)$, (A.1), (A.5) and (A.6).

Let w_1 and w_2 be the solutions of

$$\begin{gathered} (p(t) w_1')' = \lambda p(t) f(0) \\ w_1(a) = 0 = w_1(b) \end{gathered}$$

and

$$\begin{gathered} (p(t) w_2')' = \lambda p(t) g(0) \\ w_2(a) = 0 = w_2(b), \end{gathered}$$

respectively. Then there exists $\tilde{K} > 0$ such that

$$w_i(t) \leq \lambda \tilde{K} r(t); \quad i = 1, 2,$$

where $r(t)$ is as before.

Now (u, v) is a positive solution of (1.1) if and only if $\tilde{u} = u + w_1$ and $\tilde{v} = v + w_2$ satisfy:

$$\begin{gathered}(p(t)\tilde{u}')' = -\lambda p(t)\tilde{f}(\tilde{v} - w_2)\\ (p(t)\tilde{v}')' = -\lambda p(t)\tilde{g}(\tilde{u} - w_1)\\ \tilde{u}(a) = 0 = \tilde{u}(b), \qquad \tilde{v}(a) = 0 = \tilde{v}(b),\end{gathered} \tag{4.1}$$

respectively, and $\tilde{u} > w$, $\tilde{v} > w_2$ on (a, b).

Here

$$\tilde{f}(s) = \begin{cases} f(s) - f(0) & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases}$$

and

$$\tilde{g}(s) = \begin{cases} g(s) - g(0) & \text{if } s \geq 0 \\ 0 & \text{if } s < 0. \end{cases}$$

For $(u_1, v_1) \in C[a, b] \times C[a, b]$, let $(u, v) = A(u_1, v_1)$ be the solution of

$$\begin{gathered}(p(t)u')' = -\lambda p(t)\tilde{f}(v_1 - w_2)\\ (p(t)v')' = -\lambda p(t)\tilde{g}(u_1 - w_1)\\ u_1(a) = 0 = u_1(b), \qquad v_1(a) = 0 = v_1(b).\end{gathered} \tag{4.2}$$

Let $\mathbf{K}$ be the cone defined by

$$\mathbf{K} = \{(u, v) \in C[a, b] \times C[a, b] : u(t) \geq |u|_0 r(t), v(t) \geq |v|_0 r(t)\}.$$

Then $A : \mathbf{K} \to \mathbf{K}$ is completely continuous. Let the Green's function $K(s, t)$, constants $C, \bar{c}$ and δ (< 1) be as before. Let

$$h(r) > C(m) := \frac{2\tilde{K}}{C}\left[\frac{\bar{c}\delta^\alpha}{C2^\alpha m}\right]^{-1/(1-\alpha)}. \tag{4.3}$$

Choose γ so that

$$\frac{2r\tilde{K}}{Ch(r)} < \gamma < \left(\frac{\bar{c}\delta^\alpha}{C2^\alpha m}\right)^{1/1-\alpha} r.$$

Let $I = ((\gamma/(\bar{c}h(\gamma\delta/2))), (r/(C\bar{h}(r))))$. Then $I \neq \varnothing$, since $\gamma < r$ and hence

$$\frac{\gamma}{\bar{c}h(\gamma\delta/2)} \leq \frac{\gamma}{\bar{c}(\gamma\delta/(2r))^\alpha h(r)} < \frac{\gamma^{1-\alpha} m(2r)^\alpha}{\bar{c}\delta^\alpha \bar{h}(r)}$$
$$< \left(\frac{\bar{c}\delta^\alpha}{C2^\alpha m} r^{1-\alpha}\right) \frac{m(2r)^\alpha}{\bar{c}\delta^\alpha \bar{h}(r)} = \frac{r}{C\bar{h}(r)}.$$

We claim that (1.1) has two positive solutions for $\lambda \in I$. To prove our claim first we prove:

Lemma 4.1. *If $(u, v) \leq A(u, v)$, then $|(u, v)|_0 \neq r$.*

Proof. Suppose that $(u, v) \leq A(u, v)$. Then

$$u(x) \leq \lambda \int_a^b K(x, y)\tilde{f}(v - w_2)(y)\,dy$$

and

$$v(x) \leq \lambda \int_a^b K(x, y)\tilde{g}(u - w_1)(y)\,dy.$$

Assume that $|(u, v)|_0 = |u|_0 = r$. Then

$$|u|_0 \leq \lambda C\tilde{f}(|v|_0) \leq \lambda C\tilde{f}(|u|_0) \leq \lambda C\bar{h}(|u|_0)$$

or $(r/(C\bar{h}(r))) \leq \lambda$, a contradiction with $\lambda \in I$. Similarly, if $|(u, v)|_0 = |v|_0 = r$, we obtain a contradiction.

Next we prove:

Lemma 4.2. *If $(u, v) \geq A(u, v)$, then $|(u, v)|_0 \neq \gamma$, R where $R > r$.*

Proof. Suppose that $(u, v) \geq A(u, v)$ and $|(u, v)|_0 = |u|_0 = \gamma$. Now

$$v(x) \geq \lambda \int_c^d K(x, y)\tilde{g}(u - w_1)(y)\,dy,$$

and

$$\begin{aligned} u(y) - w_1(y) &\geq |u|_0 r(y) - \lambda\tilde{K} r(y) \\ &= (\gamma - \lambda\tilde{K}) r(y) \\ &\geq \frac{\gamma}{2}\delta; \quad y \in [c, d], \end{aligned}$$

since

$$\begin{aligned}\frac{\gamma}{2} - \lambda\tilde{K} &> \frac{r\tilde{K}}{Ch(r)} - \lambda\tilde{K} \\ &> \left(\frac{r}{Ch(r)} - \lambda\right)\tilde{K} \\ &> \left(\frac{r}{Ch(r)} - \frac{r}{C\bar{h}(r)}\right)\tilde{K} \geq 0.\end{aligned}$$

Hence, we obtain

$$\gamma = |u|_0 \geq |v|_0 \geq \lambda\bar{c}\tilde{g}\left(\frac{\gamma\delta}{2}\right) \geq \lambda\bar{c}h\left(\frac{\gamma\delta}{2}\right),$$

or

$$\frac{h(\gamma(\delta/2))}{\gamma} \leq \frac{1}{\lambda\bar{c}} \tag{4.4}$$

which is a contradiction since $\lambda \in I$.

Thus, $|u|_0 \neq \gamma$. Also, if $|u|_0 = R \gg r$, then repeating the above steps in establishing (4.4) we obtain $(h(R(\delta/2))/R) \leq (1/(\lambda\bar{c}))$. But since $\lim_{x\to\infty}(h(x)/x) = \infty$, there exists $R \gg r$ such that $|u|_0 \neq R$. Similarly, Lemma 4.2 can be proven in the case when $|(u,v)|_0 = |v|_0$.

Now we establish Theorem 1.3.

Proof of Theorem 1.3. From Lemmas 4.1, 4.2 and standard fixed point theorems, it follows that (4.1) has solutions $(\tilde{u}_1, \tilde{v}_1)$ and $(\tilde{u}_2, \tilde{v}_2)$ with $|(\tilde{u}_1, \tilde{v}_1)|_0 > r$ and $\gamma < |(\tilde{u}_2, \tilde{v}_2)|_0 < r$.

Now suppose that $|\tilde{u}_2|_0 \geq |\tilde{v}_2|_0$. Then

$$\begin{aligned}u_2(x) &= \tilde{u}_2(x) - w_1(x) \\ &\geq |\tilde{u}_2|_0 r(x) - \lambda\tilde{K}r(x) \\ &\geq (\gamma - \lambda\tilde{K})r(x) \\ &\geq \left(\frac{2r\tilde{K}}{Ch(r)} - \lambda\tilde{K}\right)r(x) \\ &\geq \left(\frac{2r}{Ch(r)} - \frac{r}{C\bar{h}(r)}\right)\tilde{K}r(x) > 0,\end{aligned}$$

for $x \in (a,b)$. Using $\tilde{v}_2(x) = \lambda \int_a^b K(x,y)\tilde{g}(\tilde{u}_2 - w_1)$ and the fact that $\tilde{u}_2(y) - w_1(y) \geq (\gamma\delta/2)$; $y \in [c,d]$, we deduce that

$$|\tilde{v}_2|_0 \geq \lambda \bar{c}\tilde{g}\left(\frac{\gamma\delta}{2}\right) \geq \lambda \bar{c} h\left(\frac{\gamma\delta}{2}\right) \geq \lambda \bar{c}\left(\frac{\gamma\delta}{2r}\right)^{\alpha} h(r) > \lambda \tilde{K},$$

see (4.3). This implies

$$\tilde{v}_2(x) - w_2(x) \geq |\tilde{v}_2|_0 r(x) - \lambda \tilde{K} r(x) > (\lambda \tilde{K} - \lambda \tilde{K}) r(x) = 0.$$

Similarly, one can prove that $\tilde{u}_1 > w_1(x)$ and $\tilde{v}_1 > w_2(x)$. Hence, the result. □

5. Examples. In this section we discuss examples satisfying the hypotheses of Theorems 1.1–1.3.

Example 5.1. Let

$$f(u,v) = u^2v^2 + u + v - 1,$$
$$g(u,v) = u^4v^4 + u + v - 1.$$

Clearly (A.2) is satisfied. Further, $f(u,v) \geq v - 1$ and $g(u,v) \geq u - 1$ and $h^*(z) = z^4 + 2z - 1$ and thus (A.3) is satisfied, and hence all the hypotheses of Theorem 1.1 are satisfied.

Example 5.2. Let

$$f(u,v) = u^{1/2}v^{1/4} + u^{1/2} + v^{1/2} - 1$$
$$g(u,v) = u^{1/3}v^{1/8} + u^{1/3} + v^{1/8} - 1.$$

Clearly (A.2) is satisfied. Further $f(u,v) \geq v^{1/2} - 1$ and $g(u,v) \geq u^{1/3} - 1$ and $\tilde{h}(z) = z^{3/4} + 2z^{1/2} - 1$ and thus (A.4) is satisfied and hence all the hypotheses of Theorem 1.2 are satisfied.

Example 5.3. Let

$$f(x) = \begin{cases} ax^{1/2} - 1; & x \in [0,1], \\ ax^{\beta} - 1; & x > 1, \end{cases}$$

and

$$g(x) = \begin{cases} (ka)x^{1/3} - 1; & x \in [0,1], \\ (ka)x^{\beta_1} - 1; & x > 1, \end{cases}$$

where $a > 0$, $k \geq 1$, $\beta > 1$ and $\beta_1 > 1$. Then clearly (A.5) is satisfied and taking $r = 1$ we have $h(x) = ax^{1/2}$, $\bar{h}(x) = kax^{1/3}$, and hence $h(1) = a$ and $\bar{h}(1) = ka$. Thus, for $\alpha = (1/2)$, (A.6) is satisfied, and since $m = k$, if $a > C(m) = (2\tilde{K}/C)[\bar{C}\gamma^\alpha/(C2^\alpha k)]^{-1/(1-\alpha)}$ then all the hypotheses of Theorem 1.3 are satisfied.

Example 5.4. Next we discuss an example for the single equation case (or when $f(x) = g(x)$). In this case Theorem 1.3 holds if (A.1) is satisfied, $f : [0,\infty) \to \mathbf{R}$ is a continuous, nondecreasing function such that $f(0) < 0$, $\lim_{x\to\infty}(f(x)/x) = \infty$, there exists $r > 0$ and $0 < \alpha < 1$ such that $\tilde{f}(x) \geq (x/r)^\alpha \tilde{f}(r)$ for $x \in [0,r]$ where $\tilde{f}(x) = f(x) - f(0)$ and if

$$\tilde{f}(r) > \frac{2\tilde{K}}{C}\left[\frac{\bar{c}\delta^\alpha}{C2^\alpha}\right]^{-1/(1-\alpha)}. \tag{5.1}$$

Now consider $f(x) = a(x^{3/2} + x^{1/4}) - 1$; $a > 0$. Taking $r = 1$ we have $\tilde{f}(x) = a(x^{3/2} + x^{1/4}) \geq 2a(x^{3/2}.x^{1/4})^{1/2} = (x/1)^{7/8}.\tilde{f}(1)$.

Hence, if a is large enough, $\tilde{f}(1)(= 2a)$ will satisfy (5.1), and all the necessary requirements will be satisfied for Theorem 1.3 to hold.

REFERENCES

1. W. Allegretto, P. Nistri and P. Zecca, *Positive solutions of elliptic non-positone problems*, Differential Integral Equations **5** (1992), 95–101.

2. A. Ambrosetti, D. Arcoya and B. Buffoni, *Positive solutions for some semi-positone problems via bifurcation theory*, Differential Integral Equations **7** (1994), 655–663.

3. V. Anuradha, H. Dang and R. Shivaji, *Existence results for superlinear semipositone BVP's*, Proc. Amer. Math. Soc. **124** (1996), 757–763.

4. V. Anuradha, S. Dickens and R. Shivaji, *Existence results for non-autonomous elliptic boundary value problem*, Electron. J. Differential Equations **4** (1994), 1–10.

5. A. Castro and R. Shivaji, *Non-negative solutions for a class of radially symmetric non-positone problems*, Proc. Amer. Math. Soc. **106** (1989).

6. A. Castro and R. Shivaji, *Non-negative solutions for a class of non-positone problems*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), 291–302.

7. H. Dang and K. Schmitt, *Existence of positive radial solutions for semilinear elliptic equations in annular domains*, Differential Integral Equations **71** (1994), 747–758.

8. M.A. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff, Groningen, The Netherlands, 1964.

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