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ON BOUNDARY CONDITIONS FOR STURM-LIOUVILLE DIFFERENTIAL OPERATORS IN THE DIRECT SUM SPACES

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ABSTRACT. Sturm-Liouville (S-L) boundary value problems on any finite number of intervals are studied in the setting of the direct sum of the L^2_w-spaces of functions defined on each of the separate intervals. The interplay between these L^2_w-spaces is of critical importance. This study is partly motivated by the occurrence of (S-L) problems with coefficients that have a singularity in the interior of the basic interval. In the one interval case, the singular self-adjoint boundary conditions are characterized in terms of certain Wronskians involving y and two linearly independent solutions of $M[y] = 0$ by Krall and Zettl in [**11**].

1. Introduction. The boundary value problems for the Sturm-Liouville (S-L) expression

$$M[y] = \frac{1}{w}[-(py')' + qy] \quad \text{on } I = (a, b),\\ -\infty \leq a < b \leq \infty$$

on two intervals are studied in the setting of the direct sum of the L^2-spaces of functions defined on each of the separate intervals by Everitt and Zettl in [**8**]. In the one interval case, the characterization of singular self-adjoint boundary conditions for Sturm-Liouville problems is identical to that in the regular case provided that y and py' are replaced by certain Wronskians involving y and two linearly independent solutions of $M[y] = 0$ has been proved by Krall and Zettl in [**11**].

Our objective in this paper is to extend the results of Krall and Zettl in [**11**] to the case of any finite number of intervals $I_r = (a_r, b_r)$, $r = 1, 2, \dots, n$. Here the interior singularities occur only at the ends of the intervals. In particular, we define a minimal and a maximal operator each associated with expressions, and characterize all self-adjoint extensions of the minimal operator in terms of "boundary

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conditions." These conditions involve the expressions on the intervals I_r, $r = 1, 2, \dots, n$.

In the regular case our conditions can be interpreted in terms of the values of the unknown function y and its quasi-derivative at all end-points.

In the singular case our conditions are given, just as in the one interval case, in terms of Wronskians involving y and two linearly independent solutions of $M_r[y] = 0$, $r = 1, 2, \dots, n$.

2. Notation and basic assumptions. Let $-\infty \le a_r < b_r \le \infty$; let I_r denote an interval with left end point a_r and right end point b_r, $r = 1, 2, \dots, n$. We use $[a_r$ to indicate a closed end-point a_r and $(a_r$ to indicate an open endpoint a_r; use of the square bracket $[a_r$ implies that $a_r \in \mathbf{R}$, the set of real numbers.

Consider Lebesgue measurable functions p_r, q_r, w_r from I_r into $\mathbf{R}$ satisfying the following basic conditions:

$$\frac{1}{p_r}, q_r, w_r \in L^2_{\text{loc}}(I_r), w_r(t) > 0, \quad \text{a.e., } r = 1, 2, \dots, n, \tag{2.1}$$

which are taken to hold throughout this paper. Differential expressions M_r, $r = 1, 2, \dots, n$ are defined by

$$M_r[y] = -(p_r y')' + q_r y \quad \text{on } I_r,\ r = 1, 2, \dots, n. \tag{2.2}$$

Let $H_r = L^2_{w_r}(I_r)$ denote, for $r = 1, 2, \dots, n$ the set (equivalence classes) of Lebesgue measurable functions f defined on I_r satisfying

$$\int_{I_r} |f(x)|^2 w_r(x)\, dx < \infty, \quad r = 1, 2, \dots, n, \tag{2.3}$$

with inner-product

$$(f, g)_r := \int_{I_r} f(x)\overline{g(x)} w(x)\, dx, \quad r = 1, 2, \dots, n, \tag{2.4}$$

and norm $\|f\| := (f, f)^{1/2}_{w_r}$, this is a Hilbert space on identifying functions which differ only on null sets. Let

$$D_r = \{f \in H_r : f, p_r f' \in AC_{\text{loc}}(I_r) \text{ and } w_r^{-1} M_r[f] \in H_r\}, \quad r = 1, 2, \dots, n.$$

Below we will denote $p_r f'$ by $f_r^{[1]}$ and call it the quasi-derivative of f. The subscript r will be omitted in most cases since it is clear from the context.

The operator T_r defined by

$$T_r f = w^{-1} M_r[f], \quad f \in D_r, \tag{2.5}$$

is called the maximal operator of M_r on I_r, $r = 1, 2, \dots, n$. It is well known, see [**14**, p. 68], that D_r is dense in H_r. Hence T_r has a uniquely defined adjoint. Let

$$T_{0,r} = T_r^* \quad \text{and} \quad D_{0,r} = \text{domain of } T_r^*,$$
$$r = 1, 2, \dots, n.$$

The operator $T_{0,r}$ is called the minimal operator of M_r on I_r.

For $f, g \in D_r$ and $\alpha, \beta \in I_r$, $r = 1, 2, \dots, n$, Green's formula is

$$\int_\alpha^\beta \{M_r[f]\bar{g} - f\overline{M_r[g]}\}\, dx = [f,g]_r(\beta) - [f,g]_r(\alpha), \tag{2.6}$$

where

$$[f,g]_r = f\bar{g}^{[1]} - f^{[1]}\bar{g}, \quad f, g \in D_r, \quad r = 1, 2, \dots, n; \tag{2.7}$$

and $y^{[1]}$ denotes $p_r y'$ for $r = 1, 2, \dots, n$.

For $f, g \in D_r$, the limits $\lim_{\beta \to b_r} [f,g]_r(\beta)$ and $\lim_{\alpha \to a_r} [f.g]_r(\alpha)$ exist and are infinite. These are denoted by $[f,g]_r(b_r)$ and $[f,g]_r(a_r)$, respectively, $r = 1, 2, \dots, n$.

For $f, g \in AC_{\text{loc}}(I_r)$, let

$$W_r(f,g) = f p_r g' - g p_r f'. \tag{2.8}$$

Choosing solutions Θ and ϕ of $M_r[y] = 0$ satisfying:

$$W_r(\theta, \phi)(x) = 1 \quad \text{for all } x \in I_r, \qquad r = 1, 2, \dots, n. \tag{2.9}$$

Note that the bilinear form $[f,g]_r$ in (2.6) can be written as

$$\begin{aligned} [f,g]_r &= f p_r \bar{g}' - \bar{g} p_r f' \\ &= (\bar{g}, p_r \bar{g}') \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ p_r f' \end{pmatrix}. \end{aligned} \tag{2.10}$$

From (2.8) and (2.9), we get

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Theta & \phi \\ p_r\Theta' & p_r\phi' \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \Theta & p_r\Theta' \\ \phi & p_r\phi' \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.11}$$

and hence the bilinear form in (2.10) can also be written as:

$$\begin{aligned} [f,g]_r &= (W_r(\bar{g},\Theta), W_r(\bar{g},\phi))\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} W_r(f,\Theta) \\ W_r(f,\phi) \end{pmatrix} \\ &= \overline{W}_r(g,\phi)W_r(f,\Theta) - \overline{W}_r(g,\Theta)W_r(f,\phi) \\ &= \det\begin{pmatrix} W_r(f,\Theta) & W_r(f,\phi) \\ W_r(\bar{g},\Theta) & W_r(\bar{g},\phi) \end{pmatrix}, \\ & \qquad r = 1,2,\dots,n; \end{aligned} \tag{2.12}$$

see [**11**] and [**12**]. Let w_r be a function which satisfies:

$$w_r > 0 \quad \text{a.e. on } I_r, \quad w_r \in L^1_{\text{loc}}(I_r), \quad r = 1,2,\dots,n. \tag{2.13}$$

The endpoint a_r is regular if it is finite and

$$p_r^{-1}, q_r, w_r \in L^1[a_r, a_r+\varepsilon] \quad \text{for some } \varepsilon > 0. \tag{2.14}$$

Similarly, the endpoint b_r is regular if (2.14) holds with the interval $[a_r, a_r+\varepsilon]$ replaced by $[b_r-\varepsilon, b_r]$. An endpoint is called singular if it is not regular. Thus, a_r is singular if it is either infinite or finite and (2.14) fails to hold for one or more of p_r^{-1}, q_r and w_r. An important distinction between a regular endpoint is the fact that at a regular endpoint c_r, all initial value problems $y(c_r) = \alpha_r$, $(p_r y')(c_r) = \beta_r$; $\alpha_r, \beta_r \in \mathbf{C}$, $r = 1,2,\dots,n$, have a unique solution. This is not true when c_r is singular, see [**6**].

Assume that a_r and b_r are singular endpoints. For any open interval (a_r, b_r) and $\lambda \in \mathbf{C}$, the conditions (2.1) imply that any solution y of

$$M_r[y] = \lambda w_r y, \quad \lambda \in \mathbf{C} \quad \text{on } I_r, \quad r = 1,2,\dots,n, \tag{2.15}$$

is in $L^2_{w_r}(a_r, b_r)$, see [**4**]. However, such a y may or may not be in $L^2_{w_r}(a_r, b_r)$. If y is in $L^2_{w_r}(a_r, \beta_r)$ for some β_r in (a_r, b_r), then this is true for all β_r in (a_r, b_r). If for some β_r in (a_r, b_r) all solutions of (2.15) are in $L^2_{w_r}(a_r, \beta_r)$, then we say that $M_r[.]$ is in the limit-circle case at a_r, or simply that a_r is LC. Otherwise, $M_r[.]$ is in the limit-point case at a_r or a_r is LP. Similarly, b_r is LC means that all solutions of (2.15) are in $L^2_{w_r}(a_r, b_r)$, $a_r < \alpha_r < b_r$, $r = 1, 2, \dots, n$. This classification is independent of λ in (2.15), see [**14**]. Otherwise, b_r is LP. The limit-point, limit-circle terminology is used for historical reasons.

The classification of the self-adjoint extensions of $T_{0,r}$ depends, in an essential way, on the deficiency index of $T_{0,r}$. We briefly recall the definition of this notion for abstract symmetric operators is a separable Hilbert space.

A linear operator A_r from a Hilbert space H_r into H_r is said to be symmetric if its domain $D(A_r)$ is dense in H_r and

$$(A_r f, g) = (f, A_r g), \quad f, g \text{ in } D(A_r), \\ r = 1, 2, \dots, n.$$

Any such operator has associated with it a pair (d_r^+, d_r^-), where each of d_r^+, d_r^- is a nonnegative integer or $+\infty$. The extended integers are called the deficiency indices of A_r and are defined as follows:

For $\lambda \in \mathbf{C}$, the set of complex numbers, let $\mathbf{R}_\lambda$ denote the range of $(A_r - \bar{\lambda} I)$, I being the identity operator. Let

$$N_{\lambda,r} = \{f \in (A_r^*) \mid A_r^* f = \lambda f\}, \quad r = 1, 2, \dots, n, \tag{2.16}$$

and with

$$\left.\begin{aligned} N_r^+ &= N_{i,r}, & N_r^- &= N_{-i,r}; \\ d_r^+ &= \dim N_r^+, & d_r^- &= \dim N_r^-, \end{aligned}\right\} \quad r = 1, 2, \dots, n. \tag{2.17}$$

The subspaces N_r^+, N_r^- are called the deficiency spaces of A_r, and the pair (d_r^+, d_r^-) are called the deficiency indices of A_r. For later use, recall the following two results.

For any $\lambda \in \mathbf{C} \backslash \mathbf{R}$, we have, from the general theory,

$$D(A_r^*) = D(A_r) \dot{+} N_{\lambda,r} \dot{+} N_{\lambda,r}^-, \quad r = 1, 2, \dots, n, \tag{2.18}$$

where $D(A_r)$, $N_{\lambda,r}$ and $N^-_{\lambda,r}$ are linearly independent subspaces and the sum is direct (which we indicate with the symbol $\dot{+}$), see [**2**].

Any self-adjoint extension S_r of the symmetric operator A_r, $r = 1, 2, \ldots, n$, satisfies

$$A_r \subset S_r = S_r^* \subset A_r^*, \quad r = 1, 2, \ldots, n,$$

and hence is completely determined by specifying its domain $D(S_r)$,

$$D(A_r) \subset D(S_r) \subset D(A_r^*).$$

This can be proved using formula (2.18), see [**1, 2, 14**].

Theorem 2.1. *The operator $T_{0,r}$ is a closed symmetric operator from H_r into H_r and*

$$T_{0,r}^* = T_r, \qquad T_r^* = T_{0,r}, \quad r = 1, 2, \ldots, n. \tag{2.19}$$

Proof. See [**14**, Section 17.4].

To relate the deficiency indices of $T_{0,r}$ to the equation

$$M_r[y] = \lambda w_r y \quad \text{on } I_r = (a_r, b_r), \quad r = 1, 2, \ldots, n, \tag{2.20}$$

observe that

$$N_{\lambda,r} = \{y \in H_r \mid T_{0,r}^* y = T_r y = w_r^{-1} M_r[y] = \lambda y,\ r = 1, 2, \ldots, n\}.$$

From this we can conclude that N_r^+, N_r^- consists of the solutions of the equation (2.20), which are in the space $L^2_{w_r}(I_r)$, for $\lambda = +i$ and $\lambda = -i$, respectively. Thus, d_r^+, d_r^- are the number of linearly independent solutions of (2.20) which are in the space H_r for $\lambda = +i$ and $\lambda = -i$, respectively. It is well known that $d_r^+ = d_r^-$, $r = 1, 2, \ldots, n$, under conditions (2.1), see [**7**, Section 9]. The common value is denoted by d_r, $r = 1, 2, \ldots, n$.

From the above discussion we see that there are only three possibilities $d_r = 0, 1, 2$, $r = 1, 2, \ldots, n$.

Some of the basic facts are summarized in:

Theorem 2.2. (a) $D_{0,r} = \{f \in D_r : [f,g](b_r) - [f,g](a_r) = 0$ *for all* $g \in D_r\}$,

(b) *If* M_r *is in the limit point case at an endpoint* c, *then* $[f,g](c) = 0$, *for all* $f, g \in D_r$, $c = a_r$ *or* $c = b_r$, $r = 1, 2, \dots, n$.

(c) *If an endpoint* c *is regular, then, for any solution* y, y *and* $y^{[1]}$ *are continuous.*

(d) *If* a_r *and* b_r *are both regular, then, for any* $\tau_{1,r}$, $\tau_{2,r}$, $\delta_{1,r}$, $\delta_{2,r}$ *in* $\mathbf{C}$, *there exists a function* f *in* D_r *such that*

$$\left.\begin{array}{ll} f(a_r) = \tau_{1,r}, & f^{[1]}(a_r) = \tau_{2,r}; \\ f(b_r) = \delta_{1,r}, & f^{[1]}(b_r) = \delta_{2,r}, \end{array}\right. \quad r = 1, 2, \dots, n,$$

(e) *If* a_r *is regular and* b_r *singular, then a function* f *from* D_r *is in* $D_{0,r}$ *if and only if the following conditions are satisfied*:

(i) $f(a_r) = 0$ *and* $f^{[1]}(a_r) = 0$;

(ii) $[f,g](b_r) = 0$ *for all* $g \in D_r$, $r = 1, 2, \dots, n$.

The analogous results hold when a_r *is singular and* b_r *is regular, see* [**8**, Proposition 1], [**9**] *and* [**14**].

Lemma 2.3. *Given* $\alpha_r, \beta_r, \tau_r$ *and* δ_r *in* $\mathbf{C}$, *there exists a* $\psi \in D_r \backslash D_{0,r}$ *such that*

$$\left.\begin{array}{ll} W_r(\psi,\Theta)(a_r) = \alpha_r, & W_r(\psi,\phi)(a_r) = \beta_r; \\ W_r(\psi,\Theta)(b_r) = \tau_r, & W_r(\psi,\phi)(b_r) = \delta_r, \end{array}\right. \quad r = 1, 2, \dots, n.$$

Furthermore, ψ *can be taken to be a linear combination of* Θ *and* ϕ *near each end point.*

Proof. The proof is similar to that in [**8**, Lemma 2].

Since $T_{0,r}$ is symmetric, it follows that if S_r is any self-adjoint extension of $T_{0,r}$, we have

$$T_{0,r} \subset S_r = S_r^* \subset T_{0,r}^* = T_r, \quad r = 1, 2, \dots, n. \tag{2.21}$$

Thus such a self-adjoint operator S_r is completely determined by its domain $D(S_r)$. From (2.21) we have

$$D_{0,r} \subset D(S_r) \subset D_r, \quad r = 1, 2, \dots, n. \tag{2.22}$$

To specify $D(S_r)$, we start with formula (2.18) applied to $T_{0,r}$:

$$D_r = D_{0,r} \dot{+} N_r^+ \dot{+} N_r^-, \quad r = 1, 2, \dots, n. \tag{2.23}$$

Let H be the direct sum

$$H = \bigoplus_{r=1}^{n} H_r = \bigoplus_{r=1}^{n} L^2_{w_r}(a_r, b_r). \tag{2.24}$$

Elements of H will be denoted by $\underset{\sim}{f} = \{f_1, \dots, f_n\}$ with $f_1 \in H_1, \dots, f_n \in H_n$.

Remark. When $I_i \cap I_j = \varnothing$, $i \neq j$, $i, j = 1, 2, \dots, n$, the direct sum space $\oplus_{r=1}^{n} L^2_{w_r}(I_r)$ can be naturally identified with the space $L^2_{w_r}(\cup_{r=1}^{n} I_r)$, where $w = w_r$ on the interval I_r, $r = 1, \dots, n$. This remark is of particular significance when $\cup_{r=1}^{n} I_r$ may be taken as a single interval, see [**8**].

We now establish by [**8, 9, 11**] and [**13**] some further notation

$$D_0(M) = \bigoplus_{r=1}^{n} D_0(M_r), \qquad D(M) = \bigoplus_{r=1}^{n} D(M_r); \tag{2.25}$$

$$T_0(M)f = (T_0(M_1)f_1, \dots, T_0(M_n)f_n), \quad f_1 \in D(M_1), \dots, f_n \in D(M_n). \tag{2.26}$$

Also,

$$T(M)f = (T(M_1)f_1, \dots, T(M_n)f_n), \quad f_1 \in D(M_1), \dots, f_n \in D(M_n), \tag{2.27}$$

$$(2.28)\qquad \begin{aligned} [\underset{\sim}{f}, \underset{\sim}{g}] &= \sum_{r=1}^{n} \{[f_r, g_r]_r(b_r) - [f_r, g_r]_r(a_r)\}, \quad \underset{\sim}{f}, \underset{\sim}{g} \in D(M), \\ (\underset{\sim}{f}, \underset{\sim}{g}) &= \sum_{r=1}^{n} (f_r, g_r), \end{aligned}$$

where $\underset{\sim}{f} = \{f_1, \dots, f_n\}$, $\underset{\sim}{g} = \{g_1, \dots, g_n\}$, and $(\cdot, \cdot)_r$ is the inner product defined in (2.4).

Note that $T_0(M)$ is a closed symmetric operator in H.

3. The characterization of self-adjoint domains. In [**11**] Krall and Zettl characterized the singular self-adjoint boundary conditions for Sturm-Liouville problems in terms of Wronskians involving y and two linearly independent solutions of $M[y] = 0$ for some one interval case. In this section we generalize the results of the characterization of self-adjoint domains in [**11**] for separate intervals $I_r = (a_r, b_r)$, $r = 1, 2, \dots, n$.

We summarize a few additional properties of T_0 in the form of a lemma.

Lemma 3.1. *We have*

(a) $T_0^* = \oplus_{r=1}^{n} T_{0,r}^* = \oplus_{r=1}^{n} T_r$. *In particular,*

$$D(T_0^*) = D = \bigoplus_{r=1}^{n} D_r,$$

(b) $N^+ = \oplus_{r=1}^{n} N_r^+$, $N^- = \oplus_{r=1}^{n} N_r^-$,

(c) *The deficiency indices* (d^+, d^-) *of* T_0 *given by*

$$d^+ = \bigoplus_{r=1}^{n} d_r^+, \qquad d^- = \bigoplus_{r=1}^{n} d_r^-,$$

(d) $D = D_0 \dotplus N^+ \dotplus N^-$.

Proof. Part (a) follows immediately from the definition of the operator $T_0(M)$ and from the general definition of an adjoint operator.

The other parts are either direct consequences of part (a) or follow immediately from the definitions.

Since $d_j^+ = d_j^-$, $j = 1, 2, \ldots, n$, we have $d^+ = d^- = d$. Also, the possible values of d are

$$0 \le d \le 2n. \tag{3.1}$$

If S_r, $r = 1, 2, \ldots, n$ are self-adjoint extensions of $T_{0,r}$,

$$S = \bigoplus_{r=1}^{n} S_r, \tag{3.2}$$

is a self-adjoint extension of $T_0(M)$, see [**8**] and [**9**].

The next result is a straightforward extension of Theorem 4 in [**14**, Section 18.1]; see also [**3**] and [**9**].

Theorem 3.2. *If the operator S with domain $D(S)$ is a self-adjoint extension of T_0, then there exist $\underset{\sim}{\psi}_j \subset D(S) \subset D$, $j = 1, 2, \ldots, d$, satisfying the following conditions*:

(i) $\underset{\sim 1}{\psi}, \ldots, \underset{\sim d}{\psi}$ *are linearly independent modulo D_0*;

(ii)

$$[\underset{\sim j}{\psi}, \underset{\sim k}{\psi}] = \sum_{r=1}^{n} \{[\psi_{jr}, \psi_{kr}](b_r) - [\psi_{jr}, \psi_{kr}](a_r)\} = 0, \quad j, k = 1, 2, \ldots, d,$$

(iii) $D(S)$ *consists precisely of those $\underset{\sim}{f}$ in D which satisfy*

$$[\underset{\sim}{f}, \underset{\sim j}{\psi}] = \sum_{r=1}^{n} \{[f_r, \psi_{jr}](b_r) - [f_r, \psi_{jr}](b_r)\} = 0, \quad j = 1, 2, \ldots, d, \ldots . \tag{3.3}$$

Conversely, given $\underset{\sim j}{\psi} \in D$, $j = 1, 2, \ldots, d$, which satisfy conditions (i) and (ii), then the set $D(S)$ defined by (iii) is the domain of a self-adjoint extension of T_0.

Proof. The proof entirely similar to that of [**14**, Theorem 18] and [**9**, Theorem 1.1] and is therefore omitted.

Remark. It is well known from [**14**] that no boundary condition is needed at a limit-point end-point. On the other hand, a boundary condition is needed for each limit-circle end-point.

The self-adjoint extensions are determined by boundary conditions imposed at the endpoints of each of the intervals I_r. The type of these boundary conditions depends on the nature of the problem in the interval I_r. There are four possibilities for each r, $r = 1, 2, \dots, n$.

Case (i). Assume both endpoints a_r and b_r are regular endpoints. In this case, if we put

$$\bar{\psi}_{jr}^{[1]}(a_r) = (-1)^k \alpha_{jk}^r, \qquad \bar{\psi}_{jr}^{[2-k]}(b_r) = (-1)^{(k-1)} \beta_{jk}^r, \quad j, k = 1, 2, \quad r = 1, 2, \dots, n, \tag{3.4}$$

we have by (2.7) and (3.3) that the boundary conditions on the functions $y_r \in D(M_r)$ are

$$B_r(y_r, I_r) = M^r Y(a_r) + N^r Y(b_r) = 0, \quad r = 1, 2, \dots, n, \tag{3.5}$$

where

$$M_r = (\alpha_{jk}^r), \qquad N^r = (\beta_{jk}^r), \quad j, k = 1, 2, \quad r = 1, 2, \dots, n,$$

are 2×2 matrices over $\mathbf{C}$, $Y(\cdot) = (y, p_r y')^\top(\cdot)$, $\top$ for transpose, and $\alpha_{jk}^r, \beta_{jk}^r$ are complex numbers satisfying

$$M^r J (M^r)^* = N^r J (N^r)^*, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

The above boundary conditions determine the domains of self-adjoint extensions of $T_0(M_r)$ for each r, see [**11**] and [**14**] for more details.

In the other three cases, the self-adjoint extensions S_r of $T_0(M_r)$, $r = 1, 2, \dots, n$, are determined by boundary conditions in terms of certain Wronskians involving y and two linearly independent solutions of

$$M_r[y] = 0 \quad \text{on } I_r, \quad r = 1, 2, \dots, n, \tag{3.7}$$

at a singular endpoint.

Case (ii). Assume both endpoints a_r and b_r are singular and LC. By (2.12), (3.3) and Lemma 2.3, if we put

(3.8)
$$\left.\begin{aligned} \overline{W}_r(\psi_{jr}, \phi) &= \beta_{j1}^r, & \overline{W}_r(\psi_{jr}, \Theta) &= -\beta_{j2}^r, \\ \overline{W}_r(\psi_{jr}, \phi) &= -\alpha_{jr}^r, & \overline{W}_r(\psi_{jr}, \Theta) &= \alpha_{j2}^r, \end{aligned}\right\} \; j=1,2;\; r=1,\dots,n.$$

Then the boundary conditions in this case on the functions $y_r \in D(M_r)$ are:

$$B_r(y_r, I_r) = M^r Y(a_r) + N^r Y(b_r) = 0, \quad r = 1, 2, \dots, n, \tag{3.9}$$

which determine the domains of self-adjoint extensions of $T_0(M_r)$ for each r, where

$$M^r = (\alpha_{jk}^r), \qquad N^r = (\beta_{jk}^r), \quad j, k = 1, 2;\; r = 1, 2, \dots, n,$$

are 2×2 matrices over $\mathbf{C}$ satisfying

$$M^r J (M^r)^* = N^r J (N^r)^*, \tag{3.10}$$

and

$$Y(\cdot) = (W_r(y_r, \Theta), W_r(y_r, \phi))^\top (\cdot),$$

$\top$ for transposed matrix.

Case (iii). (a) Assume the left endpoint a_r is regular and the right endpoint b_r is singular and LC. The boundary conditions in this case on the functions $y_r \in D(M_r)$ are

$$\begin{gathered} B_r(y_r, I_r) = M^r Y(a_r) + N^r Y(b_r) = 0, \\ r = 1, 2, \dots, n, \end{gathered} \tag{3.11}$$

but where

$$Y(a_r) = (y, p_r y')^\top (a_r), \tag{3.12}$$

$$Y(b_r) = (W_r(y, \Theta), W_r(y, \phi))^\top (b_r), \quad r = 1, 2, \dots, n, \tag{3.13}$$

and the matrices M^r, N^r, satisfying

$$M^r J(M^r)^* = N^r J(N^r)^*.$$

(b) If a_r is singular and LC and b_r is regular, then let

$$\begin{aligned} Y(a_r) &= (W_r(y, \Theta), W_r(y, \phi))^\top (a_r), \\ Y(b_r) &= (y, p_r y')^\top (b_r), \end{aligned} \qquad r = 1, 2, \dots, n,$$

and the rest is the same as in Case (iii) (a).

Case (iv). Assume one endpoint is LP and the other is either regular or singular LC.

(a) Suppose a_r is LP. Then the boundary conditions in this case on the functions $y_r \in D(M_r)$ are

$$B_r(y_r, I_r) = M^r Y(a_r) + N^r Y(b_r) = 0, \quad r = 1, 2, \dots, n, \tag{3.14}$$

with $M^r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$\begin{aligned} Y(b_r) &= (y, p_r y')^\top (b_r), && \text{if } b_r \text{ is regular}, \\ Y(b_r) &= (W_r(y, \Theta), W_r(y, \phi))^\top (b_r) && \text{if } b_r \text{ is singular and LC}. \end{aligned}$$

(b) If b_r is LP and a_r is regular or singular LC, then the boundary conditions in this case on the functions $y_r \in D(M_r)$ are

$$B_r(y_r, I_r) = M^r Y(a_r) + N^r Y(b_r) = 0, \quad r = 1, 2, \dots, n, \tag{3.15}$$

with $N^r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$\begin{aligned} Y(a_r) &= (y, p_r y')^\top (a_r), && \text{if } a_r \text{ is regular,} \\ Y(a_r) &= (W_r(y,\Theta), W_r(y,\phi))^\top (a_r), && \text{if } a_r \text{ is singular and LC.} \end{aligned}$$

Next the characterization of all self-adjoint extensions of $T_0(M)$ in terms of boundary conditions featuring $L^2_{w_r}(a_r, b_r)$-solutions of the equation (3.7) for any n intervals $I_r = (a_r, b_r)$, $r = 1, 2, \dots, n$, is covered by the following theorem.

Theorem 3.3. *Let $T_0(M)$ be the minimal operator with deficiency indices (d, d). Then the set of all $y = (y_r) \in D(M)$ such that*

$$\sum_{r=1}^{n} B_r(y, I_r) = 0 \tag{3.16}$$

is the domain of self-adjoint extension S of $T_0(M)$ where $B_r(y, I_r)$ takes one of the forms (3.5), (3.9), (3.11), (3.14) *and* (3.15), *respectively, depending on the nature of the problem in the interval I_r.*

Conversely, let S be a self-adjoint extension of the minimal operator $T_0(M)$ with deficiency indices (d, d). Then $D(S)$ is the set of $y \in D(M)$ satisfying (3.16).

Proof. The proof follows from the results for the case of a single interval; see [**8, 11**] and [**14**].

4. Discussion. In this final section we consider the following discussion about the results in Section 2. First we discuss the possibility of the self-adjoint extensions which are not expressible as a direct sum of self-adjoint extensions in the separate intervals $I_r = (a_r, b_r)$, $r = 1, 2$. We will refer to self-adjoint extensions of $T_0(M)$ which do not arise in (3.2) as "new self-adjoint extensions"; see [**8**] for more details.

In (3.1), the only possible value of the deficiency index d for the two intervals are 0,1,2,3 and 4, so we have the following cases.

Case 1. $d = 0$. This can only occur when all four endpoints are LP. In this case, $T_0|$ is itself adjoint and has no proper self-adjoint extensions.

Case 2. $d = 1$. We must have three LP endpoints and one LC or regular. There are no new self-adjoint extensions, i.e., all self-adjoint extensions of T_0 can be obtained by forming direct sums of the self-adjoint extensions of $T_{0,1}$ and $T_{0,2}$. These are obtained as in the one interval case. In other words, the conditions of Theorem 3.2 reduce to the known self-adjointness conditions on the interval with singular LC or regular endpoint.

Case 3. $d = 2$. There must be two LP endpoints. Each of the other two may be LC or regular.

(i) If both endpoints are from the same interval, say I_r, then

$$S = T_{0,r} \oplus S_2,$$

where S_2 is a self-adjoint extension of $T_{0,2}$. The conditions of Theorem 3.2 reduce to those for determining the extensions of $T_{0,2}$ on I_2, i.e.,

$$M^2 Y(a_2) + N^2 Y(b_2) = 0,$$

where

$$\begin{aligned} &Y(\cdot) = (y, p_2 y')^\top(\cdot) && \text{at a regular endpoint} \\ &Y(\cdot) = (W_2(y,\Theta), W_2(y,\phi))^\top(\cdot), && \text{at singular endpoints} \end{aligned}$$

and M^2, N^2 are 2×2 matrices over $\mathbf{C}$ satisfying

$$M^2 J (N^2)^* = N^2 J (N^2)^*.$$

(ii) If there is one LP and one LC or regular endpoint from each interval, then "maxing" can occur and we get new self-adjoint extensions of T_0. For the sake of definiteness, assume that the endpoints a_1 and b_2 are limit-points, a_2 and b_1 are regular or singular LC, then

$$M^2 Y(a_2) + N^1 Y(b_1) = 0,$$

where

$$\begin{aligned} &Y(a_2) = (y, p_2 y')^\top(a_2) && \text{if } a_2 \text{ is regular} \\ &Y(a_2) = (W_2(y,\Theta), W_2(y,\phi))^\top(a_2) && \text{if } a_2 \text{ is singular and LC.} \end{aligned}$$

Similarly at the point b_1.

Case 4. $d = 3$. Here we must have either $d_1 = 2$, $d_2 = 1$ or $d_1 = 1$, $d_2 = 2$.

We assume the former holds. The latter is entirely similar. Thus we must have either a_1, b_1, a_2 are regular or singular LC and b_2 is LP, or a_1, b_1, b_2 are regular or singular LC and a_2 is LP. Again, for definiteness, we assume the former holds. In this case only the term involving b_2 (which LP) in (3.3) is zero for all $f \in D(M)$. Using the notation from Case 3, "the boundary condition" $\sim$ (3.3) becomes

$$M^1 Y(a_1) + N^1 Y(b_1) + M^2 Y(a_2) = 0,$$

where

$$\begin{array}{ll} Y(a_r) = (y, p_r y')^\top (a_r), & \text{if } a_r \text{ is regular} \\ Y(a_r) = (W_r(y, \Theta), W_r(y, \phi))^\top (a_r) & \text{if } a_r \text{ is singular LC, } r = 1, 2. \end{array}$$

Case 5. $d = 4$. This means that $d_1 = 2 = d_2$. Therefore, each one of four endpoints a_1, b_1, a_2 and b_2 is either regular or singular LC. In this case the boundary conditions in Theorem 3.2 take the form

$$\sum_{r=1}^{2} \{M^r Y(a_r) + N^r Y(b_r)\} = 0,$$

where

$$\begin{array}{ll} Y(\cdot) = (y, p_r y')^\top (\cdot) & \text{at regular endpoints,} \\ Y(\cdot) = (W_r(y, \oplus), W_r(y, \phi))^\top (\cdot) & \text{at singular LC endpoints.} \end{array}$$

We refer to [**8**] for more details.

Secondly, we show that the characterization of the singular self-adjoint boundary condition is identical to that in the regular case provided that y and py' are replaced by certain Wronskians involving y and two linearly independent solutions of $M_r[y] = 0$, $r = 1, 2$.

In Case 2, $d = 1$, there are three LP endpoints and one regular or singular LC. In this case all self-adjoint extensions of T_0 can be obtained by forming a direct sum of the self-adjoint extensions of $T_{0,1}$ and $T_{0,2}$.

(a) Assume that b_1 is regular and the other three points a_1, a_2, b_2 are LP-endpoints. In this case, the condition (3.3) becomes

$$\begin{aligned}([\underset{\sim}{y}, \underset{\sim}{\psi_1}])_{a_r}^{b_r} &= \sum_{r=1}^{n}([y_r, \psi_{1r}])_{a_r}^{b_r} \\ &= [y_1, \psi_{11}](b_1) \\ &= y_1(b_1)\bar{\psi}_{11}^{[1]}(b_1) - \bar{\psi}_{11}^{[1]}(b_1)y_1(b_1) = 0.\end{aligned} \tag{4.1}$$

If b_1 is regular, then by (3.4) we get (4.1) can be rewritten as

$$\beta_{11}^1 y_1(b_1) + \beta_{12}^1 y_1^{[1]}(b_1) = 0. \tag{4.2}$$

From Theorem 3.2 (i), we have that not both β_{11}^1 and β_{12}^1 can be zero since this would imply, by Theorem 2.2 that $\underset{\sim}{\psi_1} = (\psi_{11}, \psi_{12}) \in D_0$. Condition (ii) in Theorem 3.2 becomes

$$\beta_{11}^1 \bar{\beta}_{12}^1 - \bar{\beta}_{11}^1 \beta_{12}^1 = 0. \tag{4.3}$$

Since β_{11}^1 can be taken to be real, (4.2) just means that both β_{11}^1 and β_{12}^1 must be real. To summarize, we can say that if b_1 is regular and a_1, a_2, b_2 are LP endpoints, then all self-adjoint domains are determined by boundary conditions (4.2) where β_{11}^1 and β_{12}^1 are real and cannot both be zero. Also, the boundary conditions at a regular endpoint a_1 are all of the form:

$$\alpha_{11}^1 y_1(a_1) + \alpha_{12}^1 y_1^{[1]}(a_1) = 0, \tag{4.4}$$

where α_{11}^1 and α_{12}^1 are real and cannot both be zero.

Similarly, when each of the endpoints a_2 and b_2 is regular, then the boundary conditions are all of the form

$$\alpha_{11}^2 y_2(a_2) + \alpha_{12}^2 y_2^{[1]}(a_2) = 0; \quad a_1, b_1, b_2 \text{ are LP}, \tag{4.5}$$

$$\beta_{11}^2 y_2(b_2) + \beta_{12}^2 y_2^{[1]}(b_2) = 0; \quad a_1, b_1, a_2 \text{ are LP}, \tag{4.6}$$

respectively.

(b) Assume that b_1 is singular LC and the other three points are LP endpoints. Using (2.12), (2.3) and Lemma 2.3, we can express condition (3.3) of Theorem 3.2 as

$$
\begin{aligned}
([\underset{\sim}{y}, \underset{\sim}{\psi}_1])_{a_r}^{b_r} &= \sum_{r=1}^{2} ([y_r, \psi_{1r}])_{a_r}^{b_r} \\
&= [y_1, \psi_{11}](b_1) \qquad (4.7)\\
&= (\bar{W}_1(\psi_{11}, \phi) W_1(y_1, \Theta) - \bar{W}_1(\psi_{11}, \Theta) W_1(y_1, \phi))(b_1) = 0.
\end{aligned}
$$

Set

$$\beta_{11}^1 = \bar{W}_1(\psi_{11}, \phi)(b_1), \qquad \beta_{12}^1 = -\bar{W}_1(\psi_{11}, \Theta)(b_1). \tag{4.8}$$

Note that for fixed Θ and ϕ a given $\underset{\sim}{\psi}_1 \in D$ determined β_{11}^1 and β_{12}^1 by (4.8). Conversely, by Lemma 2.3, given β_{11}^1 and β_{12}^1 in $\mathbf{C}$, there exist a $\psi \in D$ such that (4.8) holds. Thus, the "boundary conditions" (3.3) can be expressed as:

$$\beta_{11}^1 W_1(y_1, \Theta)(b_1) + \beta_{12}^1 W_1(y_1, \phi)(b_1) = 0. \tag{4.9}$$

Again, by Theorem 3.2, β_{11}^1 and β_{12}^1 cannot both be zero.

With identification (4.8), Condition (ii) again becomes (4.3) and reduces to requiring both β_{11}^1 and β_{12}^1 to be real.

In summary, we can say that if the points a_1, a_2, b_2 are LP endpoints and b_1 is singular LC, then all self-adjoint domains are determined by "boundary conditions" of the form (4.9) where β_{11}^1 and β_{12}^1 real and cannot both be zero.

Remark. Assume that a_1, a_2 and b_2 are LP endpoints. Comparing (4.9) with (4.2), note that when $y_1(b_1)$ is replaced by $W_1(y_1, \Theta)(b_1)$ and $y_1^{[1]}(b_1)$ is replaced by $W_1(y_1, \phi)(b_1)$, then the singular case when the endpoint b_1 is singular LC is an exact parallel of the case when b_1 is regular.

Again, when a_1 is singular LC and the points b_1, a_2, b_2 are LP endpoints, all self-adjoint domains are determined by "boundary conditions":

$$\alpha_{11}^1 W_1(y_1, \Theta)(a_1) + \alpha_{11}^1 W_1(y_1, \phi)(a_1) = 0, \tag{4.10}$$

where α_{11}^1 and α_{12}^1 are real and cannot both be zero.

Similarly, when each of the points a_2 and b_2 is singular LC and the other three endpoints are LP endpoints, then the boundary conditions are all of the form:

$$\alpha_{11}^2 W_2(y_2,\Theta)(a_2)+\alpha_{12}^2 W_2(y_2,\phi)(a_2)=0;\quad a_1,b_1,b_2 \text{ are } LP,$$
$$\beta_{11}^2 W_2(y_2,\Theta)(b_2)+\beta_{12}^2 W_2(y_2,\phi)(b_2)=0;\quad a_1,b_1,a_2 \text{ are } LP,$$

respectively.

We refer to [**11**] for more details in the one interval case.

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