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MODULES FOR WHICH HOMOGENEOUS MAPS ARE LINEAR

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ABSTRACT. Given an R-module V, the near-ring of homogeneous maps $\mathcal{M}_R(V)$ is the set of maps $\{f : V \to V \mid f(rv) = rf(v)$ for all $r \in R$ and $v \in V\}$ endowed with point-wise addition and composition of functions as multiplication. Modules with the property that $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ when R is commutative and Noetherian, and V is finitely generated, are characterized. Commutative Noetherian rings with the property that $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ for all uniform modules, V, are also classified.

1. Introduction. Let R be a commutative Noetherian ring with identity and V a nonzero unital R-module. The set of maps $\mathcal{M}_R(V) := \{f : V \to V \mid f(rv) = rf(v)$ for all $r \in R$ and $v \in V\}$ is a right near-ring under point-wise addition and composition of functions, and the elements are called *homogeneous maps*. This near-ring has been the subject of several investigations. See, for example, [**3**] and [**4**]. We write functions on the left of the elements on which they act; therefore $\mathcal{M}_R(V)$ satisfies the right distributive law. Recall that an R-module V is *uniform* if for any nonzero R-submodules M and N, $M \cap N \neq \langle 0\rangle$. In the third section, we will see in particular that modules over Dedekind domains are rather well behaved, since V uniform implies in this case that $\mathcal{M}_R(V) = \mathrm{End}_R(V)$. In fact, if we restrict ourselves to domains, this property will characterize Dedekind domains. From this consideration, we conclude that the problem of determining when homogeneous maps are linear becomes significantly more interesting when we consider Noetherian rings in general.

2. When are all the homogeneous maps on a finitely generated module linear? We denote the injective hull of V by $E(V)$. Since every module can be embedded in an injective module, the fol-

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lowing structural result of Matlis is useful in our situation. (See [**6**] for an exposition of this result).

Theorem 2.1 (Matlis). *Let R be a commutative Noetherian ring. Then the following holds.*

1. *Every injective module is uniquely a direct sum of uniform injective modules.*

2. *The map $P \mapsto E(R/P)$yields a one-to-one correspondence between the prime ideals P of R and the isomorphism classes of uniform injective R-modules.*

3. *If P is a prime ideal of R, then every element of $E(R/P)$ is annihilated by some power of P.*

The module $E(R/P)$ may be regarded as an R_P-module, and the action of $r \in R$ on $E(R/P)$ is the same as the action of $r/1 \in R_P$ on $E(R/P)$ (see [**8**, Chapter 5] for details), thus multiplication by elements in $R\backslash P$ is an isomorphism on $E(R/P)$. The third part of the Matlis' theorem leads us to the following definition.

Definition 2.2. Assume I is an ideal of R. For $v \in V$, we define the I-*exponent* of v (I-*exp* v) to be the smallest nonnegative integer s such that $I^s v = 0$. If $I^n v \neq 0$ for all n, then we define I-*exp* v to be ∞. We define I^0 to be R, even if I is the zero ideal, so that I-*exp* $v \geq 1$, unless $v = 0$, in which case it equals 0.

Lemma 2.3. *Let $v \in V\backslash\{0\}$ and suppose I-exp $v < \infty$; then there exists $d \in I^m$ such that I-exp $(dv) = 1$, where $m = (I\text{-}exp\ v) - 1$.*

Proof. This is clear since $I^{m+1}v = 0$, but $I^m v \neq 0$. □

Since multiplication by $s \in R\backslash P$ acts as an isomorphism on $E(R/P)$, we have the following result.

Lemma 2.4. *Let $V \subseteq E(R/P)$ and $s \in R\backslash P$. Then, if $v \in V$, P-exp $v = P$-exp (sv).*

In the remainder of this section, R will be a commutative Noetherian ring and V a finitely generated R-module, unless stated otherwise. In the next result we show how to construct a nonlinear homogeneous map under certain conditions.

Theorem 2.5. *Suppose* $E(V) = \oplus_{i=1}^n E(R/P_i)$ *and* $P_i \subseteq P_n$ *for all* i *where* $P_1, \ldots, P_n$ *are prime ideals of* R. *Then there exists a* $\Lambda \in \mathcal{M}_R(V)$ *such that* $\Lambda(V) \subseteq E(R/P_n)$ *and* $P_n\Lambda(V) = 0$, *but* $\Lambda(V) \neq 0$. *We also have that* $\Lambda \notin \mathrm{End}_R(V)$ *if* V_{P_n} *is not* R_{P_n}*-cyclic. Conversely, if* V_{P_n} *is* R_{P_n}*-cyclic, then* $\mathcal{M}_R(V) = \mathrm{End}_R(V)$.

Proof. Let $Q := P_n$. Define $\Pi : \oplus_{i=1}^n E(R/P_i) \to E(R/Q)$ by $v_1 + \cdots + v_n \mapsto v_n$. Since $E(V)$ is an essential extension of V, we have that $V \cap E(R/Q) \neq 0$ and thus that $\Pi(V) \neq 0$. Since V is finitely generated, we can choose $a \in V$ such that Q-*exp* $\Pi(a)$ is as large as possible (use Matlis's theorem). From Lemma 2.3, we have a $d' \in Q^m$ with $m = Q\text{-}exp\ \Pi(a) - 1$ such that $Q\text{-}exp\ d'\Pi(a) = 1$. From the fact that $E(V)$ is an essential extension of V, it follows that there exists $t \in R\backslash Q$ such that $td'\Pi(a) \in V$ (note if $t \in Q$, then $td'\Pi(a) = 0$). From Lemma 2.4, $Q\text{-}exp\ td'\Pi(a) = 1$. Let $d = td'$. Since $Q\text{-}exp\ \Pi(a) \geq Q\text{-}exp\ \Pi(v)$ for all $v \in V$, and since $d \in Q^m$, we have that $Q\text{-}exp\ d\Pi(v) \leq 1$ for all $v \in V$. Let $X = R_Q a \cap V$, and suppose X is generated as an R-module by $g_i = (r_i/s_i)a$ for $i = 1, \ldots, m$. Let $s = \prod_{i=1}^m s_i$ and $\Psi = sd\Pi$. Then $\Psi(X) \subseteq V$, and from Lemma 2.4 we have $Q\text{-}exp\ \Psi(a) = 1 \geq Q\text{-}exp\ \Psi(x)$ for all $x \in X$.

Define $\Lambda : V \to V$ by

$$\Lambda(v) = \begin{cases} \Psi(v) & \text{if } v \in X \\ 0 & \text{otherwise.} \end{cases}$$

Now we show that Λ is homogeneous. If $v \in X$ and $r \in R$, then $rv \in X$ and thus $r\Lambda(v) = \Lambda(rv)$. If $v \notin X$ and $r \in Q$, then since $Q\text{-}exp\ \Psi(v) \leq 1$, $\Lambda(rv) = 0 = r\Lambda(v)$. If $v \notin X$ and $r \notin Q$, then $rv \notin X$, since $rv \in X$ implies that $v = (1/r)(rv) \in X$. So we conclude that Λ is homogeneous. Suppose V_Q is not R_Q-cyclic, and let $y \in V\backslash X$. Since X is a submodule of V, $a + y \notin X$. Thus $0 = \Lambda(a + y) \neq \Lambda(a) + \Lambda(y) = \Lambda(a) \neq 0$.

Conversely, suppose V_Q is R_Q-cyclic, and let $v_1, v_2 \in V$. Notice that, from the discussion following Theorem 2.1, we have that the natural

R-homomorphism from V to V_Q is a monomorphism. Thus there exists $x \in V$ such that $v_1 = (r_1/s_1)x$ and $v_2 = (r_2/s_2)x$. So for $f \in \mathcal{M}_R(V)$ we have $s_1 s_2 f(v_1 + v_2) = f(s_2 r_1 x + s_1 r_2 x) = s_1 s_2(f(v_1) + f(v_2))$, from which we conclude that $f(v_1 + v_2) = f(v_1) + f(v_2)$. □

We notice that if V_Q is locally R_Q-cyclic (with Q as in the previous proof and V not necessarily finitely generated), then the argument in the last part of the previous proof will show that $\mathcal{M}_R(V) = \mathrm{End}_R(V)$, where we define V_Q to be *locally R_Q-cyclic* if there exists for each $v_1, v_2 \in V_Q$ an element x in V_Q such that $v_i = (r_i/s_i)x$ for some $r_i/s_i \in R_Q$.

Corollary 2.6. *Suppose V is a finitely generated R-module and $E(V) = E(R/P) \oplus \cdots \oplus E(R/P)$, where P is a prime ideal of R. Then $P = \sqrt{(0 : V)}$. Also, $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ if and only if V_P is R_P-cyclic.*

Proof. This follows since $\sqrt{(0 : V)} = P$ from Matlis's theorem and the remarks following it. □

Corollary 2.7. *Suppose V is a finitely generated uniform R-module. Then $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ if and only if V_P is R_P-cyclic, where $P = \sqrt{(0 : V)}$.*

Proof. The result follows from Corollary 2.6. □

Example 2.8. Let $R = k[x, y]$, where k is any field, and let $V = \langle x^2 y, xy^2 \rangle / \langle x^3 y \rangle$. Then routine calculations show that V is uniform and that $\sqrt{(0 : V)} = \langle x \rangle$. It is clear that $V_{\langle x \rangle}$ is $R_{\langle x \rangle}$-cyclic ($V_{\langle x \rangle}$ is generated by $(xy^2)/1$), and so we conclude that $\mathcal{M}_R(V) = \mathrm{End}_R(V)$.

Quite often one only knows that the injective hull of a module exists, and not much more about it. In the next few results, we therefore develop an alternative way of determining whether or not a module satisfies the hypothesis of Theorem 2.5, and also a method for determining the primes that are involved.

Definition 2.9. Let V be an R-module (not necessarily finitely generated), and let P be a prime ideal of R. We say P is an *associated prime ideal* of V, $P \in \operatorname{Ass} V$, if there exists a $v \in V$ such that $(0 : v) := \operatorname{Ann}_R(v) = P$ (see [**7**, Definition 9.32]). We will denote the maximal members of $\operatorname{Ass} V$ (which might not be maximal ideals of R) by Max-Ass V.

Lemma 2.10. *Let V be an R-module. Then each maximal member of the nonempty set $\theta := \{\operatorname{Ann}_R(v) \mid v \in V\backslash\{0\}\}$ is prime, and thus belongs to* $\operatorname{Ass} V$. *In fact, the collection of maximal members of θ is* Max-Ass V.

Proof. See [**7**, Lemma 9.34]. □

Proposition 2.11. *Let V be an R-module of finite uniform dimension* (*but not necessarily finitely generated*). *If $E(V) = \oplus_{i=1}^{n} E(R/P_i)$, then* $\operatorname{Ass} V = \{P_1, \dots, P_n\}$.

Proof. For each i, $i = 1, 2, \dots, n$, choose $v_i \in E(R/P_i) \cap (V\backslash\{0\})$, and $d_i \in R$ such that P_i-*exp* $d_i v_i = 1$. Then $P_i \subseteq (0 : dv_i)$, but from the remarks following Theorem 2.1 we also have reverse containment and thus equality. Thus $\operatorname{Ass} V \supseteq \{P_1, \dots, P_n\}$. To show that $\operatorname{Ass} V \subseteq \{P_1, \dots, P_n\}$, let $P \in \operatorname{Ass} V$. Since $P \in \operatorname{Ass} V$, there exists $v \in V$ such that $(0 : v) = P$. Suppose $v = x_1 + \cdots + x_n$ with $x_i \in E(R/P_i)$. Then $P = (0 : v) = \sqrt{(0 : v)} = \sqrt{\cap_{i=1}^{i=n}(0 : x_i)} = \cap_{x_i \neq 0} P_i$. Thus $P = P_j$ for some j (see [**7**, Lemma 3.55]). □

Proposition 2.12. *Suppose V is an R module such that* Max-Ass V *has only one element*; *then $\mathcal{M}_R(V) = \operatorname{End}_R(V)$ if and only if V_P is R_P-cyclic, where* $\{P\} =$ Max-Ass V.

Proof. From Proposition 2.11 we have that V satisfies the hypothesis of Corollary 2.6. □

In the remaining results in this section we will show that $\mathcal{M}_R(V) = \operatorname{End}_R(V)$ if and only if V_P is R_P-cyclic for all P in Max-Ass V.

Definition 2.13. Submodules $X_1, \dots, X_n$ of an R-module V are called a system of partial components for V if:

1. $\text{Max-Ass}\, X_i$ has only one element for each i (thus multiplication by elements in $R\backslash\text{Max-Ass}\, X_i$ is a monomorphism on X_i);

2. for each i there exists $\alpha_i \in R\backslash\text{Max-Ass}\, X_i$, such that $\alpha_i V \subseteq X_i$, and for each $v \in V\backslash\{0\}$, $\alpha_j^2 v \neq 0$ for at least one j.

Proposition 2.14. *Suppose* $E(V) = \oplus_{i=1}^m E(R/P_i)$ *and* $\text{Max-Ass}\, V = \{Q_1, \dots, Q_n\}$. *Then* $X_j := V \cap \oplus_{P_i \subseteq Q_j} E(R/P_i)$, $j = 1, \dots, n$, *is a system of partial components for* V.

Proof. Since $X_j \subseteq \oplus_{P_i \subseteq Q_j} E(R/P_i)$, $E(X_j) \subseteq \oplus_{P_i \subseteq Q_j} E(R/P_i)$. In order to obtain equality, we show that $\oplus_{P_i \subseteq Q_j} E(R/P_i)$ is an essential extension of X_j. If W is a nonzero submodule of $\oplus_{P_i \subseteq Q_j} E(R/P_i)$, we have that $W \cap V \neq \varnothing$, since $E(V)$ is an essential extension of V. But then $W \cap V \subseteq X_j$ implies that $W \cap X_j \neq \varnothing$. From the summands that appear in $E(X_j)$, we conclude that $\text{Max-Ass}\, X_j$ has only one element.

For each Q_j choose $\beta_j \in \cap_{P_i \not\subseteq Q_j} P_i \backslash Q_j$ ([**7**, Lemma 3.55]). From Matlis's theorem, there exist positive integers n_j such that $\alpha_j := \beta_j^{n_j}$ is such that $\alpha_j V \subseteq X_j$ since $\beta_j^n v = 0$ for sufficiently large n if $v \in E(R/P)$ and $\beta_j \in P$ and since V is finitely generated. The remaining properties of a system of partial components follow from the fact that multiplication by an element in the complement of the prime ideal Q is an isomorphism on $E(R/P)$ if $P \subseteq Q$. □

Proposition 2.15. *Suppose* $X_1, \dots, X_n$ *is a system of partial components. Then* $\mathcal{M}_R(X_i) = \text{End}_R(X_i)$ *if and only if* $(X_i)_{Q_i}$ *is* R_{Q_i}*-cyclic, where* $\{Q_i\} = \text{Max-Ass}\, X_i$.

Proof. Since $\text{Max-Ass}\, X_i$ has only one element, the result follows from Proposition 2.12. □

Proposition 2.16. *Suppose* V *has a system of partial components* $X_1, \dots, X_n$. *Then* $\mathcal{M}_R(V) = \text{End}_R(V)$ *if and only if* $\mathcal{M}_R(X_i) = \text{End}_R(X_i)$ *for each* i.

Proof. $\Rightarrow$. Suppose $\mathcal{M}_R(X_j) \neq \mathrm{End}_R(X_j)$ for some j, and that $\{Q_i\} = \text{Max-Ass}\, X_i$ for each $i = 1, \ldots, n$. Then from Theorem 2.5 we have a nonlinear homogeneous map Λ_j on X_j such that $Q_j \Lambda_j(X_j) = 0$. Let α_i be as in Definition 2.13. Define $\Psi_j : V \to V$ by $\Psi_j(v) = \Lambda_j(\alpha_j v)$. In order to verify that Ψ_j is nonlinear, observe that $\Lambda_j(x+y) - \Lambda_j(x) - \Lambda_j(y) \neq 0$ implies that $\Lambda_j(\alpha_j(x+y)) - \Lambda_j(\alpha_j x) - \Lambda_j(\alpha_j y) \neq 0$.

$\Leftarrow$. Suppose $\mathcal{M}_R(X_i) = \mathrm{End}_R(X_i)$ for each i, but $\mathcal{M}_R(V) \neq \mathrm{End}_R(V)$. Suppose f is a homogeneous map on V such that $\gamma := f(v+w) - f(v) - f(w) \neq 0$ for some $v, w \in V$. Let α_i be as in Definition 2.13. Then $\alpha_j^2 \gamma \neq 0$ for some j. But $\alpha_j^2 \gamma = (\alpha_j f)(\alpha_j v + \alpha_j w) - (\alpha_j f)(\alpha_j v) + (\alpha_j f)(\alpha_j w) = 0$, since $\alpha_j V \subseteq X_j$ and $(\alpha_j f)(X_j) \subseteq X_j$. □

Theorem 2.17. *Suppose V has a system of partial components $X_1, \ldots, X_n$. Then $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ if and only if $(X_i)_{Q_i}$ is R_{Q_i}-cyclic for each i, where $\{Q_i\} = \text{Max-Ass}\, X_i$.*

Proof. This result follows from Propositions 2.15 and 2.16. □

Example 2.18. Let J_i be Q_i-primary for $i = 1, \ldots, m$, in the ring R. Also suppose that $Q_i \not\subseteq Q_j$ if $i \neq j$, and let $V = \oplus_{i=1}^n R/J_i$. Since $Q_i^{n_i} \subseteq J_i$ for some n_i (to see this, first note that the radical of J_i is Q_i, and then recall that in a Noetherian ring some power of the radical of an ideal is contained in the ideal), and since $(J_i : r)$ is Q_i-primary if $r \notin J_i$ ([**7**, Lemma 4.14]) and thus contained in Q_i, we have from Lemma 2.10 that $\text{Max-Ass}\, R/J_i = \{Q_i\}$ because, if $r \notin Q_i$, then $(J_i : r) = Q_i$. Now choose $\beta_i \in \cap_{j \neq i} Q_j \backslash Q_i$. Also let n be large enough such that $\beta_i^n \in J_i$ if $i \neq j$. Let $\alpha_i := \beta_i^n$. Then since $(J_i : \alpha_i^2) = J_i$ [, Lemma 4.14], the α_i and $X_i := R/J_i$ satisfy the properties as stated in Definition 2.13. Since each R/J_i is cyclic as an R-module, we conclude from Theorem 2.17 that $\mathcal{M}_R(V) = \mathrm{End}_R(V)$. □

Theorem 2.19. $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ *if and only if V_P is R_P-cyclic for all P in* $\text{Max-Ass}\, V$.

Proof. Let X_i and Q_i be as in Proposition 2.14. From Matlis's theorem we have that if $a \notin X_i$, then $ra = 0$ for some $r \notin Q_i$; thus we conclude that $V_{Q_i} = (X_i)_{Q_i}$. Notice that, from Proposition 2.11, we have that the Q_i are precisely the members of Max-Ass V. Now simply use Theorem 2.17 to obtain the result. $\square$

3. Homogeneous maps on uniform modules. In this section we apply some of the previous results in order to classify all commutative Noetherian rings with the property that $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ for all uniform modules V. So in this section we will assume that R is commutative and Noetherian.

From the results in [**2**] describing when $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ and when $\mathcal{M}_R(V)$ is a ring, where R is a Dedekind domain, it follows that if $\mathcal{M}_R(V)$ is a ring and if V is also uniform, then $\mathcal{M}_R(V) = \mathrm{End}_R(V)$. This is not the case for arbitrary Noetherian rings. In fact, let $R = \mathbf{Z}_2[x,y]/\langle x,y\rangle^2$, $V = \langle x,y\rangle/\langle x^2,y^2\rangle$; then, since $\mathrm{Ann}_R v \subseteq \mathrm{Ann}_R w$ implies that $w \in Rv$ for all $v, w \in V$, we have from $\mathrm{Ann}_R v \subseteq \mathrm{Ann}_R f(v)$ that $f(v) \in Rv$ for all $v \in V$ and $f \in \mathcal{M}_R(V)$. Thus $\mathcal{M}_R(V)$ is a ring, since if $f_i(v) = r_i v$ for i from 1 to 3, then $f_3(f_2+f_1)(v) = r_3(r_2+r_1)v = r_3r_2v + r_3r_1v = f_3f_2(v) + f_3f_1(v)$. But since V is not cyclic, $\mathcal{M}_R(V) \neq \mathrm{End}_R(V)$. It is not hard to verify that V is also uniform. $\square$

Lemma 3.1. *$\mathcal{M}_R(V) = \mathrm{End}_R(V)$ for all uniform modules V if and only if the dimension of $(PR_P)^{n-1}/(PR_p)^n$ as an R_P/PR_P vector space is less than or equal to 1 for all $n \geq 1$ and for all prime ideals P of R.*

Proof. $\Rightarrow$. Suppose there exist a prime ideal P and a positive integer m such that $\dim_{R_P/PR_P}(PR_P)^{m-1}/(PR_p)^m \geq 2$. Since $E(R/P) \simeq E(R_P/PR_P)$ as R-modules ([**8**, Proposition 5.6]), and since all submodules of $E(R/P)$ are uniform, it follows from Corollary 2.7 that it is enough to find a finitely generated R-submodule V of $E(R_P/PR_P)$ such that V_P is not R_P-cyclic. Let $A_n := \{x \in E(R_P/PR_P) \mid (PR_P)^n x = 0\}$. Then $A_n/A_{n-1} \simeq PR_P^{n-1}/PR_P^n$ as R_P/PR_P-vector spaces ([**8**, Chapter 5, p. 133]). Since $\dim_{R_P/PR_P}(PR_P)^{m-1}/(PR_p)^m \geq 2$ we have that A_m is not cyclic as an R_P-module, and thus $\mathcal{M}_R(A_m) \neq \mathrm{End}_R(A_m)$.

$\Leftarrow$. Suppose V is uniform, and $\dim_{R_P/PR_P}(PR_P)^{n-1}/(PR_p)^n \leq 1$ for all $n \geq 1$. Since V is uniform, we have that $E(V) = E(R/P)$ for some prime P. Then again, from Nakayama's lemma [**7**, Proposition 9.3], we have that each A_i is cyclic as an R_P-module for each i. From Matlis's theorem we have that $E(V) = \cup_{i=1}^{\infty} A_i$, and so from the remarks following Theorem 2.5 we have that $\mathcal{M}_R(V) = \mathrm{End}_R(V)$, since V_P is locally R_P-cyclic. □

Theorem 3.2. *$\mathcal{M}_R(V) = \mathrm{End}_R(V)$ for all uniform modules V if and only if the maximal ideal of R_P is principal for each prime ideal P of R.*

Proof. Suppose $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ for all uniform modules V; then, from Lemma 3.1, we have that $\dim_{R_P/PR_P}(PR_P)^{n-1}/(PR_p)^n \leq 1$ for all $n \geq 1$, for all prime ideals P of R. This holds in particular for $n = 2$. But then we have from an application of Nakayama's lemma that PR_P is principal. The converse follows trivially from Lemma 3.1. □

Corollary 3.3. *Suppose R is an Artinian commutative ring. Then $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ for all uniform modules V if and only if R_P is a principal ideal ring for all primes P.*

Proof. From Hopkins-Levitzki we have that Artinian implies Noetherian. The result now follows since the following are equivalent for local Artinian rings (see [**1**, Proposition 8.8]):

(a) every ideal is principal;

(b) the maximal ideal M of A is principal;

(c) $\dim_{A/M} M/M^2 \leq 1$. □

Corollary 3.4. *Suppose R is a (Noetherian) domain. Then $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ for all uniform modules V if and only if R is a Dedekind domain.*

Proof. $\Leftarrow$. If R is a Dedekind domain, then R_P is a discrete valuation ring (see [**1**, Proposition 9.3]), and in particular a principal ideal domain for each P, and thus the result follows from Theorem 3.2.

$\Rightarrow$. Suppose $\mathcal{M}_R(V) = \mathrm{End}_R(V)$ for all uniform modules V. Then from Theorem 3.2, we have that R_P is a local domain with principal maximal ideal and thus of dimension one by applying the principal ideal theorem, and therefore a discrete valuation ring. Thus R is a Dedekind domain. □

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