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METRIC SINGULARITIES

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ABSTRACT. This paper proves an existence and uniqueness theorem for geodesics through a metric singular point, where the dimension of the isotropic subspace at the singular point may exceed one. The second part of this paper proves an orbit separation theorem for orbits through the singular point of a smooth vector field. An approximation to the individual orbits is defined in terms of the derivatives of the vector field at the singular point. We prove a theorem that there exists a unique orbit for the vector field corresponding to each approximation.

1. Introduction. Let M denote a smooth manifold, and let g denote a smooth section in the bundle of symmetric two tensors on M. A point where g_p is degenerate is a *metric singular point* or a *metric singularity*.

When the dimension of the isotropic subspace

$$I(p) = T_pM \cap T_pM^{\perp}$$

at the metric singularity is one there is an existence and uniqueness theorem for geodesics through the singular point, see [**16**]. Section 2 in the present paper is concerned with the case where the dimension of the isotropic subspace exceeds one. Using the theorem of Section 2, an existence and uniqueness theorem for geodesics through the metric singular point is proven, see Theorem 2.6. The existence and uniqueness is formulated in terms of a new differentiable structure at the metric singularity.

The second part of this paper proves an existence and uniqueness theorem for orbits through a singular point of a smooth vector field. The orbit is approximated by (3.0) where the $\gamma^{\mathbf{p},q,\mathbf{r}}$ are defined in terms of a $v \in \mathbf{R}^s$ and the derivatives of the vector field at the origin. Here

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s is the dimension of the stable manifold. We prove a theorem that, corresponding to each approximation (3.0) there exists a unique orbit for the vector field, see Theorem 3.2. Theorem 3.2 is needed to prove the existence and uniqueness of geodesics in the main Theorem 2.6.

If the singular point is a sink we prove that every orbit tending to the singular point is approximated by a curve (3-0) for a suitable choice of $v \in \mathbf{R}^s$.

The main motivation for proving the theorem of Section 3 comes from the study of metric singularities. These have been studied in the papers cited in the bibliography.

Metric singularities occur naturally for timelike minimal surfaces, see, e.g., [**4**] or [**14**]. For instance, one can prove that there exist real analytic type changing minimal surfaces in Minkowski space.

In [**14**] it is shown how one can parametrize such surfaces near the metric singularity. In [**19**] it is proven that every real analytic, zero mean curvature immersion arises in this way. The main tool in the proof of this result is to show the existence of isothermal coordinates for type changing surfaces with everywhere isotropic tangency, meaning that the isotropic subspace is everywhere tangent to the singular set.

One can also show the existence of isothermal coordinates when there is isotropic transversality. In [**19**] we also study type changing surfaces of constant mean curvature $\neq 0$. When there is everywhere isotropic tangency, one can naturally introduce harmonic functions. It is then possible to characterize harmonic morphisms as the horizontally conformal harmonic mappings, see [**19**]. Questions from extrinsic geometry such as normal parallel translation in the presence of metric singularities are also treated here.

Existence and uniqueness of geodesics for type changing metrics have been studied in [**6, 16**] and [**17**].

There are applications of the notion type changing metric in general relativity, see [**7, 13**] and [**22**].

2. Metric singularities. In this section let M denote a smooth n dimensional manifold with a smooth symmetric two tensor g. Also let p denote a singular point for g, that is, g_p is degenerate. Define the isotropic space at p,

$$I(p) = T_pM \cap T_pM^{\perp}.$$

Now let $v_1, \dots, v_n$ denote a basis at p with

$$\text{span}\,\{v_1, \dots, v_I\} = I(p),$$

$1 \leq I \leq n$. We call this an adapted basis. A frame

$$X_1, \dots, X_n : U \longrightarrow TU$$

defined on an open neighborhood U of p is adapted to $v_1, \dots, v_n$ provided

$$X_i(p) = v_i, \quad i = 1, \dots, n.$$

Definition 2.1. (M, g) satisfies the isotropy condition at p with respect to $v_1, \dots, v_n$ provided

$$X_1(p)[g(X_i, X_j)] = X_i(p)[g(X_1, X_j)]$$

for all $i, j \in \{1, \dots, I\}$ for one and hence any frame adapted to $v_1, \dots, v_n$. If this holds, v_1 is called *geodesic*.

Notice that this condition holds trivially if the dimension of $I(p)$ is one.

Definition 2.2. (M, g) satisfies the augmented isotropy condition at p with respect to $v_1, \dots, v_n$ provided

$$X_k(p)[g(X_i, X_j)] = X_i(p)[g(X_k, X_j)]$$

for all $i, j, k \in \{1, \dots, I\}$ where $X_1, \dots, X_n$ is any frame adapted to $v_1, \dots, v_n$.

We shall now prove that every isotropic vector at p is geodesic if (M, g) satisfies the augmented isotropy condition.

Proposition 2.3. *If (M,g) satisfies the augmented isotropy condition at p with respect to an adapted basis, then (M,g) satisfies the augmented isotropy condition of p with respect to any adapted basis.*

Proof. Let (M,g) satisfy the augmented isotropy condition at p with respect to an adapted basis $v_1,\dots,v_n$, and let $X_1,\dots,X_n : U \to TU$ be a frame adapted to $v_1,\dots,v_n$. Also let $Y_1,\dots,Y_n : U \to TU$ be a frame adapted to some other adapted basis. We can then write

$$Y_i = \sum_{q=1}^{n} \alpha_i^q X_q$$

for some smooth functions α_i^q on U. Notice that

$$\alpha_i^q(p) = 0,$$

when $1 \le i \le I$, $I+1 \le q \le n$. When $i,j,k \in \{1,\dots,I\}$, we then find that

$$\begin{aligned} Y_i(p)[g(Y_j,Y_k)] &= \sum_{q=1}^{I} \alpha_i^q \sum_{l=1}^{I} \sum_{m=1}^{I} \alpha_j^l \alpha_k^m X_q(p)[g(X_l,X_m)] \\ &= \sum_{m=1}^{n} \alpha_k^m X_m(p)\bigg[g\bigg(\sum_{l=1}^{n} \alpha_j^l X_l, \sum_{q=1}^{n} \alpha_i^q X_q\bigg)\bigg] \\ &= Y_k(p)[g(Y_j,Y_i)] \end{aligned}$$

and the proposition follows.

Now define

$$\begin{aligned} G : H_+ = \{(x_1,\dots,x_n) \mid x_1 > 0\} &\longrightarrow H_+ \\ (v_1,\dots,v_n) &\longmapsto v_1^2(1,v_2,\dots,v_n) \end{aligned}$$

and

$$\begin{aligned} \Phi : H_+ \times H_\pm &\longrightarrow H_+ \times H_\pm \\ (v,y) &\longmapsto (G(v),(1/y_1)(1,\dots,y_n)) \end{aligned}$$

where

$$H_\pm = \{(x_1,\dots,x_n) \mid x_1 \neq 0\}.$$

Φ has an inverse Θ.

Definition 2.4. A smooth curve

$$X : \left]0, t^+\right[\longrightarrow TM$$

is resolvent differentiable if there exists $s^+ > 0$ such that

$$\phi^* \circ X(s) \in H_+ \times H_\pm, \quad s \in \left]0, s^+\right[$$

and $\Theta \circ \phi^* \circ X$ is the restriction of a smooth curve through 0 and

$$\pi_1\left(\frac{d}{dt}(\Theta \circ \phi^* \circ X)(0)\right) > 0$$
$$\pi_{n+1}\left(\frac{d}{dt}(\Theta \circ \phi^* \circ X)(0)\right) \neq 0.$$

This is independent of the choice of tangent bundle chart ϕ^*, where (U, ϕ) is a chart around p adapted to $v_1, \dots, v_n$, i.e., $\partial_i(p) = v_i$.

Definition 2.5. Two resolvent differentiable curves X_i $:]0, t_i^+[\to TM$, $i = 1, 2$, are tangential provided

$$\frac{d}{dt}(\Theta \circ \phi^* \circ X_1)(0) = \frac{d}{dt}(\Theta \circ \phi^* \circ X_2)(0).$$

Tangentiality is independent of the choice of chart (U, ϕ) adapted to $v_1, \dots, v_n$. Tangential is an equivalence relation. An equivalence class is called a resolvent tangent vector. The set of equivalence classes is denoted

$$T_p(\Xi, T, M).$$

Define an injective mapping

$$T_\Xi \phi^* : T_p(\Xi, TM) \longrightarrow \mathbf{R}^{2n}$$
$$[X] \longmapsto \frac{d}{dt}(\Theta \circ \phi^* \circ X)(0).$$

Notice that when (U,ϕ) and (V,ψ) are adapted to the same frame $v_1,\dots,v_n$, then

$$T_\Xi\phi^*(u)=T_\Xi\psi^*(u),\quad \forall\, u\in T_p(\Xi,T,M).$$

Now define

$$f=f^\phi=\det\{g(\partial_i,\partial_j)\}$$
$$\lambda^\phi=\frac{1}{I!}\frac{\partial^I f^\phi}{\partial x_1^I(p)}.$$

Notice that $\lambda^\phi=\lambda^\psi=\lambda$. We shall assume $\lambda\neq 0$. Let G^{km} denote the complement to g_{km} in the matrix $\{g_{ij}\}$ and define

$$\tilde\Gamma_{ij}^k=\sum_{m=1}^n G^{km}[ij,m]$$
$$[ij,m]=\frac{1}{2}\left\{\frac{\partial g_{im}}{\partial x_j}+\frac{\partial g_{jm}}{\partial x_i}-\frac{\partial g_{ij}}{\partial x_m}\right\}.$$

Define

$$b_m^k=-\frac{1}{(I-2)!}\frac{\partial^{I-1}}{\partial x_1^{I-2}\partial x_m}\tilde\Gamma_{11}^k(0)$$
$$\alpha_m^k=-\frac{2}{(I-1)!}\frac{\partial^{I-1}}{\partial x_1^{I-1}}\tilde\Gamma_{1m}^k(0),$$

for $k,m\in\{1,\dots,n\}$. We shall see later that these constants are independent of the choice of chart adapted to $v_1,\dots,v_n$. These constants give rise to a linear map M with matrix representation

$$\begin{pmatrix}
1 & 0 & 0 & \cdot & 0\\
0 & 0 & (a_{I+1}^2+(2/3)b_{I+1}^2)/\lambda & \cdot & (a_n^2+(2/3)b_n^2)/\lambda\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & (a_{I+1}^I+(2/3)b_{I+1}^I)/\lambda & \cdot & (a_n^I+(2/3)b_n^I)/\lambda\\
0 & 0 & (2/3) & \cdot & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdot & (2/3)\\
0 & 1 & 0 & \cdot & 0\\
0 & 0 & 3(a_{I+1}^2+(2/3)b_{I+1}^2)/(2\lambda) & \cdot & 3(a_n^2+(2/3)b_n^2)/(2\lambda)\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 3(a_{I+1}^I+(2/3)b_{I+1}^I)/(2\lambda) & \cdot & 3(a_n^I+(2/3)b_n^I)/(2\lambda)\\
0 & 0 & 1 & \cdot & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdot & 1
\end{pmatrix}$$

in the standard basis in $\mathbf{R}^{2n}$. We can now define the space of initial velocities for geodesics through p, namely,

$$T_p^G(\Xi, T, M) = \left\{ u \in T_p(\Xi, TM) | z = T_\Xi \phi^*(u),\ \frac{3}{2} = z_{n+1} z_1^2,\ z \in \operatorname{Im} M \right\}.$$

It will be convenient to have

$$\tau(t) = t^3, \quad t \in \mathbf{R}.$$

If (M, g) satisfies the isotropy condition at p with respect to $v_1, \dots, v_n$ where $v_{I+1}, \dots, v_n$, $I \geq 2$, are orthonormal we can prove

Theorem 2.6. *Given $u \in T_p^G(\Xi, TM)$ there exists a geodesic $\gamma :]0, t^+[\to M$ such that $\gamma' \circ \tau$ is resolvent differentiable with resolvent tangent vector u.*

If $\beta :]0, s^+[\to M$ is a geodesic such that $\beta' \circ \tau$ is resolvent differentiable with resolvent tangent vector u, then $\beta = \gamma$ on their common domain of definition.

Proof. Notice that the first $I - 2$ differentials of $\tilde{\Gamma}_{ij}^k$ vanishes at $\phi(p) = 0$ where (U, ϕ) is a chart adapted to $v_1, \dots, v_n$. This means we can write using Einstein's summation convention

$$\tilde{\Gamma}_{ij}^k = \Omega_{ij}^{k, i_1, \dots, i_{I-1}}(x) x_{i_1} \cdots x_{i_{I-1}}.$$

Here

$$\begin{aligned} \Omega_{11}^{1,1,\dots,1} &= \frac{1}{(I-1)!} \frac{\partial^{I-1}}{\partial x_1^{I-1}} \tilde{\Gamma}_{11}^1(0) \\ &= \sum_{\sigma \in S_I, \sigma(1) = I} \frac{\partial g_{2\sigma(2)}}{\partial x_1} \cdots \frac{\partial g_{I\sigma(I)}}{\partial x_1} \frac{1}{2} \frac{\partial g_{11}}{\partial x_I} \\ &\quad + (-1)^{I+1} \sum_{\substack{\sigma \in S_I, \\ \sigma(1) = I}} \frac{\partial g_{2\sigma(2)}}{\partial x_1} \cdots \frac{\partial g_{I\sigma(I)}}{\partial x_1} \frac{1}{2} \frac{\partial g_{1I}}{\partial x_1} g_{I+1,I+1} \cdots g_{nn} \\ &= \frac{1}{2} \lambda. \end{aligned}$$

Also, for $2 \leq k \leq I$, $1 \leq m \leq I$,

$$\begin{aligned}
&\sum \Omega_{11}^{k,1,\dots,m,\dots,1} \\
&\quad = \frac{1}{(I-2)!}\frac{\partial^{I-1}}{\partial x_1^{I-2}\partial x_m}\tilde{\Gamma}_{11}^k(0) \\
&\quad = \frac{1}{2}\det\begin{pmatrix} (\partial g_{11}/\partial x_m) & \cdots & (\partial g_{11}/\partial x_1) & \cdots & (\partial g_{1I}/\partial x_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (\partial g_{1I}/\partial x_m) & \cdots & (\partial g_{1I}/\partial x_1) & \cdots & (\partial g_{II}/\partial x_1) \end{pmatrix} \\
&\qquad \cdot g_{I+1,I+1}\cdots g_{nn} \\
&\quad = \begin{cases} 0 & k \neq m \\ -\lambda/2 & k = m, \end{cases}
\end{aligned}$$

using the isotropy condition. This constant vanishes when $I+1 \leq k \leq n$. Similarly

$$\begin{aligned}
\Omega_{1i}^{k,1,\dots,1} &= \frac{1}{(I-1)!}\frac{\partial^{I-1}}{\partial x_1^{I-1}}\tilde{\Gamma}_{1i}^k(0) \\
&= \frac{1}{2}\det\begin{pmatrix} (\partial g_{11}/\partial x_1) & \cdots & (\partial g_{11}/\partial x_i) & \cdots & (\partial g_{1I}/\partial x_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (\partial g_{1I}/\partial x_1) & \cdots & (\partial g_{1I}/\partial x_i) & \cdots & (\partial g_{II}/\partial x_1) \end{pmatrix} \\
&\quad \cdot g_{I+1,I+1}\cdots g_{nn} \\
&= \begin{cases} 0 & i \neq k, \\ \lambda/2 & i = k, \end{cases}
\end{aligned}$$

for $2 \leq k \leq I$, $1 \leq i \leq I$, while it vanishes for $I+1 \leq k \leq n$. Now define a vector field

$$\begin{aligned}
Y : \phi(U) \times \mathbf{R}^n &\longrightarrow \mathbf{R}^n \times \mathbf{R}^n \\
(x,y) &\longmapsto (fy, -\tilde{\Gamma}_{ij}^k y_i y_j e_k),
\end{aligned}$$

where $e_1, \dots, e_n$ is the standard basis in $\mathbf{R}^n$. It is proportional to the geodesic spray. It gives rise to the vector field

$$Z(v,y) = y_1 D\Theta(Y(\Phi(v,y))), \quad (v,y) \in H_+ \times H_\pm.$$

This vector field is the restriction of a smooth vector field also denoted Z defined on an open neighborhood of the origin. In fact, here

$$\begin{aligned} Z(v,y) &= v_1^{2(I-1)} \begin{pmatrix} (1/2)v_1\lambda + F_1(v) \\ -v_n\lambda + y_n\lambda + F_n(v) \\ y_1\Omega_{11}^{1,1,\cdots,1}(0) + W_1(v,y) \\ y_n\Omega_{11}^{1,1,\cdots,1}(0) - \sum_{m=2}^{n}\sum_{l=1}^{I-1}\Omega_{11}^{n,1\ldots m\ldots 1}(0)v_m \\ -\sum_{i=2}^{n} 2\Omega_{1i}^{n,1,\ldots 1}(0)y + i + W_n(v,y) \end{pmatrix} \\ &= v_1^{2(I-1)}\mathcal{Z}(v,y), \end{aligned}$$

where $F_i(0)=0$, $DF_i(0)=0$, $W_i(0,0)=0$, $DW_i(0,0)=0$. So

$$L = D\mathcal{Z}_0 = \begin{pmatrix} \lambda/2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & 0 & \lambda/2 & 0 & \cdots & 0 \\ 0 & \lambda/2 & \cdots & * & 0 & -\lambda/2 & \cdots & * \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \lambda/2 & * & 0 & 0 & -\lambda/2 & * \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda/2 \end{pmatrix}$$

$-\lambda$ is an eigenvalue of algebraic multiplicity $n-I$, 0 and $-(3/2)\lambda$ is an eigenvalue of algebraic multiplicity $I-1$ and $\lambda/2$ is an eigenvalue of geometric multiplicity $n-I+2$. We can assume $\lambda<0$ and then there is a smooth $n-I+2$-dimensional stable manifold $W^s(\mathcal{Z},0)$ for $\mathcal{Z}$ through 0. Define

$$\begin{aligned} \pi : \mathbf{R}^{2n} &\longrightarrow \mathbf{R}^{n-I+2} \\ (v,y) &\longmapsto (v_1, y_1, y_{I+1}, \ldots, y_n). \end{aligned}$$

The restriction of π to $W^s(\mathcal{Z},0)$ is a local diffeomorphism on an open neighborhood of 0 with inverse

$$\rho : W \longrightarrow W^s(\mathcal{Z},0).$$

Now define a vector field

$$W(x) = \pi\circ\mathcal{Z}\circ\rho(x), \quad x\in W$$

which has

$$DW_0 = \lambda/2 \text{ id}.$$

We can now use the blowing up construction via the map

$$\eta : \mathbf{R}^n \backslash \{0\} \longrightarrow \mathbf{R}^n$$
$$x \longmapsto x \frac{\|x\| - 1}{\|x\|}.$$

The restriction of this map to the exterior E of the unit sphere is a diffeomorphism with inverse ζ. Furthermore,

$$V(x) = \frac{1}{\|x\| - 1} D\zeta(W(\eta(x)))$$

is the restriction of a smooth vector field on an open neighborhood of the unit sphere with flow Φ^V and

$$\langle V(x), x \rangle = \lambda/2, \quad x \in S^{n+1-I}.$$

Now define

$$b : S^{n+1-I} \cap \{(w_1, w_{n+1}, w_{n+I}, \dots, w_{2n}) \mid w_1 > 0, w_{n+1} > 0\} = S \longrightarrow$$
$$T = \mathbf{R}^{n-I+2} \cap \left\{ z_1 > 0, z_{n+1} > 0, z_{n+1} z_1^2 = \frac{3}{2} \right\}$$
$$w \longmapsto (3/2 w_{n+1} w_1^2)^{1/3} w$$

with inverse

$$a : T \longrightarrow S, \quad z \longmapsto z/\|z\|.$$

Now suppose we are given a $u \in T_p^G(\Xi, TM)$. Define

$$w = a(\pi(T_\Xi \phi^*(u)))$$

and a reparametrization function

$$\tau_*(t) = \int_0^t \pi_{n+1} \circ \beta(s) f \circ G \circ \beta_1(s) / (\|\Phi_w^V(s)\| - 1) \pi_1 \circ \beta(s)^{2(I-1)} \, ds$$
$$\beta(s) = \rho \circ \eta \circ \Phi_w^V(s).$$

Since

$$\frac{d}{ds}(\pi_{n+1} \circ \beta)(0) = \pi_{n+1} \circ D\rho((\lambda/2)w) = (\lambda/2)w_{n+1}$$

we can write

$$\pi_{n+1} \circ \beta(s) = sk_1(s), \quad k_1(0) = (\lambda/2)w_{n+1}.$$

Doing this for the other functions in the integrand defining τ_*, we find that

$$\begin{aligned}\tau_*(t) &= t^3 k(t)\\ k(0) &= \lambda^3 w_{n+1} w_1^2/12.\end{aligned}$$

So τ_* is invertible on a small interval from 0. Substituting $s = \tau_*(v)$ in

$$\tau^{-1}(t) = \int_0^t \frac{1}{3} s^{-2/3}\, ds,$$

we find that $\tau_*^{-1} \circ \tau$ is smooth at 0 with

$$\frac{d}{dt}(\tau_*^{-1} \circ \tau)(0) = (12/(\lambda^3 w_{n+1} w_1^2))^{1/3}.$$

Notice that the restriction of

$$\Phi \circ \beta \circ \tau_*^{-1}$$

to small positive reals is the tangent vector field of a geodesic $\gamma :]0, t^+[\to \phi(U)$ which after reparameterization with τ has resolvent tangent

$$\begin{aligned}\frac{d}{dt}(\beta \circ \tau_*^{-1} \circ \tau)(0) &= D\rho_0(\lambda/2w)(12/(\lambda^3 w_{n+1} w_1^2))^{1/3}\\ &= D\rho_0(b(w)) = D\rho_0(\pi(T_\Xi \phi^*(u)))\\ &= T_\Xi \phi^*(u),\end{aligned}$$

because $D\rho_0 = M$. So γ is the local representation of a geodesic also denoted γ such that $\gamma' \circ \tau$ has resolvent tangent u. We have proven the existence part of the theorem.

To prove the uniqueness statement using Theorem 3.2, let $\beta :]0, s^+[\to \phi(U)$ be the local representative for a geodesic which is resolvent differentiable with resolvent tangent u. Define

$$k(t) = \pi_1 \circ \Theta \circ \beta' \circ \tau(t)^{2(I-1)}\tau'(t)\lambda t/(2\pi_{n+1} \circ \Theta \circ \beta' \circ \tau(t) f \circ \beta \circ \tau(t))$$

which is the restriction of a smooth function on an open interval I around 0 with $k(0) = 1$. Then, for $t > 0$ in this interval,

$$\frac{d}{dt}(\Theta \circ \beta' \circ \tau)(t) = \frac{1}{t\lambda/2}\mathcal{Z}(\Theta \circ \beta' \circ \tau(t))k(t).$$

Let $h : 0 \in J \to I$ denote a solution to

$$h'(t) = \frac{1}{t}h(t)\frac{1}{k(h(t))}, \quad t \neq 0, \quad t \in I,$$

which is at least C^4 at 0 with $h'(0) = 1$. We find it by applying the method outlined in the proof of Theorem 2.2.

Define

$$y(t) = \Theta \circ \beta' \circ \tau(t)$$

and

$$x_*(t) = y(h(t)), \quad t \in I, \quad t \geq 0.$$

Let L denote a linear isomorphism such that

$$\mathcal{Z}^* = L \circ \mathcal{Z} \circ L^{-1}$$

has its differential at 0 in Jordan canonical form. By construction $L \circ x_*$ is an integral curve for

$$\frac{1}{t\lambda/2}\mathcal{Z}^*.$$

Since y is smooth and h is C^4, we can write

$$x(t) = L \circ x_*(t) = wt + ct^2 + dt^3 + t^{3+e}f(t),$$

where $w = L \circ T_\Xi \phi^*(u)c$, $d \in \mathbf{R}^{2n}$ and f is a continuous function with $f(0) = 0$. By Theorem 2.2, there is a continuous function $f_* : [0, b_*[\to \mathbf{R}^{2n}$ such that $f_*(0) = 0$ and

$$\begin{aligned} z(t) &= H_w(t) + t^{3+\varepsilon}f_*(t) \\ &= wt + c_*t^2 + d_*t^3 + t^{3+\varepsilon}f_*(t) \end{aligned}$$

is an integral curve for $1/(t\lambda/2)\mathcal{Z}^*$.

We claim that $c_* = c$ and $d_* = d$. To see this, differentiate

$$\frac{\lambda}{2} t x'(t) = \mathcal{Z}^*(x(t))$$

twice to obtain

$$(D\mathcal{Z}_0^* - \lambda \,\mathrm{id})x''(0) = -D^2\mathcal{Z}_0^*(x'(0), x'(0))$$

and three times to obtain

$$\begin{aligned}\left(D\mathcal{Z}_0^* - \frac{3}{2}\lambda \,\mathrm{id}\right)x'''(0) &= -3D^2\mathcal{Z}_0^*(x'(0), x''(0)) \\ &\quad - D^3\mathcal{Z}_0^*(x'(0), x'(0), x'(0)).\end{aligned}$$

It is logically equivalent that

$$\begin{aligned}(D\mathcal{Z}_0^* - \lambda \,\mathrm{id})z''(0) &= -D^2\mathcal{Z}_0^*(z'(0), z'(0)) \\ \left(D\mathcal{Z}_0^* - \frac{3}{2}\lambda \,\mathrm{id}\right)z'''(0) &= -3D^2\mathcal{Z}_0^*(z'(0), z''(0)) \\ &\quad - D^3\mathcal{Z}_0^*(z'(0), z'(0), z'(0)).\end{aligned} \tag{2.1}$$

By definition of $H_w(t)$, we have

$$\begin{aligned}-\frac{\lambda}{2}z''(0) &= -D^2\mathcal{Z}_0^*(z'(0), z'(0)) \\ -\lambda z'''(0) &= -3D^2\mathcal{Z}_0^*(z'(0), z''(0)) \\ &\quad - D^3\mathcal{Z}_0^*(z'(0), z'(0), z'(0)).\end{aligned} \tag{2.2}$$

Now write

$$\begin{aligned}z''(0) &= b_1 + b_2 \\ D^2\mathcal{Z}_0^*(z'(0), z'(0)) &= c_1 + c_2\end{aligned}$$

where

$$b_1, c_1 \in \ker\left(D\mathcal{Z}_0^* - \frac{\lambda}{2}\mathrm{id}\right)$$

and b_2 and c_2 belong to the direct sum A of the generalized eigenspaces corresponding to 0, $-\lambda$ and $-(3/2)\lambda$. By (2.1) and (2.2)

$$(D\mathcal{Z}_0^* - \lambda\,\mathrm{id})(b_2) = -c_2 = -\frac{\lambda}{2}b_2;$$

hence

$$\left(D\mathcal{Z}_0^* - \frac{\lambda}{2}\mathrm{id}\right)(b_2) = 0.$$

Since $\lambda/2$ is not an eigenvalue of the restriction of $D\mathcal{Z}_0^*$ to A, we conclude that $b_2 = 0$, hence $c_2 = 0$. Now

$$\begin{aligned} z''(0) = b_1 &= (D\mathcal{Z}_0^* - \lambda\,\mathrm{id})^{-1}(-c_1) \\ &= (D\mathcal{Z}_0^* - \lambda\,\mathrm{id})^{-1}(-D^2\mathcal{Z}_0^*(z'(0), z'(0))) \\ &= (D\mathcal{Z}_0^* - \lambda\,\mathrm{id})^{-1}(-D^2\mathcal{Z}_0^*(x'(0), x'(0))) \\ &= x''(0). \end{aligned}$$

Thus $c = c_*$. The proof that $d = d_*$ is completely similar. Hence $x = z$ is unique by Theorem 3.2.

Now define

$$\tau_*(t) = \int_0^t 2\pi_{n+1} \circ z^*(s) f \circ G \circ z_1^*(s)/\pi_1 \circ z^*(s)^{2(I-1)} \lambda s\, ds$$
$$z^* = L^{-1} \circ z.$$

Then

$$\begin{aligned} &\frac{d}{dt}(\tau \circ h \circ \tau_*^{-1})(t) \\ &\qquad = \tau'(h(\tau_*^{-1}(t)))\frac{1}{\tau_*^{-1}(t)} h(\tau_*^{-1}(t)) \frac{1}{k(h(\tau_*^{-1}(t)))\tau_*'(\tau_*^{-1}(t))} \\ &\qquad = 1. \end{aligned}$$

Hence $\tau \circ h = \tau_*$, which is defined in terms of the unique z^*. So

$$\Theta \circ \beta' \circ \tau \circ h(t) = x_*(t).$$

Hence,

$$\beta = \Phi \circ z^* \circ \tau_*^{-1}$$

is unique. The theorem follows.

Example 2.7. Let $M = \mathbf{R}^2$ equipped with the symmetric two tensor field

$$g = (ax_1 + bx_2)dx_1^2 + 2bx_1 dx_1 dx_2 + cx_2 dx_2^2$$

where $a, c \in \mathbf{R}$, $b \neq 0$. The origin is a singular point and $\partial_1(0), \partial_2(0)$ an adapted frame.

Then $\partial_1(0)$ is geodesic and

$$\lambda < 0.$$

Theorem 2.6 applies to show existence and uniqueness of geodesics at the origin.

3. Orbit separation theorem. Let

$$A : U \to \mathbf{R}^n$$

denote a smooth vector field defined on an open neighborhood U of the origin in $\mathbf{R}^n$ with $A(0) = 0$. We can assume that $L = DA_0$ is in Jordan canonical form

$$\operatorname{diag}(A_1, \dots, A_{k_1}, A_{k_1+1}, \dots, A_{k_2}, A_{k_2+1}, \dots, A_{k_3}, A_{k_3+1}, \dots, A_k)$$

where

$$A_i = \begin{pmatrix} \lambda_i & \mu_i & \cdot & \cdot & 0 & 0 & 0 & 0 \\ -\mu_i & \lambda_i & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \delta & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & \delta & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \delta & 0 & \lambda_i & \mu_i \\ 0 & 0 & \cdot & \cdot & 0 & \delta & -\mu_i & \lambda_i \end{pmatrix}$$

and $\delta > 0$, $\mu_i \neq 0$, $\lambda_i < 0$ when $i = 1, \dots, k_1$ and $\lambda_i \geq 0$, $i = k_2 + 1, \dots, k_3$ and

$$A_i = \begin{pmatrix} \lambda_i & 0 & \cdot & \cdot & 0 \\ \delta & \lambda_i & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \lambda_i \end{pmatrix}$$

where $\lambda_i < 0$ when $i = k_1+1, \dots, k_2$ and $\lambda_i \geq 0$ when $i = k_3+1, \dots, k$. We shall assume that DA_0 has at least one eigenvalue with negative real part.

The aim of the present section is to prove the orbit separation Theorem 3.2. This theorem relies on definition (3.0), which defines a curve γ through the origin in $\mathbf{R}^n$. The coefficients in the definition depend on $v \in \mathbf{R}^{k_2}$. Theorem 3.2 says that γ is a very good approximation to an orbit of the vector field A. Moreover, γ approximates a unique integral curve of A. It is this uniqueness statement that is used to prove the uniqueness statement of Theorem 2.6.

Now define

$$\begin{aligned} \lambda_* &= \max_{i \in \{1, \dots, k_2\}} \lambda_i \\ \mathbf{p} &= (p_1, \dots, p_{k_2}), \quad p_i \in \mathbf{N}_0 \\ \mathbf{r} &= (r_1, \dots, r_{k_1}), \quad r_i \in \mathbf{Z} \\ \mathbf{s} &= (\mathbf{p}, q, \mathbf{r}) \\ \alpha_i &= \lambda_i / \lambda_*, \quad i \in \{1, \dots, k\} \\ \beta_i &= \mu_i / \lambda_*, \quad i \in \{1, \dots, k_1\} \cup \{k_2 + 1, \dots, k_3\}. \end{aligned}$$

We shall also need

$$\begin{aligned} \mathbf{a} &= (\alpha_1, \dots, \alpha_{k_2}) \\ \mathbf{b} &= (\beta_1, \dots, \beta_{k_1}) \end{aligned}$$

and

$$\alpha_* = \max\{\{|\alpha_i|\}_{i \in \{1, \dots, k\}} \cup \{|\alpha_i| + |\beta_i|\}_{i \in \{1, \dots, k_1\} \cup \{k_2+1, \dots, k_3\}}\}.$$

If $p_i \geq [\alpha_*/\alpha_i] + 1$, $i \in \{1, \dots, k_2\}$, then

$$\mathbf{pa} = p_1 \alpha_1 + \cdots + p_{k_2} \alpha_{k_2} > \alpha_*.$$

Consider then the nonempty finite set

$$\{\mathbf{pa} > \alpha_* \mid p_i \leq [\alpha_*/\alpha_i] + 1, |\mathbf{p}| = p_1 + \cdots + p_{k_2} \geq 1\}.$$

It has a minimum α. Take $\varepsilon \in]0, 1[$ with

$$2\varepsilon \in]0, \alpha - \alpha^*[$$

and choose $p \in \mathbf{N}$ subject to

$$p \leq \alpha_* + 2\varepsilon < p + 1.$$

Consider also

$$B : U \longrightarrow \mathbf{R}^n, \quad B(x) = A(x) - L(x).$$

By repeated use of the standard trick from singularity theory, we can write using the summation convention

$$\begin{aligned} B(x) &= B_*(x) + B_{**}(x) \\ B_*(x) &= \sum_{k=2}^{p} B_{j_1,\dots,j_k} x_{j_1} \cdots x_{j_k} \\ B_{**}(x) &= B_{j_1,\dots,j_{p+1}}(x) x_{j_1} \cdots x_{j_{p+1}} \end{aligned}$$

for suitable real constants $B_{j_1,\dots,j_k}$ and smooth functions $B_{j_1,\dots,j_{p+1}}$ defined on a possibly smaller open set U henceforth also denoted U.

We can assume with an appropriate choice of $\delta > 0$ that

$$(\alpha_* + \delta/|\lambda_*|)/(\alpha_* + \varepsilon) < 1.$$

Now define a continuous curve through the origin in $\mathbf{R}^n$,

$$\gamma : [0, +\infty[\longrightarrow \mathbf{R}^n$$

$$\begin{aligned} \gamma(t) = \sum_{q=0}^{np_*} \sum_{\substack{|\mathbf{p}| \geq 1 \\ \mathbf{pa} \leq \alpha_*}} \sum_{|\mathbf{r}| \leq p} t^{\mathbf{pa}} \ln^q t \, (\operatorname{Re} \gamma^{\mathbf{s}} \cos(\mathbf{br} \ln t) \\ + \operatorname{Im} \gamma^{\mathbf{s}} \sin(\mathbf{br} \ln t)) \end{aligned} \tag{3.0}$$

$$\gamma_j^{\mathbf{s}} \in \mathbf{C}, \quad j = 1, \dots, n, \quad p_* = 2^{2p+3}.$$

We shall now embark on a definition of the $\gamma^{\mathbf{s}}$. We shall define them inductively using induction on $|\mathbf{p}|$.

Given $(v_1, \dots, v_{k_2}) \in \mathbf{R}^{k_2}$ we shall first define $\gamma^{\mathbf{s}}$ for $|\mathbf{p}| = 1$. To this end, let

$$\begin{aligned} m_s &= \dim A_s \\ I_s &= \{m_1 + \cdots + m_{s-1} + 1, \dots, m_1 + \cdots + m_s\} \end{aligned}$$

and take $2j+1 \in I_s$, $s \in \{1,\dots,k_1\}$. So I_s is the set of row indices, corresponding to the Jordan block A_s. Let $\mathbf{p} = (\mathbf{r},0) = e_s$ where $e_1,\dots,e_{k_2}$ is the canonical basis in $\mathbf{R}^{k_2}$ and define

$$\gamma^{\mathbf{s}}_{2j+1} = \begin{cases} \dfrac{\delta^q}{q!\lambda_*^q}(v_{2j+1-2q}+iv_{2j+2-2q}) & 0 \le 2q \le 2j-(m_1+\cdots+m_{s-1}) \\ 0 & 2q > 2j-(m_1+\cdots+m_{s-1}) \end{cases}$$

$$\gamma^{\mathbf{s}}_{2j+2} = \begin{cases} \dfrac{d^q}{q!\lambda_*^q}(v_{2j+2-2q}-iv_{2j+1-2q}) & 0 \le 2q \le 2j-(m_1+\cdots+m_{s-1}) \\ 0 & 2q > 2j-(m_1+\cdots+m_{s-1}). \end{cases}$$

For $j \in I_s$, $s \in \{k_1+1,\dots,k_2\}$, let $\mathbf{p} = e_s$ and $\mathbf{r} = 0$ and define

$$\gamma^{\mathbf{s}}_j = \begin{cases} \dfrac{\delta^q}{q!\lambda_*^q}v_{j-q} & 0 \le q \le j-(m_1+\cdots+m_{s-1})-1 \\ 0 & q \ge j-(m_1+\cdots+m_{s-1}). \end{cases}$$

Let $\gamma^{\mathbf{s}}_i = 0$ for all other choices of $i \in \{1,\dots,k\}$ and $\mathbf{p}, q, \mathbf{r}$ with $|\mathbf{p}| = 1$. Assuming $\gamma^{\mathbf{s}}$ has been defined when $|\mathbf{p}| \le l$, $\mathbf{pa} \le \alpha_*$, $l \ge 1$ and

$$\gamma^{\mathbf{s}} = 0$$

for $|\mathbf{r}| \ge |\mathbf{p}|+1$ and $q > n(2^{|\mathbf{p}|}+2^{|\mathbf{p}|+1}-2)$, define for these values of $\mathbf{p}$, $\zeta^{\mathbf{p},q,\mathbf{r}}_j = 0$ if $|\gamma^{\mathbf{s}}_j| = 0$ and otherwise $\zeta^{\mathbf{p},q,\mathbf{r}}_j \in [0,2\pi[$ by

$$\cos\zeta^{\mathbf{p},q,\mathbf{r}}_j - i\sin\zeta^{\mathbf{p},q\mathbf{r}}_j = \gamma^{\mathbf{s}}_j/|\gamma^{\mathbf{s}}_j|.$$

This leads us to the next definition for $\mathbf{p}, q, \mathbf{r}$ with $|\mathbf{p}| = l+1$, namely,

$$\begin{aligned} F^{\mathbf{s}}_j = \sum_{k=2}^{l} \sum_{q_1+\cdots+q_k=q} \sum_{\mathbf{p}_1+\cdots+\mathbf{p}_k=\mathbf{p}} \sum_{\mathbf{r}_1\pm\cdots\pm\mathbf{r}_k=\mathbf{r}} &|\gamma^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}|\cdots|\gamma^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k}| \\ \cdot B^j_{j_1\cdots j_k}\frac{1}{2^{k-1}}&(\cos(\zeta^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}\pm\cdots\pm\zeta^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k}) \\ &- i\sin(\zeta^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}\pm\cdots\pm\zeta^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k})). \end{aligned}$$

Proposition 3.1. *$F^{\mathbf{p},q,\mathbf{r}} = 0$ for $|\mathbf{r}| \ge |\mathbf{p}|+1$ or $q > n(2^{|\mathbf{p}|}+2^{|\mathbf{p}|+1}-4)$.*

Proof. We shall prove this by induction on $|\mathbf{p}| \geq 2$. If $|\mathbf{p}| = 2$, then there exists no $\mathbf{p}_1, \dots, \mathbf{p}_k$ such that

$$\mathbf{p} = \mathbf{p}_1 + \cdots + \mathbf{p}_k, \quad |\mathbf{p}_i| \geq 1$$

for $k \geq 3$. Take $\mathbf{p}_1, \mathbf{p}_2$ such that $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}$. Then

$$|\mathbf{p}_1| = |\mathbf{p}_2| = 1;$$

hence,

$$\gamma^{\mathbf{p}_i, q_i, \mathbf{r}_i} = 0, \quad q_i > n.$$

If $q_1 + q_2 = q > 2n$, then one of the $q_i > n$; hence,

$$F^{\mathbf{s}} = 0, \quad q > 2n.$$

If $\mathbf{r}_1 \pm \mathbf{r}_2 = \mathbf{r}$ and $|\mathbf{r}| \geq 3$, then

$$|\mathbf{r}_i| \geq 2$$

for $i = 1$ or 2, hence $\gamma^{\mathbf{p}_i, q_i, \mathbf{r}_i} = 0$, so

$$F^{\mathbf{s}} = 0, \quad |\mathbf{r}| \geq 3.$$

This proves the proposition for $|\mathbf{p}| = 2$. Assuming the validity of the proposition for $|\mathbf{p}| \leq l - 1$, $l \geq 3$, consider $\mathbf{s}$ with $|\mathbf{p}| = l$. Then there exists no $\mathbf{p}_1, \dots, \mathbf{p}_k$ such that

$$\mathbf{p}_1 + \cdots + \mathbf{p}_k = \mathbf{p}$$

for $k \geq l + 1$. Consider then $\mathbf{p}_1, \dots, \mathbf{p}_i$, $2 \leq i \leq l$ such that

$$\mathbf{p}_1 + \cdots + \mathbf{p}_i = \mathbf{p}.$$

If $q = q_1 + \cdots + q_i > n(2^{|\mathbf{p}|} + 2^{|\mathbf{p}|+1} - 4)$, then there exists $j \in \{1, \dots, i\}$ such that

$$q_j > n(2^{|\mathbf{p}_j|+1} + 2^{|\mathbf{p}_j|} - 2);$$

hence, $\gamma^{\mathbf{p}_j, q_j, \mathbf{r}_j} = 0$ and this implies that

$$F^{\mathbf{s}} = 0, \quad q > n(2^{|\mathbf{p}|} + 2^{|\mathbf{p}|+1} - 4).$$

If $\mathbf{r}_1 \pm \cdots \pm \mathbf{r}_i = \mathbf{r}$ and $|\mathbf{r}| \geq l+1$, then there exists a $j \in \{1, \ldots, i\}$ such that

$$|\mathbf{r}_j| > |\mathbf{p}_j|;$$

hence, $\gamma^{\mathbf{p}_j, q_j, \mathbf{r}_j} = 0$. We deduce that

$$F^s = 0, \quad |\mathbf{r}| \geq l+1 = |\mathbf{p}| + 1.$$

The proposition follows.

By definition, $F^{\mathbf{s}} = 0$ when $|\mathbf{p}| = 1$.

We shall now define $\gamma^{\mathbf{s}}$ when $|\mathbf{p}| = l+1$, $\mathbf{pa} \leq \alpha_*$. Consider initially

$$\begin{gathered} j = m_1 + \cdots + m_{s-1} + 1 \\ s \in \{1, \ldots, k_1\} \cup \{k_2 + 1, \ldots, k_3\}. \end{gathered}$$

We are thus considering a Jordan block A_s with nonzero imaginary part of the eigenvalue. j is the row number in A corresponding to the first row of the Jordan block of A_s. We shall now define $\gamma_j^{\mathbf{s}}, \gamma_{j+1}^{\mathbf{s}}$. There are two cases (i) and (ii) below to consider.

(i) $a = \mathbf{pa} - i\mathbf{br} \neq \alpha_s \pm i\beta_s$. Define

$$\xi_{r,q} = a^{q-r+1}$$

and

$$\begin{aligned} &\sum_{q=r+1}^{np_*} (-1)^{q-r} q!/(r! a^{q-r+1}) \begin{pmatrix} \alpha_s & \beta_s \\ -\beta_s & \alpha_s \end{pmatrix} \begin{pmatrix} \gamma_j^{\mathbf{s}} \\ \gamma_{j+1}^{\mathbf{s}} \end{pmatrix} \\ &\quad + \sum_{q=r}^{np_*} (-1)^{q-r} q!/(r!(a^{q-r+1}\lambda_*) \begin{pmatrix} F_j^{\mathbf{s}} \\ F_{j+1}^{\mathbf{s}} \end{pmatrix} = - \begin{pmatrix} a_j^{\mathbf{s}} \\ a_{j+1}^{\mathbf{s}} \end{pmatrix}. \end{aligned} \tag{3.1}$$

Due to assumption (i) and Proposition 3.1, the linear system

$$\begin{pmatrix} \alpha_s/\xi_{r,r} - 1 & \beta_s/\xi_{r,r} \\ -\beta_s/\xi_{r,r} & \alpha_s/\xi_{r,r} - 1 \end{pmatrix} \begin{pmatrix} \gamma_j^{\mathbf{s}} \\ \gamma_{j+1}^{\mathbf{s}} \end{pmatrix} = \begin{pmatrix} a_j^{\mathbf{s}} \\ a_{j+1}^{\mathbf{s}} \end{pmatrix} \tag{3.2}$$

can be solved for $\gamma_j^{\mathbf{s}}, \gamma_{j+1}^{\mathbf{s}}$ when $q = np_*$ and by descending induction on q. For the subsequent rows $j = m_1 + \cdots + m_{s-1} + 2i + 1 \leq m_1 + \cdots + m_s$,

$i \geq 1$, define

$$\begin{aligned}(3.3)\qquad &\sum_{q=r+1}^{np_*}(-1)^{q-r}q!/(r!a^{q-r+1})\begin{pmatrix}\alpha_s & \beta_s\\ -\beta_s & \alpha_s\end{pmatrix}\begin{pmatrix}\gamma_j^{\mathbf{s}}\\ \gamma_{j+1}^{\mathbf{s}}\end{pmatrix}\\ &+\sum_{q=r}^{np_*}(-1)^{q-r}q!/(r!a^{q-r+1}\lambda_*)\left(\begin{pmatrix}F_j^{\mathbf{s}}\\ F_{j+1}^{\mathbf{s}}\end{pmatrix}+\delta\begin{pmatrix}\gamma_{j-2}^{\mathbf{s}}\\ \gamma_{j-1}^{\mathbf{s}}\end{pmatrix}\right)\\ &=-\begin{pmatrix}a_j^{\mathbf{s}}\\ a_{j+1}^{\mathbf{s}}\end{pmatrix}.\end{aligned}$$

Now define $\gamma_j^{\mathbf{s}}, \gamma_{j+1}^{\mathbf{s}}$ by increasing induction on j from $m_1+\cdots+m_{s-1}+3$ and decreasing induction on q from np_* using (3.2).

(ii) If $a = \alpha_s \pm i\beta_s$, $j = m_1 + \cdots + m_{s-1} + 1$, define

$$-\begin{pmatrix}a_j^{\mathbf{s}}\\ a_{j+1}^{\mathbf{s}}\end{pmatrix}=\sum_{q=r}^{np_*}(-1)^{q-r}q!/(r!a^{q-r+1}\lambda_*)\begin{pmatrix}F_j^{\mathbf{s}}\\ F_{j+1}^{\mathbf{s}}\end{pmatrix}$$

and consider the linear system

(3.4)

$$\begin{pmatrix}\alpha_s/a-1 & \beta_s/a & 0 & 0 & \cdot & 0\\ -\beta_s/a & \alpha_s/a-1 & 0 & 0 & \cdot & 0\\ -np_*\alpha_s/a^2 & -np_*\beta_s/a^2 & \alpha_s/a-1 & \beta_s/a & \cdot & 0\\ np_*\beta_s/a^2 & -np_*\alpha_s/a^2 & -\beta_s/a & \alpha_s/a-1 & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ (-1)^{np_*+1}(np_*)!\beta_s/a^{np_*+1} & (-1)^{np_*}(np_*)!\alpha_s/a^{np_*+1} & \cdot & \cdot & \cdot & \alpha_s/a-1\end{pmatrix}$$

$$\cdot\begin{pmatrix}\gamma_j^{\mathbf{p},np_*,\mathbf{r}}\\ \gamma_{j+1}^{\mathbf{p},np_*,\mathbf{r}}\\ \gamma_j^{\mathbf{p},np_*-1,\mathbf{r}}\\ \gamma_{j+1}^{\mathbf{p},np_*-1,\mathbf{r}}\\ \cdot\\ \gamma_{j+1}^{\mathbf{p},0,r}\end{pmatrix}=\begin{pmatrix}a^{\mathbf{p},np_*,\mathbf{r}}\\ a_{j+1}^{\mathbf{p},np_*,\mathbf{r}}\\ a_j^{\mathbf{p},np_*-1,\mathbf{r}}\\ a_{j+1}^{\mathbf{p},np_*-1,\mathbf{r}}\\ \cdot\\ a_{j+1}^{\mathbf{p},0,r}\end{pmatrix}.$$

By Proposition 3.1, $a_j^{\mathbf{p},np_*,\mathbf{r}} = a_{j+1}^{\mathbf{p},np_*,\mathbf{r}} = 0$. Letting $\gamma_{j+1}^{\mathbf{p},0,\mathbf{r}} = 0$, we obtain a linear system where the rank of the total matrix is equal to the rank of the coefficient matrix and the number of variables. The system then has a unique solution which defines the $\gamma_j^{\mathbf{s}}, \gamma_{j+1}^{\mathbf{s}}$. For $j = m_1 + \cdots + m_{s-1} + 2i + 1 \leq m_1 + \cdots + m_s$, $i \geq 1$ define

$$-\begin{pmatrix} a_j^{\mathbf{p},r,\mathbf{r}} \\ a_{j+1}^{\mathbf{p},r,\mathbf{r}} \end{pmatrix} = \sum_{q=r}^{np_*}(-1)^{q-r}q!/(r!a^{q-r+1}\lambda_*)\left(\begin{pmatrix} F_j^{\mathbf{s}} \\ F_{j+1}^{\mathbf{s}} \end{pmatrix} + \delta\begin{pmatrix} \gamma_{j-2}^{\mathbf{s}} \\ \gamma_{j-1}^{\mathbf{s}} \end{pmatrix}\right)$$

and proceed as above to define $\gamma_j^{\mathbf{s}}, \gamma_{j+1}^{\mathbf{s}}$ by increasing induction on j from $m_1 + \cdots + m_{s-1} + 3$ using (3.4). Consider then

$$j = m_1 + \cdots m_{s-1} + 1,$$
$$s \in \{k_1 + 1, \dots, k_2\} \cup \{k_3 + 1, \dots, k\}.$$

So now we are looking at a Jordan block eigenvalue, a real number. Again j is the row number in A corresponding to the first row of the Jordan block A_s. We shall define $\gamma_j^{\mathbf{s}}$. Once again there will be two cases (i) and (ii) to consider.

(i) $a \neq \alpha_s$. Define

$$-a_j^{\mathbf{p},r,\mathbf{r}} = \sum_{q=r+1}^{np_*}(-1)^{q-r}\alpha_s q!/(r!a^{q-r+1})\gamma_j^{\mathbf{s}} + \sum_{q=r}^{np_*}(-1)^{q-r}q!/(r!\lambda_* a^{q-r+1})F_j^{\mathbf{s}}. \tag{3.5}$$

Due to assumption (i),

$$(\alpha_s/a - 1)\gamma_j^{\mathbf{s}} = a_j^{\mathbf{s}} \tag{3.6}$$

can be solved for $\gamma_j^{\mathbf{s}}$ when $q = np_*$ and by descending induction on q. For $j = m_1 + \cdots + m_{s-1} + i \leq m_1 + \cdots + m_s$, $i \geq 2$, define

$$-a_j^{\mathbf{s}} = \sum_{q=r+1}^{np_*}(-1)^{q-r}q!\alpha_s/(r!a^{q-r+1})\gamma_j^{\mathbf{s}} + \sum_{q=r}^{np_*}(-1)^{q-r}q!/(r!a^{q-r+1}\lambda_*)(F_j^{\mathbf{s}} + \delta\gamma_{j-1}^{\mathbf{s}}). \tag{3.7}$$

Now define $\gamma_j^{\mathbf{s}}$ by decreasing induction on q from np_* and increasing induction on j from $m_1 + \cdots + m_{s-1} + 2$ using (2.6).

(ii) If $\alpha_s = a$, $j = m_1 + \cdots + m_{s-1} + 1$, define for $r = np_* - 1$,

$$\gamma_j^{\mathbf{p},r+1,\mathbf{r}} = a^2/((r+1)\alpha_s)a_j^{\mathbf{p},r,\mathbf{r}}, \tag{3.8}$$

and by descending induction in r. Here

$$\begin{aligned} a_j^{\mathbf{p},r,\mathbf{r}} = &\sum_{q=r+2}^{np_*} (-1)^{q-r}\alpha_s q!/(r!a^{q-r+1})\gamma_j^{\mathbf{s}} \\ &+ \sum_{q=r}^{np_*} (-1)^{q-r} q!/(r!a^{q-r+1}\lambda_*)(F_j^{\mathbf{s}} + \delta\gamma_{j-1}^{\mathbf{s}}) \end{aligned}$$

with $\delta = 0$. Finally, let $\gamma_j^{\mathbf{p},0,\mathbf{r}} = 0$. For $j = m_1 + \cdots + m_{s-1} + i \leq m_1 + \cdots + m_s$, $i \geq 2$, define $\gamma_j^{\mathbf{p},r+1,\mathbf{r}}$ by decreasing induction on r from $np_* - 1$ and increasing induction in j from $m_1 + \cdots + m_{s-1} = 2$ from (3.8). Finally let $\gamma_j^{\mathbf{p},0,\mathbf{r}} = 0$. This defines $\gamma^{\mathbf{s}}$ for $\mathbf{p}$ with $|\mathbf{p}| = l + 1$. Tracing back definitions, we see that

$$\gamma^{\mathbf{s}} = 0$$

for $|\mathbf{r}| \geq |\mathbf{p}| + 1$ or $q > n(2^{|\mathbf{p}|} + 2^{|\mathbf{p}|+1} - 2)$ using Proposition 3.1. Since the $\gamma^{\mathbf{s}}$ depend only on v we shall also use the notation

$$\gamma(t) = H_v(t), \quad t \geq 0.$$

There are no resonances in the stable part if

$$\begin{aligned} a = \mathbf{pa} - i\mathbf{br} \neq \alpha_s \pm i\beta_s, \quad s \in \{1, \dots, k_1\} \\ a = \mathbf{pa} - i\mathbf{br} \neq \alpha_s, \quad s \in \{k_1 + 1, \dots, k_2\} \end{aligned}$$

for all $\mathbf{p}, \mathbf{r}$ with $\mathbf{pa} \leq \alpha_*$ and $|\mathbf{r}| \leq p$.

Theorem 3.2. (1) *Given $v \in \mathbf{R}^{k_2}$, then there exists a unique continuous map*

$$f : [0, b[\longrightarrow \mathbf{R}^n, \quad f(0) = 0$$

such that $t \mapsto x(\exp(\lambda_* t))$, $t \in (1/\lambda_*)\ln(]0, b[)$

$$x(t) = H_v(t) + t^{\alpha_* + \varepsilon} f(t)$$

is an integral curve for A.

(2) *When all the eigenvalues of L have negative real parts,l there are no resonances in the stable part and $\alpha_* + \varepsilon > \operatorname{trace}(L)/\lambda_*$. If y is an integral curve for A such that $y(t) \to 0$ as $t \to +\infty$, then there exists $w \in \mathbf{R}^n$ and a continuous map*

$$f : [0, b[\, \longrightarrow \mathbf{R}^n, \quad f(0) = 0$$

such that

$$y\left(\frac{1}{\lambda_*} \ln t\right) = H_w(t)_t^{\alpha_* + \varepsilon} f(t), \quad t \in \,]0, b[\,.$$

Remark 3.3. Uniqueness in (1) means that if

$$f_* : [0, b_*[\, \longrightarrow \mathbf{R}^n, \quad f_*(0) = 0$$

is a continuous map such that

$$t \longmapsto x_*(\exp(\lambda_*, t)), t \in \frac{1}{\lambda_*} \ln(]0, b_*])$$
$$x_*(t) = H_v(t) + t^{\alpha_* + \varepsilon} f_*(t)$$

is an integral curve for A, then $f = f_*$ on their common domain of definition.

Remark 3.4. (2) means that every integral curve for A with $y(t) \to 0$ as $t \to +\infty$ has a tangent vector w that characterizes it.

Proof. Before outlining the strategy of the proof we need some definitions. For $u \in \mathbf{N}$, let E_u denote the space of continuous functions

$$f : [0, b[\, \times B_a(v) \longrightarrow \overline{B_c(0)}$$
$$f(0, w) = 0, \quad \forall\, w \in B_a(v)$$

such that f_t is C^u for all $t \in [0, b[$ and for all $j \in 1, \dots, u$,

$$(t, w) \longmapsto D_2^j f(t, w)$$

is continuous with

$$|D_2^j f(t,w)| \le d_j, \quad \forall\,(t,w) \in [0,b[\times B_a(v).$$

Here $B_a(v)$ denotes the open ball of radius a around v in the max norm $|\ |$. We can assume $a, b, c > 0$ chosen to render

$$H_w(t) + t^{\alpha_* + \varepsilon} f \in U$$

for all $t \in [0,b[$, $f \in \overline{B_c(0)}$ and $w \in B_a(v)$. We shall specify the d_j shortly.

Given $f \in E_u$, define for $t \in]0,b[$

$$S(f)(t,w) = \frac{1}{t^{\alpha_*+\varepsilon}}\left(\int_0^t \frac{1}{\lambda_* s} A(H_w(s) + s^{\alpha_*+\varepsilon} f(s,w))\,ds - H_w(t)\right)$$
$$S(f)(0,w) = 0.$$

The strategy of the proof is to show that with appropriate choices of the d_j, S maps E_u into E_u and is in fact a contraction. This implies the existence of a unique fixed point f, since E_u is a complete metric space. This provides the unique continuous map mentioned in (1) of the statement of the theorem.

Now

$$\begin{aligned} &B_*(H_w(s) + s^{\alpha_*+\varepsilon} f) \\ &\quad = \sum_{q=0}^{np_*} \sum_{\mathbf{pa} \le \alpha_*} \sum_{|\mathbf{r}| \le p} s^{\mathbf{pa}} \ln^q s (\operatorname{Re} F^{\mathbf{s}} \cos(\mathbf{br} \ln s) + \operatorname{Im} F^{\mathbf{s}} \sin(\mathbf{br} \ln s)) \\ &\quad\quad + s^{\alpha_*+2\varepsilon} G_*(s,f,w) \end{aligned}$$

and

$$B_{**}(H_w(s) + s^{\alpha_*+\varepsilon} f) = s^{\alpha_*+2\varepsilon} G_{**}(s,f,w).$$

Now define

$$G : [0,b[\times \overline{B_c(0)} \times B_a(v) \longrightarrow \mathbf{R}^n$$
$$G(s,f,w) = G_*(s,f,w) + G_{**}(s,f,w).$$

Notice that there exist integers

$$K_j^a \in \mathbf{N}, \quad a = (a_1, \dots, a_j) \in \mathbf{N}^j$$

such that

$$D^l(G_s \circ (f_s \times \mathrm{id})) = \sum_{j=1}^{l} \sum_{a_1+\cdots+a_j=l} K_j^a \\ D^j G_s \circ (D^{a_1}(f_s \times \mathrm{id}), \dots, D^{a_j}(f_s \times \mathrm{id})).$$

By shrinking $a, b, c > 0$ we can assume that

$$|D^j G_s(f, w)| < k_j$$
$$|G(s, f, w)| < C$$
$$(s, f, w) \in [0, b[\times B_c(0) \times B_a(v)$$

$j = 1, \dots, u + 1$. To ensure the contraction property of S, choose $d_1, \dots, d_u$ and $b > 0$ such that

$$\frac{1}{|\lambda_*|(\alpha_* + 2\varepsilon)} b^\varepsilon \sum_{j=1}^{l} \sum_{a_1+a_j=l} K_j^a k_j (d_{a_1} + 1) \cdots (d_{a_j} + 1) d_l \\ + ((\alpha_* + \delta)/|\lambda_*|) d_l/(\alpha_* + \varepsilon) < \mu_l d_l, \quad \mu_l \in]0, 1[$$

and

$$\frac{1}{|\lambda_*|(\alpha_* + 2\varepsilon)} b^\varepsilon \sum_{j=1}^{l} \sum_{a_1+\cdots+a_j=l} K_j^a (k_{j+1}(d_{a_1} + 1) \cdots (d_{a_j} + 1) \\ + k_j((d_{a_2} + 1) \cdots (d_{a_j} + 1) + \cdots + (d_{a_1} + 1) \cdots (d_{a_{j-1}} + 1))) \\ + ((\alpha_* + \delta)/|\lambda_*|)/(\alpha_* + \varepsilon) = \zeta_l \in]0, 1[$$
$$C b^\varepsilon/(|\lambda_*|(\alpha_* + 2\varepsilon)) + (\alpha_* + (\delta/|\lambda_*|)) c/(\alpha_* + \varepsilon) < c$$
$$k_1 b^\varepsilon/(|\lambda_*|(\alpha_* + 2\varepsilon)) + (\alpha_* + (\delta/|\lambda_*|))/(\alpha_* + \varepsilon) \le \zeta \in]\max\{\zeta_l, \mu_l\}, 1[\,.$$

We shall now compute expressions for the coordinates of $S(f)(t)$ and show that lots of terms will cancel out, due to our definition of the $\gamma^{\mathbf{s}}$. In fact, we shall show that coefficients to $t^{\mathbf{pa}}$ with $\mathbf{pa} \le \alpha_*$ vanish. We are thus able to show that

$$|S(f)(t)| < c$$

and afterwards that S is a contraction.

For $j = m_1 + \cdots + m_{\sigma-1} + 2i + 1$, $\sigma \in \{1, \ldots, k_1\} \cup \{k_2 + 1, \ldots, k_3\}$, $i \geq 1$, we find using Proposition 3.1 suppressing evaluation in w,

$$
\begin{aligned}
t^{\alpha_*+\varepsilon} S(f)_j(t) &= \int_0^t \frac{1}{\lambda_* s} A_j(H_w(s) + s^{\alpha_*+\varepsilon} f(s))\, ds - H_w^j(t) \\
&= \int_0^t \frac{1}{\lambda_* s} (B_{*j} + B_{**j} + L_j)(H_w(s) + s^{\alpha_*+\varepsilon} f(s))\, ds - H_w^j(t) \\
&= \int_0^t \frac{1}{\lambda_*} \Bigg(\sum_{q=0}^{np_*} \sum_{\mathbf{pa} \leq \alpha_*} \sum_{|\mathbf{r}| \leq p} s^{\mathbf{pa}-1} \ln^q s \\
&\quad \cdot (\operatorname{Re} F_j^{\mathbf{s}} \cos(\mathbf{br} \ln s) + \operatorname{Im} F_j^{\mathbf{s}} \sin(\mathbf{br} \ln s)) \\
&\quad + s^{\alpha_*+2\varepsilon-1} G_j(s, f(s), w) + (\lambda_\sigma/s) H_w^j(s) \\
&\quad + (\mu_\sigma/s) H_w^{j+1}(s) + (\delta/s) H_w^{j-2}(s) \\
&\quad + \lambda_\sigma s^{\alpha_*+\varepsilon-1} f_j(s) + \mu_\sigma s^{\alpha_*+\varepsilon-1} f_{j+1}(s) \\
&\quad + \delta s^{\alpha_*+\varepsilon-1} f_{j-2}(s) \Bigg)\, ds - H_w^j(t) \\
&= \frac{1}{\lambda_*} \sum_{q=0}^{np_*} \sum_{\mathbf{pa} \leq \alpha_*} \sum_{|\mathbf{br}| \leq p} \sum_{r=0}^{q} t^{\mathbf{pa}} \\
&\quad \cdot (\operatorname{Re} F_j^{\mathbf{s}} (\cos(\mathbf{br} \ln t) \operatorname{Re} \xi_{r,q} - \sin(\mathbf{br} \ln t) \operatorname{Im} \xi_{r,q}) \\
&\quad + \operatorname{Im} F_j^{\mathbf{s}} (\cos(\mathbf{br} \ln t) \operatorname{Im} \xi_{r,q} + \sin(\mathbf{br} \ln t) \operatorname{Re} \xi_{r,q})) \\
&\quad \cdot (-1)^{r-q} q!/(|a|^{2(q-r+1)r!}) \ln^r t \\
&\quad + \int_0^t \frac{1}{\lambda_*} (s^{\alpha_*+\varepsilon-1} G_j(s, f(s), w) + (\lambda_\sigma/s) H_w^j(s) \\
&\quad + (\mu_\sigma/s) H_w^{j+1}(s) + (\delta/s) H_w^{j-2}(s) \\
&\quad + (\lambda_\sigma s^{\alpha_*+\varepsilon-1} f_j(s) + \mu_\sigma s^{\alpha_*+\varepsilon-1} f_{j+1}(s) \\
&\quad + \delta s^{\alpha_*+\varepsilon-1} f_{j-2}(s))\, ds - H_w^j(t),
\end{aligned}
$$

which becomes

$$
\begin{aligned}
&\sum_{r=0}^{np_*} \sum_{\mathbf{pa} \leq \alpha_*} \sum_{|\mathbf{r}| \leq p} t^{\mathbf{pa}} \ln^r t \\
&\cdot \Bigg(\Bigg(\sum_{q=r}^{np_*} \Bigg(\frac{1}{\lambda_*} (\operatorname{Re} F_j^{\mathbf{s}} \operatorname{Re} \xi_{r,q} + \operatorname{Im} F_j^{\mathbf{s}} \operatorname{Im} \xi_{r,q})
\end{aligned}
$$

$$
\begin{aligned}
&+ (\alpha_\sigma \operatorname{Re}\gamma_j^{\mathbf{s}} + \beta_\sigma \operatorname{Re}\gamma_{j+1}^{\mathbf{s}} + (\delta/\lambda_*)\operatorname{Re}\gamma_{j-2}^{\mathbf{s}})\operatorname{Re}\xi_{r,q}\\
&+ (\alpha_\sigma \operatorname{Im}\gamma_j^{\mathbf{s}} + \beta_\sigma \operatorname{Im}\gamma_{j+1}^{\mathbf{s}} + (\delta/\lambda_*)\operatorname{Im}\gamma_{j-2}^{\mathbf{s}})\operatorname{Im}\xi_{r,q}\Big)\\
&\cdot (-1)^{q-r} q!/(r!|a|^{2(q-r+1)}) - \operatorname{Re}\gamma_j^{\mathbf{s}}\Big)\cos(\mathbf{br}\ln t)\\
&+ \Bigg(\sum_{q=r}^{np_*}\Bigg(\frac{1}{\lambda_*}(-\operatorname{Re}F_j^{\mathbf{s}}\operatorname{Im}\xi_{r,q} + \operatorname{Im}F_j^{\mathbf{s}}\operatorname{Re}\xi_{r,q})\\
&\quad - (\alpha_\sigma \operatorname{Re}\gamma_j^{\mathbf{s}} + \beta_\sigma \operatorname{Re}\gamma_{j+1}^{\mathbf{s}} + (\delta/\lambda_*)\operatorname{Re}\gamma_{j-2}^{\mathbf{s}})\operatorname{Im}\xi_{r,q}\\
&\quad + (\alpha_\sigma \operatorname{Im}\gamma_j^{\mathbf{s}} + \beta_\sigma \operatorname{Im}\gamma_{j+1}^{\mathbf{s}} + (\delta/\lambda_*)\operatorname{Im}\gamma_{j-2}^{\mathbf{s}})\operatorname{Re}\xi_{r,q}\Bigg)\\
&\quad \cdot (-1)^{q-r} q!/(r!|a|^{2(q-r+1)}) - \operatorname{Im}\gamma_j^{\mathbf{s}}\Bigg)\sin(\mathbf{br}\ln t)\Bigg)\\
&+ \int_0^t \frac{1}{\lambda_*}(s^{\alpha_*+2\varepsilon-1}G_j(s,f(s),w) + \lambda_\sigma s^{\alpha_*+\varepsilon-1}f_j(s)\\
&\qquad + \mu_\sigma s^{\alpha_*+\varepsilon-1}f_{j+1}(s) + \delta s^{\alpha_*+\varepsilon-1}f_{j-2}(s))\,ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
t^{\alpha_*+\varepsilon}S(f)_{j+1}(t) = &\int_0^t \frac{1}{\lambda_*}(s^{\alpha_*+2\varepsilon-1}G_{j+1}(s,f(s),w)\\
&\quad - \mu_\sigma s^{\alpha_*+\varepsilon-1}f_j(s) + \lambda_\sigma s^{\alpha_*+\varepsilon-1}f_{j+1}(s)\\
&\qquad + \delta s^{\alpha_*+\varepsilon-1}f_{j-1}(s))\,ds\\
&+ \sum_{r=0}^{np_*}\sum_{\mathbf{pa}\le\alpha_*}\sum_{|\mathbf{r}|\le p} t^{\mathbf{pa}}\ln^r\\
&\cdot t\Bigg(\Bigg(\sum_{q=r}^{np_*}\Bigg(\frac{1}{\lambda_*}(\operatorname{Re}F_{j+1}^{\mathbf{s}}\operatorname{Re}\xi_{r,q} + \operatorname{Im}F_{j+1}^{s}\operatorname{Im}\xi_{r,q})\\
&\quad + (-\beta_\sigma \operatorname{Re}\gamma_j^{\mathbf{s}} + \alpha_\sigma \operatorname{Re}\gamma_{j+1}^{\mathbf{s}} + (\delta/\lambda_*)\operatorname{Re}\gamma_{j-1}^{\mathbf{p},q\mathbf{r}})\operatorname{Re}\xi_{r,q}\\
&\quad + (-\beta_\sigma \operatorname{Im}\gamma_j^{\mathbf{s}} + \alpha_\sigma \operatorname{Im}\gamma_{j+1}^{\mathbf{s}} + (\delta/\lambda_*)\operatorname{Im}\gamma_{j-1}^{\mathbf{s}})\operatorname{Im}\xi_{r,q}\Bigg)\\
&\cdot (-1)^{q-r} q!/(r!|a|^{2(q-r+1)}) - \operatorname{Re}\gamma_{j+1}^{\mathbf{s}}\Bigg)\cos(\mathbf{br}\ln t)
\end{aligned}
$$

$$+\Bigg(\sum_{q=r}^{np_*}\bigg(\frac{1}{\lambda_*}(-\mathrm{Re}\,F^{\mathbf{s}}_{j+1}\mathrm{Im}\,\xi_{r,q}+\mathrm{Im}\,F^{\mathbf{s}}_{j+1}\mathrm{Re}\,\xi_{r,q})$$
$$-(-\beta_\sigma\mathrm{Re}\,\gamma^{\mathbf{s}}_j+\alpha_\sigma\mathrm{Re}\,\gamma^{\mathbf{s}}_{j+1}+(\delta/\lambda_*)\mathrm{Re}\,\gamma^{\mathbf{s}}_{j-1})\mathrm{Im}\,\xi_{r,q}$$
$$+(-\beta_\sigma\mathrm{Im}\,\gamma^{\mathbf{s}}_j+\alpha_\sigma\mathrm{Im}\,\gamma^{\mathbf{s}}_{j+1}+(\delta/\lambda_*)\mathrm{Im}\,\gamma^{\mathbf{s}}_{j-1})\mathrm{Re}\,\xi_{r,q}\bigg)$$
$$\cdot(-1)^{q-r}q!/(r!|a|^{2(q-r+1)})-\mathrm{Im}\,\gamma^{\mathbf{s}}_{j+1}\Bigg)\sin(\mathbf{br}\ln t)\Bigg).$$

Using the definition of the $\gamma^{\mathbf{s}}$ we see that

(3.9)

$$t^{\alpha_*+\varepsilon}S(f)_j(t)=\int_0^t\frac{1}{\lambda_*}(s^{\alpha_*+2\varepsilon-1}G_j(s,f(s),w)+\lambda_\sigma s^{\alpha_*+\varepsilon-1}f_j(s)$$
$$+\mu_\sigma s^{\alpha_*+\varepsilon-1}f_{j+1}(s)+\delta s^{\alpha_*+\varepsilon-1}f_{j-2}(s))\,ds$$
$$t^{\alpha_*+\varepsilon}S(f)_{j+1}(t)=\int_0^t\frac{1}{\lambda_*}(s^{\alpha_*+2\varepsilon-1}G_j(s,f(s),w)-\mu_\sigma s^{\alpha_*+\varepsilon-1}f_j(s)$$
$$+\lambda_\sigma s^{\alpha_*+\varepsilon-1}f_{j+1}(s)+\delta s^{\alpha_*+\varepsilon-1}f_{j-1}(s))\,ds.$$

To see this when $j=m_1+\cdots+m_{\sigma-1}+1$, $\sigma\in\{1,\ldots,k_1\}$ and $\mathbf{p}=(\mathbf{r},0)=e_\sigma$, $r=0$ in the summation above, compute

$$\sum_{q=r}^{np_*}((\alpha_\sigma\mathrm{Re}\,\gamma^{\mathbf{s}}_j+\beta_\sigma\mathrm{Re}\,\gamma^{\mathbf{s}}_{j+1})\mathrm{Re}\,\xi_{r,q}$$
$$+(\alpha_\sigma\mathrm{Im}\,\gamma^{\mathbf{s}}_j+\beta_\sigma\mathrm{Im}\,\gamma^{\mathbf{s}}_{j+1})\mathrm{Im}\,\xi_{r,q})(-1)^{q-r}q!/(r!|a|^{2(q-r+1)}-\mathrm{Re}\,\gamma^{\mathbf{s}}_j$$
$$=\frac{1}{|a|^2}((\alpha_\sigma v_j+\beta_\sigma v_{j+1})\alpha_\sigma+(\alpha_\sigma v_{j+1}-\beta_\sigma v_j)(-\beta_\sigma))-v_j=0.$$

The coefficient to $\sin(\mathbf{br}\ln t)$ in $t^{\alpha_*+\varepsilon}S(f)_j(t)$ and the coefficients to $\cos(\mathbf{br}\ln t)$ and $\sin(\mathbf{br}\ln t)$ in $t^{\alpha_*+\varepsilon}S(f)_{j+1}(t)$ give three similar equations.

In the case that $j=m_1+\cdots+m_{\sigma-1}+2i+1$, $\sigma\in\{1,\ldots,k_1\}$, $i\geq 1$

and $\mathbf{p} = (\mathbf{r}, 0) = e_\sigma$, $0 \le 2r \le j - 1 - (m_1 + \cdots m_{\sigma-1})$ we find

$$\begin{aligned}
&\sum_{q=r}^{np_*} (-1)^{q-r} q!/(r!|a|^{2(q-r+1)}) \\
&\qquad \cdot (\alpha_\sigma \operatorname{Re}\gamma_j^{\mathbf{s}} + \beta_\sigma \operatorname{Re}\gamma_{j+1}^{\mathbf{s}} + (\delta/\lambda_*)\operatorname{Re}\gamma_{j-2}^{\mathbf{s}})\operatorname{Re}\xi_{r,q} \\
&\qquad + (\alpha_\sigma \operatorname{Im}\gamma_j^{\mathbf{s}} + \beta_\sigma \operatorname{Im}\gamma_{j+1}^{\mathbf{s}} + (\delta/\lambda_*)\operatorname{Im}\gamma_{j-2}^{\mathbf{s}})\operatorname{Im}\xi_{r,q} - \operatorname{Re}\gamma_j^{\mathbf{s}} \\
&\quad = \sum_{q=r+1}^{(j-1-(m_1+\cdots+m_{s-1}))/2} (-1)^{q-r}/(r!|a|^{2(q-r+1)}) \\
&\qquad \cdot (\delta/\lambda_*)^q (v_{j-2q}(\alpha_\sigma \operatorname{Re}\xi_{r,q} - \beta_\sigma \operatorname{Im}\xi_{r,q}) \\
&\qquad + v_{j+1-2q}(\beta_\sigma \operatorname{Re}\xi_{r,q} + \alpha_\sigma \operatorname{Im}\xi_{r,q}) \\
&\qquad - |a|^2 (v_{j-2q}\operatorname{Re}\xi_{r,q-1} + v_{j+1-2q}\operatorname{Im}\xi_{r,q-1})) = 0.
\end{aligned}$$

We have used that

$$\begin{aligned}
\alpha_\sigma \operatorname{Re}\xi_{r,q} - \beta_\sigma \operatorname{Im}\xi_{r,q} &= \operatorname{Re}\xi_{r,q-1}|a|^2 \\
\beta_\sigma \operatorname{Re}\xi_{r,q} + \alpha_\sigma \operatorname{Im}\xi_{r,q} &= \operatorname{Im}\xi_{r,q-1}|a|^2.
\end{aligned}$$

To verify the first of these equations, observe that

$$\begin{aligned}
\operatorname{Re}\xi_{r,q} &= \sum_{2p=0}^{q-r+1} \binom{q-r+1}{2p} \alpha_\sigma^{q-r+1-2p}(-1)^p \beta_\sigma^{2p} \\
\operatorname{Im}\xi_{r,q} &= -\sum_{2p+1=1}^{q-r+1} \binom{q-r+1}{2p+1} \alpha_\sigma^{q-r-2p}(-1)^p \beta_\sigma^{2p+1}.
\end{aligned}$$

Now for $q - r$ even we find

$$\begin{aligned}
&(\alpha_s^2 + \beta_s^2)\operatorname{Re}\xi_{r,q-1} \\
&\quad = \sum_{2p=0}^{q-r} \binom{q-r}{2p} \alpha_\sigma^{q-r+2-2p}(-1)^p \beta_\sigma^{2p} \\
&\qquad + \sum_{2p=0}^{q-r} \binom{q-r}{2p} \alpha_\sigma^{q-r-2p}(-1)^p \beta_\sigma^{2p+2} \\
&\quad = \alpha_\sigma^{q-r+2} + (-1)^{(q-r/2)} \beta_\sigma^{q-r+2}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{p=2}^{q-r}\left(\binom{q-r}{2p}-\binom{q-r}{2p-2}\right)\alpha_\sigma^{q-r+2-2p}(-1)^p\beta_\sigma^{2p}\\
&=\alpha_\sigma^{q-r+2}+(-1)^{(q-r/2)}\beta_\sigma^{q-r+2}\\
&+\sum_{2p=2}^{q-r}\left(\binom{q-r+1}{2p}-\binom{q-r+1}{2p-1}\right)\alpha_\sigma^{q-r+2-2p}(-1)^p\beta_\sigma^{2p}\\
&=\alpha_\sigma \mathrm{Re}\,\xi_{r,q}-\beta_\sigma \mathrm{Im}\,\xi_{r,q}.
\end{aligned}
$$

The verification of the case $q-r$ odd is similar and so is the verification of the second equation. The coefficient to $\sin(\mathbf{br}\ln t)$ in $t^{\alpha_*+\varepsilon}S(f)_j(t)$ and the coefficients to $\cos(\mathbf{br}\ln t)$ and $\sin(\mathbf{br}\ln t)$ in $t^{\alpha_*+\varepsilon}S(f)_{j+1}(t)$ give three similar verifications.

For $j=m_1+\cdots+m_\sigma+i$, $i\geq 2$, $\sigma\in\{k_1+1,\dots,k_2\}\cup\{k_3+1,\dots,k\}$ we compute

$$
\begin{aligned}
t^{\alpha_*+\varepsilon}S(f)_j(t)=&\int_0^t\frac{1}{\lambda_*}(s^{\alpha_*+2\varepsilon-1}G_j(s,f(s),w)\\
&\qquad+\lambda_\sigma \mathbf{s}^{\alpha_*+\varepsilon-1}f_j(s)+\delta s^{\alpha_*+\varepsilon-1}f_{j-1}(s))\,ds\\
&+\sum_{r=0}^{np_*}\sum_{\mathbf{pa}\leq\alpha_*}\sum_{|\mathbf{r}|\leq p}t^{\mathbf{pa}}\ln^r t\\
&\cdot\Bigg(\Bigg(\sum_{q=r}^{np_*}\Bigg(\frac{1}{\lambda_*}(\mathrm{Re}\,F_j^{\mathbf{s}}\mathrm{Re}\,\xi_{r,q}+\mathrm{Im}\,F_j^{\mathbf{s}}\mathrm{Im}\,\xi_{r,q})\\
&\qquad+(\alpha_\sigma\mathrm{Re}\,\gamma_j^{\mathbf{s}}+(\delta/\lambda_*)\mathrm{Re}\,\gamma_{j-1}^{\mathbf{s}})\mathrm{Re}\,\xi_{r,q}\\
&\qquad+(\alpha_\sigma\mathrm{Im}\,\gamma_j^{\mathbf{s}}+(\delta/\lambda_*)\mathrm{Im}\,\gamma_{j-1}^{\mathbf{s}})\mathrm{Im}\,\xi_{r,q}\Bigg)\\
&\quad\cdot(-1)^{q-r}q!/(|a|^{2(q-r+1)}r!)-\mathrm{Re}\,\gamma_j^{\mathbf{s}}\Bigg)\cos(\mathbf{br}\ln t)\\
&+\Bigg(\sum_{q=r}^{np_*}\Bigg(\frac{1}{\lambda_*}(-\mathrm{Re}\,F_j^{\mathbf{s}}\mathrm{Im}\,\xi_{r,q}+\mathrm{Im}\,F_j^{\mathbf{s}}\mathrm{Re}\,\xi_{r,q})\\
&\qquad-(\alpha_\sigma\mathrm{Re}\,\gamma_j^{\mathbf{s}}+(\delta/\lambda_*)\mathrm{Re}\,\gamma_{j-1}^{\mathbf{s}})\mathrm{Im}\,\xi_{r,q}\\
&\qquad+(\alpha_\sigma\mathrm{Im}\,\gamma_j^{\mathbf{s}}+(\delta/\lambda_*)\mathrm{Im}\,\gamma_{j-1}^{\mathbf{s}})\mathrm{Re}\,\xi_{r,q}\Bigg)
\end{aligned}
$$

$$\cdot (-1)^{q-r} q!/(|a|^{2(q-r+1)} r!) - \operatorname{Im} \gamma_j^{\mathbf{p},r,\mathbf{r}}\Big) \sin(\mathbf{br} \ln t)\Big).$$

For $j = m_1 + \cdots + m_{s-1} + 1$, the expression for $t^{\alpha_* + \varepsilon} S(f)_j(t)$ is given by the above with $\delta = 0$. Using the definition of the $\gamma^{\mathbf{s}}$ we see that

$$t^{\alpha_*+\varepsilon} S(f)_j(t) = \int_0^t \frac{1}{\lambda_*}(s^{\alpha_*+2\varepsilon-1} G_j(s, f(s), w) \\ + \lambda_\sigma s^{\alpha_*+\varepsilon-1} f_j(s) + \delta s^{\alpha_*+\varepsilon-1} f_{j-1}(s))\, ds.$$

For $j = m_1 + \cdots + m_{\sigma-1}$ the expression for

$$t^{\alpha_*+\varepsilon} S(f)_j(t)$$

is given by the above with $\delta = 0$. Introducing the change of variables $s^{\alpha_*+\varepsilon} = v$ in (3.9) and (3.10), it follows that

$$S(f)_j(t) \longrightarrow 0$$

as $t \to 0$. So $S(f)$ is a continuous map from $[0, b[\times B_a(v)$ to $\mathbf{R}^n$. From (3.9) we deduce

$$\begin{aligned} |S(f)_j(t)| &\leq \frac{1}{t^{\alpha_*+\varepsilon}|\lambda_*|} \int_0^t (s^{\alpha_*+2\varepsilon-1} C \\ &\qquad + (|\lambda_s| + |\mu_s|) s^{\alpha_*+\varepsilon-1} c + \delta s^{\alpha_*+\varepsilon-1} c)\, ds \\ &\leq \frac{1}{t^{\alpha_*+\varepsilon}} \bigg(\frac{C}{(\alpha_* + 2\varepsilon)|\lambda_*|} t^{\alpha_*+2\varepsilon} \\ &\qquad + \frac{\alpha_*}{\alpha_* + \varepsilon} t^{\alpha_*+\varepsilon} c + \delta/(|\lambda_*|(\alpha_* + \varepsilon)) t^{\alpha_*+\varepsilon} c \bigg) < c. \end{aligned}$$

Similarly (3.10) produces $|S(f)_j(t)| < c$. So

$$S(f) : [0, b[\times B_a(v) \longrightarrow B_c(0).$$

From the definition of $S(f)$ it is clear that $S(f)_t$ is C^u for all $t \in [0, b[$ and for all $j \in \{1, \ldots, u\}$,

$$(t, w) \longmapsto D_2^j S(f)(t, w)$$

is continuous at (t,w) with $t \neq 0$. Introducing the change of variables $s^{\alpha_*+\varepsilon} = v$ in (3.9) and (3.10), it follows that this is true for $t=0$ too.

From the first equation in (3.9) we compute for $t \in]0,b[$,

$$\begin{aligned}
|D_2^l S(f)_j(t,w)| = \frac{1}{t^{\alpha_*+\varepsilon}} \bigg| \int_0^t \frac{1}{\lambda_*} \bigg(& s^{\alpha_*+2\varepsilon-1} \sum_{j=1}^{l} \sum_{a_1+\cdots+a_j=l} \\
& \cdot K_j^a D_2^j G(D_2^{a_1}(f\times \mathrm{id}), \dots, D_2^{a_j}(f \times \mathrm{id}) \bigg) \\
& + s^{\alpha_*+\varepsilon-1}(\delta D_2^l f_{j-2}(s,w) \\
& + \lambda_\sigma D_2^l f_j(s,w) + \mu_\sigma D_2^l f_{j+1}(s,w))\, ds \bigg| \\
\le \frac{1}{t^{\alpha_*+\varepsilon}} \int_0^t \frac{1}{|\lambda_*|} & s^{\alpha_*+2\varepsilon-1} \sum_{j=1}^{l} \sum_{a_1+\cdots+a_j=l} \\
& \cdot K_j^a k_j (d_{a_1}+1)\cdots(d_{a_j}+1) \\
& + s^{\alpha_*+\varepsilon-1}(\delta/|\lambda_*| d_l + |\alpha_\sigma| d_l + |\beta_\sigma|\, d_l)\, ds \\
\le \frac{1}{|\lambda_*|(\alpha_*+2\varepsilon)} b^\varepsilon & \sum_{j=1}^{l} \sum_{a_1+\cdots+a_j=l} \\
& \cdot K_j^a k_j (d_{a_1}+1)\cdots(d_{a_j}+1) \\
& + ((\delta/|\lambda_*|) + \alpha_*) d_l/(\alpha_*+\varepsilon) < \mu_l d_l.
\end{aligned}$$

The second equation in (3.9) and (3.10) also produces

$$|D_2^l S(f)_j(t,w)| < \mu_l d_l, \quad l = 1, \dots, u.$$

We deduce that

$$S : E_u \longrightarrow E_u$$

is a mapping from a complete metric space to itself. (3.9) gives us for $f_1, f_2 \in E_u$ that

$$\begin{aligned}
&|S(f_1)_j(t,w) - S(f_2)_j(t,w)| \\
&\quad \le \frac{1}{t^{\alpha_*+\varepsilon}|\lambda_*|} \int_0^t (s^{\alpha_*+2\varepsilon-1} |G_j(s,f_1(s,w),w) - G_j(s,f_2(s,w),w)| \\
&\qquad + s^{\alpha_*+\varepsilon-1}(\delta + |\lambda_\sigma| + |\mu_\sigma|)|f_1^j(s,w) - f_2^j(s,w)|)\, ds
\end{aligned}$$

$$\le (k_1 b^\varepsilon/(|\lambda_*|(\alpha_* + 2\varepsilon)) + (\alpha_* + (\delta/|\lambda_*|))/(\alpha_* + \varepsilon))d(f_1, f_2)$$
$$\le \zeta d(f_1, f_2)$$
$$d(f_1, f_2) = \max_{j=0,\dots,u} \cdot \sup_{\substack{t\in[0,b[\\ w\in B_a(v)}} \{|D_2^j f_1(t,w) - D_2^j f_2(t,w)|\}.$$

(3.10) likewise produces

$$|S(f_1)_j(t,w) - S(f_2)_j(t,w)| \le \zeta d(f_1, f_2).$$

(3.9) also gives

$$\begin{aligned}
&|D_2^l S(f_1)_j(t,w) - D_2^l S(f_2)_j(t,w)| \\
&\quad\le \frac{1}{t^{\alpha_*+\varepsilon}|\lambda_*|}\int_0^t s^{\alpha_*+2\varepsilon-1} \\
&\qquad \Bigg(\sum_{j=1}^{l}\sum_{a_1+\alpha_j=l} K_j^a |D_2^j G_{(s,f_1(s,w),w)}(D_2^{a_1}(f_1\times \mathrm{id}),\dots,D_2^{a_j}(f_1\times\mathrm{id})) \\
&\qquad\quad - D_2^j G_{(2,f_2(s,w),w)}(D_2^{a_1}(f_2\times\mathrm{id}),\dots,D_2^{a_j}(f_2\times\mathrm{id})))| \\
&\qquad\quad + s^{\alpha_*+\varepsilon-1}(\delta + |\lambda_\sigma| + |\mu_\sigma|)d(f_1,f_2)\Bigg)\,ds \\
&\quad\le \frac{1}{t^{\alpha_*+\varepsilon}}\int_0^t s^{\alpha_*+2\varepsilon-1}\frac{1}{|\lambda_*|}\sum_{j=1}^{l} \\
&\qquad\qquad \cdot \sum_{a_1+\cdots+a_j=l} K_j^a (k_{j+1} d(f_1,f_2)(d_{a_1}+1)\cdots(d_{a_j}+1) \\
&\qquad\qquad + k_j d(f_1,f_2)((d_{a_2}+1)\cdots(d_{a_j}+1)+\cdots \\
&\qquad\qquad + (d_{a_1}+1)\cdots(d_{a_{j-1}}+1))) \\
&\qquad\qquad + s^{\alpha_*+\varepsilon-1}((\delta/|\lambda_*|)+\alpha_*)d(f_1,f_2)\,ds \\
&\quad\le \Bigg(\frac{1}{|\lambda_*|(\alpha_*+2\varepsilon)} b^\varepsilon \sum_{j=1}^{l}\sum_{a_1+\cdots+a_j=l} K_j^a (k_{j+1}(d_{a_1}+1)\cdots(d_{a_j}+1) \\
&\qquad\qquad + k_j((d_{a_2}+1)\cdots(d_{a_j}+1)+\cdots \\
&\qquad\qquad + (d_{a_1}+1)\cdots(d_{a_{j-1}}+1))) \\
&\qquad\qquad + ((\delta/|\lambda_*|)+\alpha_*)/(\alpha_*+\varepsilon)\Bigg)d(f_1,f_2)
\end{aligned}$$

$\leq \zeta d(f_1, f_2).$

Verify by similar computations that (3.10) gives

$$|D_2^l S(f_1)_j(t,w) - D_2^l S(f_2)_j(t,w)| \leq \zeta d(f_1, f_2).$$

We deduce that

$$d(S(f_1), S(f_2)) \leq \zeta d(f_1, f_2)$$

so S is a contraction with a unique fixed point f. Define

$$x(t) = H_v(t) + t^{\alpha_* + \varepsilon} f(t,v), \quad t \in [0, b[\,.$$

Since f is a fixed point of S, we have

$$x(t) = \int_0^t \frac{1}{\lambda_* s} A(x(s))\, ds.$$

We deduce that

$$\frac{d}{ds}(x(\exp(\lambda_* s))) = A(x(\exp(\lambda_* s))),$$

hence $s \mapsto x(\exp(\lambda_* s))$, $s \in (1/\lambda_*) \ln(]0, b[)$ is an integral curve for A. We have proven the existence and uniqueness in (1).

To prove (2) we need some important lemmas, computing the $\gamma^{\mathbf{s}}$.

Lemma 3.4. *When $j \in I_s$, $s \in \{k_1 + 1, \ldots, k_2\} \cup \{k_3 + 1, \ldots, k\}$ and $\alpha_3 \neq \mathbf{pa} - i\mathbf{br}$, then*

$$\gamma_j^{\mathbf{p},r,\mathbf{r}} = \frac{1}{\lambda_*} \sum_{q=r}^{np_*} F_j^{\mathbf{s}} (-1)^{q-r} q!/(r!(a - \alpha_s)^{q-r+1})$$

$$j = m_1 + \cdots + m_{s-1} + 1$$

$$\gamma_j^{\mathbf{p},r,\mathbf{r}} = \frac{1}{\lambda_*} \sum_{q=r}^{np_*} (F_j^{\mathbf{s}} + \delta\gamma_{j-1}^{\mathbf{s}})(-1)^{q-r} q!/(r!(a - \alpha_s)^{q-r+1})$$

$$j \neq m_1 + \cdots + m_{s-1} + 1,$$

where again $a = \mathbf{pa} - i\mathbf{br}$.

Proof. $r = np_*$ and the definition of $\gamma_j^{\mathbf{s}}$ gives

$$\gamma_j^{\mathbf{p},np_*,\mathbf{r}} = \frac{1}{\lambda_*(a-\alpha_s)} F_j^{\mathbf{p},np_*,\mathbf{r}},$$

when $j = m_1 + \cdots + m_{s-1} + 1$. We can then proceed by induction to compute

$$\begin{aligned}
\gamma_j^{\mathbf{p},r,\mathbf{r}} &= \frac{1}{(a-\alpha_s)} \sum_{q=r+1}^{np_*} \left\{ \frac{1}{\lambda_*} F_j^{\mathbf{s}} + \alpha_s \gamma_j^{\mathbf{s}} \right\} \\
&\qquad\qquad \cdot (-1)^{q-r} q!/(r! a^{q-r}) + \frac{1}{\lambda_*(a-\alpha_s)} F_j^{\mathbf{p},r,\mathbf{r}} \\
&= \frac{1}{(a-\alpha_s)} \sum_{q=r+1}^{np_*} \frac{1}{\lambda_*} F_j^{\mathbf{s}} (-1)^{q-r} q!/(r! a^{q-r}) \\
&\quad + \frac{\alpha_s}{\lambda_*} \sum_{q=r+1}^{np_*} \sum_{u=q}^{np_*} F_j^{\mathbf{p},u,\mathbf{r}} \\
&\quad \cdot (-1)^{u-r} u!/(r!(a-\alpha_s)^{u-q+1} a^{q-r}) + \frac{1}{\lambda_*(a-\alpha_s)} F_j^{\mathbf{p},r,\mathbf{r}} \\
&= \frac{1}{(a-\alpha_s)} \sum_{q=r+1}^{np_*} \frac{1}{\lambda_*} F_j^{\mathbf{s}} (-1)^{q-r} q!/(r! a^{q-r}) \\
&\quad + \frac{\alpha_s}{\lambda_*} \sum_{q=r+1}^{np_*} \sum_{u=r+1}^{q} F_j^{\mathbf{s}} \\
&\quad \cdot (-1)^{q-r} q!/(r!(a-\alpha_s)^{q-u+1} a^{u-r}) + \frac{1}{\lambda_*(a-\alpha_s)} F_j^{\mathbf{p},r,\mathbf{r}} \\
&= \frac{1}{(a-\alpha_s)} \sum_{q=r+1}^{np_*} \frac{q!}{r!\lambda_*} F_j^{\mathbf{s}} \\
&\quad \cdot (-1)^{q-r} \left(\sum_{u=r+1}^{q} \frac{\alpha_s}{(a-\alpha_s)^{q-u+1} a^{u-r}} + \frac{1}{a^{q-r}} \right) \\
&\quad + \frac{1}{\lambda_*(a-\alpha_s)} F_j^{\mathbf{p},r,\mathbf{r}} \\
&= \sum_{q=r}^{np_*} \frac{q!}{r!\lambda_*} F_j^{\mathbf{s}} (-1)^{q-r} / (a-\alpha_s)^{q-r+1}.
\end{aligned}$$

We have used that

$$\sum_{u=r+1}^{q} \frac{\alpha_s}{(a-\alpha_s)^{q-u+1}a^{u-r}} + \frac{1}{a^{q-r}}$$
$$= \frac{1}{(a-\alpha_s)} \sum_{u=r+1}^{q-1} \frac{\alpha_s}{(a-\alpha_s)^{q-u}a^{u-r}} + \frac{a}{(a-\alpha_s)a^{q-r}} = \frac{1}{(a-\alpha_s)^{q-r}},$$

and the lemma follows.

Similarly one can prove

Lemma 3.5. *When* $j = m_1 + \cdots + m_{s-1} + 2i + 1 \in I_s$, $s \in \{1, \dots, k_1\} \cup \{k_2+1, \dots, k_3\}$ *and* $\mathbf{pa} - i\mathbf{br} \neq \alpha_s \pm i\beta_s$, *then*

$$\begin{pmatrix} \gamma_j^{\mathbf{p},r,\mathbf{r}} \\ \gamma_{j+1}^{\mathbf{p},r,\mathbf{r}} \end{pmatrix} = \frac{1}{\lambda_*} \sum_{q=r}^{np_*} (-1)^{q-r} q!/r!((a-\alpha_s)^2 + \beta_s^2)^{q-r+1}$$
$$\cdot \begin{pmatrix} a-\alpha_s & \beta_s \\ -\beta_s & a-\alpha_s \end{pmatrix}^{q-r+1} \begin{pmatrix} F_j^{\mathbf{s}} \\ F_{j+1}^{\mathbf{s}} \end{pmatrix},$$
$$j = m_1 + \cdots + m_{s-1} + 1$$
$$\begin{pmatrix} \gamma_j^{\mathbf{p},r,\mathbf{r}} \\ \gamma_{j+1}^{\mathbf{p},r,\mathbf{r}} \end{pmatrix} = \frac{1}{\lambda_*} \sum_{q=r}^{np_*} (-1)^{q-r} q!/r!((a-\alpha_s)^2 + \beta_s^2)^{q-r+1}$$
$$\cdot \begin{pmatrix} a-\alpha_s & \beta_s \\ -\beta_s & a-\alpha_s \end{pmatrix}^{q-r+1} \begin{pmatrix} F_j^{\mathbf{s}} + \delta\gamma_{j-2}^{\mathbf{s}} \\ F_{j+1}^{\mathbf{s}} + \delta\gamma_{j-1}^{\mathbf{s}} \end{pmatrix},$$
$$j \neq m_1 + \cdots + m_{s-1} + 1.$$

We claim that for $k > 0$

$$H_w(t) = H_v(kt)$$
$$w = \exp\left(L\frac{1}{\lambda_*}\ln k\right)v. \tag{3.11}$$

We have

$$H_v(t) = \sum_{\mathbf{pa}\leq\alpha_*} \sum_{q=0}^{np_*} \sum_{|\mathbf{r}|\leq p} t^{\mathbf{pa}} \ln^q t(\operatorname{Re}\gamma^{\mathbf{s}} \cos(\mathbf{br}\ln t)$$
$$+ \operatorname{Im}\gamma^{\mathbf{s}} \sin(\mathbf{br}\ln t)).$$

Compute

$$H_v(kt) = \sum_{r=0}^{np_*} \sum_{\mathbf{pa} \leq \alpha_*} \sum_{|\mathbf{r}| \leq p} t^{\mathbf{pa}} \ln^r t (\operatorname{Re} \beta^{\mathbf{p},r,\mathbf{r}} \cos(\mathbf{br} \ln t) + \operatorname{Im} \beta^{\mathbf{p},r,\mathbf{r}} \sin(\mathbf{br} \ln t)),$$

where

$$\beta^{\mathbf{p},r,\mathbf{r}} = \sum_{m=0}^{np_*-r} k^{\mathbf{pa}} \frac{(m+r)!}{r!m!} \ln^m k \gamma^{\mathbf{p},m+r,\mathbf{r}} \exp(-i\mathbf{br} \ln k). \tag{3.12}$$

For $\mathbf{p} = (\mathbf{r}, 0) = e_s$, $2j+1 \in I_s$, $s \in \{1, \dots, k_1\}$, we find

$$\begin{aligned} \operatorname{Re} \beta^{\mathbf{p},r,\mathbf{r}}_{2j+1} &= \frac{1}{r!} \left(\frac{\delta}{\lambda_*} \right)^r \sum_{2m=0}^{2j-(m_1+\cdots+m_{s-1})-2r} k^{\mathbf{pa}} \frac{1}{m!} \ln^m k \left(\frac{\delta}{\lambda_*} \right)^m \\ &\quad \cdot (v_{2j+1-2(m+r)} \cos(\mathbf{br} \ln k) + v_{2j+2-2(m+r)} \sin(\mathbf{br} \ln k)) \\ &= \frac{1}{r!} \left(\frac{\delta}{\lambda_*} \right)^r \left(\exp \left(L \frac{1}{\lambda_*} \ln k \right) v \right)_{2j+1-2r}. \end{aligned}$$

This and three similar computations show that the $\beta^{\mathbf{p},r,\mathbf{r}}$, $|\mathbf{p}| = 1$ have the values $\eta^{\mathbf{p},r,\mathbf{r}}$ that the definition procedure in the beginning of this chapter would assign them starting from

$$w = \exp \left(L \frac{1}{\lambda_*} \ln k \right) v.$$

Assume that

$$\eta^{\mathbf{p},r,\mathbf{r}} = \beta^{\mathbf{p},r,\mathbf{r}} \tag{3.13}$$

for all $\mathbf{p}$ with $|\mathbf{p}| \leq l < p$. We have shown that this is true for $l = 1$. To prove it for $|\mathbf{p}| \leq l+1$ let $j = m_1 + \cdots + m_{s-1} + 1$, $s \in \{k_1+1, \dots, k_2\} \cup \{k_3+1, \dots, k\}$. Then we find using Lemma 3.4,

$$\begin{aligned} &\beta^{\mathbf{p},r,\mathbf{r}}_j \\ &= \sum_{m=0}^{np_*-r} k^{\mathbf{pa}} \frac{(m+r)!}{r!m!} \frac{1}{\lambda_*} \sum_{q=m+r}^{np_*} \sum_{k=2}^{l} \sum_{\mathbf{r}_1 \pm \cdots \pm \mathbf{r}_k = \mathbf{r}} \sum_{\mathbf{p}_1 + \cdots + \mathbf{p}_k = \mathbf{p}} \sum_{q_1 + \cdots + q_k = q} \end{aligned}$$

$$\begin{aligned}
&\cdot B^j_{j_1\cdots j_k}|\gamma^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}|\cdots|\gamma^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k}|\\
&\cdot(\cos(\zeta^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}\pm\cdots\pm\zeta^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k})-i\sin(\zeta^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}\pm\cdots\pm\zeta^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k}))\\
&\cdot\frac{1}{2^{k-1}}\exp(-i\mathbf{b}\mathbf{r}\ln k)\frac{q!}{(m+r)!}(-1)^{q-(m+r)}\frac{1}{(a-\alpha_s)^{q-(m+r)+1}}\\
=&\frac{k^{\mathbf{p}\mathbf{a}}}{\lambda_*}\sum_{q=r}^{np_*}\sum_{m=0}^{np_*-q}\frac{(q+m)!}{q!m!}\ln^m k\\
&\cdot\sum_{k=2}^{l}\sum_{\mathbf{r}_1\pm\cdots\pm\mathbf{r}_k=\mathbf{r}}\sum_{\mathbf{p}_1+\cdots+\mathbf{p}_k=\mathbf{p}}\sum_{q_1+\cdots+q_k=q+m}\\
&\cdot B^j_{j_1\cdots j_k}\gamma^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}\cdots i^{\pm}(\gamma^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k})\\
&\cdot\frac{1}{2^{k-1}}\exp(-i\mathbf{b}\mathbf{r}\ln k)\frac{q!}{r!}(-1)^{q-r}\frac{1}{(a-\alpha_s)^{q-r+1}},
\end{aligned}$$

where i^+ is the identity and i^- is conjugation. Now by Lemma 3.4 and the induction hypothesis

$$\begin{aligned}
\eta^{\mathbf{p},r,\mathbf{r}}_j=&\frac{1}{\lambda_*}\sum_{q=r}^{np_*}\sum_{k=2}^{l}\sum_{q_1+\cdots+q_k=q}\sum_{\mathbf{p}_1+\cdots+\mathbf{p}_k=\mathbf{p}}\sum_{\mathbf{r}_1\pm\cdots\pm\mathbf{r}_k=\mathbf{r}}\\
&\cdot B^j_{j_1\cdots j_k}\eta^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}\cdots i^{\pm}(\eta^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k})\frac{1}{2^{k-1}}\frac{q!}{r!}(-1)^{q-r}\frac{1}{(a-\alpha_s)^{q-r+1}}\\
=&\frac{k^{\mathbf{p}\mathbf{a}}}{\lambda_*}\sum_{q=r}^{np_*}\sum_{k=2}^{l}\sum_{q_1+\cdots+q_k=q}\sum_{\mathbf{p}_1+\cdots+\mathbf{p}_k=\mathbf{p}}\sum_{\mathbf{r}_1\pm\cdots\pm\mathbf{r}_k=r}B^j_{j_1\cdots j_k}\frac{1}{2^{k-1}}\\
&\cdot\sum_{m_1=0}^{np_*-q_1}\frac{(m_1+q_1)!}{q_1!m_1!}\gamma^{\mathbf{p}_1,m_1+q_1,\mathbf{r}_1}_{j_1}\exp(-i\mathbf{b}\mathbf{r}_1\ln k)\\
&\cdot\sum_{m_k=0}^{np_*-q_k}\frac{(m_k+q_k)!}{q_k!m_k!}i^{\pm}(\gamma^{\mathbf{p}_k,m_k+q_k,\mathbf{r}_k}_{j_k}\exp(-i\mathbf{b}\mathbf{r}_k\ln k))\\
&\cdot\ln^{m_1+\cdots+m_k}k\frac{q!}{r!}(-1)^{q-r}\frac{1}{(a-\alpha_s)^{q-r+1}}.
\end{aligned}$$

Introducing the change of variable $m_k=m-(m_1+\cdots+m_{k_1})$ we find

that $\eta_j^{\mathbf{p},r,\mathbf{r}} = \beta_j^{\mathbf{p},r,\mathbf{r}}$ provided
(3.14)

$$\sum_{q_1+\cdots+q_k=q} B^j_{j_1\cdots j_k} \sum_{m_1=0}^{np_*-q_1} \cdots \sum_{m=m_1+\cdots+m_{k-1}}^{m_1+\cdots m_{k-1}+np_*-q_k} \frac{(m_1+q_1)!}{q_1!m_1!} \cdots \frac{(m-(m_1+\cdots+m_{k-1})+q_k)!}{q_k!(m-(m_1+\cdots+m_{k-1}))!} \gamma_{j_1}^{\mathbf{p}_1,m_1+q_1,\mathbf{r}_1} \cdots i^{\pm}(\gamma_{j_k}^{\mathbf{p}_k,m-(m_1+\cdots+m_{k-1})+q_k,\mathbf{r}_k}) = \sum_{m=0}^{np_*-q} \frac{(q+m)!}{m!q!} \sum_{q_1+\cdots+q_k=q+m} B^j_{j_1\cdots j_k} \gamma_{j_1}^{\mathbf{p}_1,q_1,\mathbf{r}_1} \cdots i^{\pm}(\gamma_{j_k}^{\mathbf{p}_k,q_k,\mathbf{r}_k})$$

for all $q \in \{r, \dots, np_*\}$ and $k \in \{2, \dots, l\}$.

To prove this claim we need

Lemma 3.6. *For all $k \geq 2$, $m \geq 1$,*

$$\frac{q!}{(q-m)!} = \sum_{m_1+\cdots+m_{k-1}\leq m} \binom{m}{m_1\cdots m_{k-1}} \frac{q_1!}{(q_1-m_1)!} \cdots \frac{q_k!}{(q_k-(m-(m_1+\cdots+m_{k-1})))!}$$

where $q = q_1 + \cdots + q_k$.

Proof. For $m = 1$ this is true. Assume the claim is true for all values $\leq m$. For $k = 2$ we find

$$\begin{aligned}\frac{q!}{(q-(m+1))!} &= \left(\frac{q!}{(q-m)!}\right)(q_1+q_2-m) \\ &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{q_1!q_2!}{(q_2-(m-i))!(q_1-i-1)!} + \frac{q_1!q_2!}{(q_1-i)!(q_2-(m-i)-1)!} \right) \\ &= \sum_{i=1}^{m} \left(\binom{m}{i-1} + \binom{m}{i} \right) \frac{q_1!q_2!}{(q_1-i)!(q_2-(m+1-i))!}\end{aligned}$$

$$+\frac{q_1!}{(q_1-(m+1))!}+\frac{q_2!}{(q_2-(m+1))!}$$
$$=\sum_{i=0}^{m+1}\binom{m+1}{i}\frac{q_1!q_2!}{(q_1-i)!(q_2-(m+1-i))!}$$

so for $k=2$ the lemma is valid. For $k\geq 3$ we find

(3.15)
$$\frac{q!}{(q-(m+1))!}=\frac{q!}{(q-m)!}(q-m)$$
$$=\sum_{\substack{m_1+\cdots+m_{k-1}\leq m\\ m_1>0}}\binom{m}{m_1-1\cdots m_{k-1}}Q$$
$$+\cdots+\sum_{m_1+\cdots+m_{k-1}\leq m}\binom{m}{m_1\cdots m_{k-1}}Q$$
$$Q=\frac{q_1!\cdots q_k!}{(q_1-m_1)!\cdots(q_k-(m+1-(m_1+\cdots+m_{k-1})))!}.$$

We claim that

(3.16)
$$\sum\nolimits_{\substack{m_1+\cdots+m_{k-1}=m+1\\ m_1>0\\ m_2=0\vee\cdots\vee m_{k-1}=0}}\binom{m}{m_1-1\cdots m_{k-1}}Q+\cdots$$
$$+\sum\nolimits_{\substack{m_1+\cdots+m_{k-1}=m+1\\ m_{k-1}>0,\\ m_1=0\vee\cdots\vee m_{k-2}=0}}\binom{m}{m_1\cdots m_{k-1}-1}Q$$
$$=\sum_{\substack{m_1+\cdots+m_{k-1}=m+1\\ \neg(m_1,\ldots,m_{k-1}>0)}}\binom{m+1}{m_1\cdots m_{k-1}}Q,$$

for all $k\geq 3$ and all $m\geq 1$. This is certainly true for $k=3$. Assume (3.16) is true for all values $\leq k$. For $m_1+\cdots+m_k=m+1$, we have

$$\sum_{\substack{m_1>0\\ m_2=0\vee\cdots\vee m_k=0}}\binom{m}{m_1-1\cdots m_k}Q+\cdots$$
$$+\sum_{\substack{m_k>0\\ m_1=0\vee\cdots\vee m_{k-1}=0}}\binom{m}{m_1\cdots m_{k-1}}Q$$

$$
\begin{aligned}
&= \sum_{m_2,\dots,m_k>0} \binom{m+1}{0 \quad m_2\cdots m_k} Q + \cdots \\
&\quad + \sum_{m_1,\dots,m_{k-1}>0} \binom{m+1}{m_1\cdots m_{k-1} \quad 0} Q \\
&\quad + \sum_{\substack{m_1>0\\ m_3=0\vee\cdots\vee m_k=0}} \binom{m}{m_1-1 \quad m_3\cdots m_k} Q + \cdots \\
&\quad + \sum_{\substack{m_1>0\\ m_2=0\vee\cdots\vee m_{k-1}=0}} \binom{m}{m_1-1 \quad m_2\cdots m_{k-1}} Q + \cdots \\
&\quad + \sum_{\substack{m_k>0\\ m_2=0\vee\cdots\vee m_{k-1}=0}} \binom{m}{m_2\cdots m_{k-1}} Q + \cdots \\
&\quad + \sum_{\substack{m_1=0\vee\cdots\vee m_{k-2}=0\\ m_k>0}} \binom{m}{m_1\cdots m_{k-2} \quad m_{k-1}} Q
\end{aligned}
$$

which by the induction hypothesis becomes

$$
\begin{aligned}
&\sum_{m_2,\dots,m_k>0} \binom{m+1}{0 \quad m_2\cdots m_k} Q + \cdots \\
&\qquad + \sum_{m_1,\dots,m_{k-1}>0} \binom{m+1}{m_1\cdots m_{k-1} \quad 0} Q \\
&\qquad + \sum_{\neg(m_2,\dots,m_k>0)} \binom{m+1}{m_2\cdots m_k} Q + \cdots \\
&\qquad + \sum_{\neg(m_1,\dots,m_{k-1}>0)} \binom{m+1}{m_1\cdots m_{k-1}} Q \\
&\quad = \sum_{\neg(m_1,\dots,m_k>0)} \binom{m+1}{m_1\cdots m_k} Q
\end{aligned}
$$

and (3.16) follows by induction. Similarly one can prove, summing over all $m_1 + \cdots + m_{k-1} \le m$,

$$\sum_{\substack{m_1>0\\ m_2=0\vee\cdots\vee m_{k-1}=0}} \binom{m}{m_1-1\cdots m_k-1} Q + \cdots$$

$$+ \sum_{\substack{m_{k-1}>0\\ m_1=0\vee\cdots\vee m_{k-2}=0}} \binom{m}{m_1\cdots m_{k-1}-1} Q$$

$$+ \sum_{\neg(m_1,\dots,m_{k-1}>0)} \binom{m}{m_1\cdots m_{k-1}} Q$$

$$= \sum_{\neg(m_1,\dots,m_{k-1}>0)} \binom{m+1}{m_1\cdots m_{k-1}} Q.$$

Now (3.15) is equal to

$$\sum_{\substack{m_1+\cdots+m_{k-1}\le m\\ m_1,\dots,m_{k-1}>0}} \Bigg(\binom{m}{m_1-1\cdots m_{k-1}} + \cdots$$

$$+ \binom{m}{m_1\cdots m_{k-1}-1} + \binom{m}{m_1\cdots m_{k-1}} \Bigg) Q$$

$$+ \sum_{\substack{m_1+m_{k-1}=m+1\\ m_1>0}} \binom{m}{m_1-1\cdots m_{k-1}} Q + \cdots$$

$$+ \sum_{\substack{m_1+m_{k-1}=m+1\\ m_{k-1}>0}} \binom{m}{m_1\cdots m_{k-1}-1} Q$$

$$+ \sum_{\substack{m_1+\cdots+m_{k-1}\le m\\ m_1>0\\ m_2=0\vee\cdots\vee m_{k-1}=0}} \binom{m}{m_1-1\cdots m_{k-1}} Q + \cdots$$

$$+ \sum_{\substack{m_1+\cdots+m_{k-1}\le m\\ m_{k-1}>0\\ m_1=0\vee\cdots\vee m_{k-2}=0}} \binom{m}{m_1\cdots m_{k-1}-1} Q$$

$$+ \sum_{\substack{m_1+\cdots+m_{k-1}\le m\\ \neg(m_1,\dots,m_{k-1}>0)}} \binom{m}{m_1\cdots m_{k-1}} Q$$

which by (3.16) and (3.17) becomes

$$\sum_{\substack{m_1+\cdots+m_{k-1}\le m+1\\ m_1,\dots,m_{k-1}>0}} \binom{m+1}{m_1\cdots m_{k-1}} Q + \sum_{\substack{m_1+\cdots+m_{k-1}=m+1\\ \neg(m_1,\dots,m_{k-1}>0)}} \binom{m+1}{m_1\cdots m_{k-1}} Q + \sum_{\substack{m_1+\cdots+m_{k-1}\le m\\ \neg(m_1,\dots,m_{k-1}>0)}} \binom{m+1}{m_1\cdots m_{k-1}} Q = \sum_{m_1+\cdots+m_{k-1}\le m+1} \binom{m+1}{m_1\cdots m_{k-1}} Q$$

and the lemma follows.

Using the lemma we can now prove (3.14). By Lemma 3.6, the righthand side of (3.14) is equal to

$$\sum_{m=0}^{np_*-q} \sum_{m_1+\cdots+m_{k-1}\le m} \frac{1}{m!} \cdot \sum_{q+1+\cdots+q_{k-1}\le q+m} \binom{m}{m_1\cdots m_{k-1}} \frac{q!}{(q_1-m_1)!}\cdots \cdot \frac{(q+m-(q_1+\cdots+q_{k-1}))!}{(q-(q_1+\cdots+q_{k-1})+m_1+\cdots+m_{k-1})!} B^j_{j_1\cdots j_k} \gamma^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}\cdots \cdot i^{\pm}(\gamma^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k})$$
$$= \sum_{m=0}^{np_*-q} \sum_{m_1+\cdots+m_{k-1}\le m} \frac{1}{m!} \cdot \sum_{\substack{q_1+\cdots+q_{k-1}\le q+m_1+\cdots+m_{k-1}\\ q_1\ge m_1,\dots,q_{k-1}\ge m_{k-1}}} \binom{m}{m_1\cdots m_{k-1}} \frac{q_1!}{(q_1-m_1)!}\cdots \cdot \frac{(q+m-(q_1+\cdots+q_{k-1}))!}{(q-(q_1+\cdots+q_{k-1})+m_1+\cdots+m_{k-1})!} B^j_{j_1\cdots j_k} \gamma^{\mathbf{p}_1,q_1,\mathbf{r}_1}_{j_1}\cdots \cdot i^{\pm}(\gamma^{\mathbf{p}_k,q_k,\mathbf{r}_k}_{j_k}).$$

Substituting $q_q^* = q_1 - m_1, \dots, q_{k-1}^* = q_{k-1} - m_{k-1}$ and renaming $q_1^*, \dots, q_{k-1}^*$ to $q_1, \dots, q_{k-1}$, this becomes

$$\sum_{m=0}^{np_*-q} \sum_{m_1+\cdots+m_{k-1}\le m} \sum_{q_1+\cdots+q_{k-1}\le q} \frac{(q_1+m_1)!}{q_1!m_1!} \cdots \frac{(q+m-(m_1+\cdots+m_{k-1})-(q_1+\cdots+q_{k-1}))!}{(q-(q_1+\cdots+q_{k-1}))!(m-(m_1+\cdots+m_{k-1}))!} B^j_{j_1\cdots j_k}\gamma_{j_1}^{\mathbf{p}_1,q_1+m_1,\mathbf{r}_1}\cdots i^{\pm}(\gamma_{j_k}^{\mathbf{p}_k,q+m-(q_1+\cdots+q_{k-1})-(m_1+\cdots+m_{k-1}),\mathbf{r}_k}),$$

which becomes the lefthand side of (3.14) with the definition $q_k = q-(q_1+\cdots+q_{k-1})$. We have shown that

$$\eta_j^{\mathbf{s}} = \beta_j^{\mathbf{s}}. \tag{3.18}$$

To prove (3.18) for $j \in I_s\backslash\{m_1+\cdots+m_{s-1}+1\}$ use induction on j to verify that

$$\begin{aligned}\beta_j^{\mathbf{p},r,\mathbf{r}} = \frac{k^{\mathbf{pa}}}{\lambda_*}\Bigg(&\sum_{m=0}^{np_*-r}\frac{1}{r!m!}\ln^m k \\ &\cdot\sum_{q=m+r}^{np_*} F_j^{\mathbf{s}} q!(-1)^{q-(r+m)}\frac{1}{(a-\alpha_s)^{q-(r+m)+1}} \\ &+\sum_{m=0}^{np_*-r}\frac{1}{r!m!}\ln k\sum_{q=m+r}^{np_*}\delta\gamma_{j-1}^{\mathbf{s}} q!(-1)^{q-(r+m)} \\ &\cdot\frac{1}{(a-\alpha_s)^{q-(r+m)+1}}\Bigg)\exp(-i\mathbf{br}\ln k).\end{aligned} \tag{3.19}$$

Also

$$\begin{aligned}\eta_j^{\mathbf{s}} &= \frac{1}{\lambda_*}\sum_{q=r}^{np_*} F_j^{\mathbf{s}} q!(-1)^{q-r}/(r!(a-\alpha_s)^{q-r+1}) \\ &+\frac{1}{\lambda_*}\sum_{q=r}^{np_*}\delta\eta_{j-1}^{\mathbf{s}} q!(-1)^{q-r}/(r!(a-\alpha_s)^{q-r+1}).\end{aligned} \tag{3.20}$$

We have already seen that the first sum in (3.19) matches the first sum in (3.20). By the induction hypothesis the second sum in (3.20) becomes

$$\frac{\delta}{\lambda_*}\sum_{q=r}^{np_*}\sum_{m=0}^{np_*-q} k^{\mathbf{pa}}\frac{(m+q)!}{q!m!}\ln^m k\gamma_{j-1}^{\mathbf{p},m+q,\mathbf{r}}$$
$$\cdot \exp(-i\mathbf{br}\ln k)q!(-1)^{q-r}/(r!(a-\alpha_s)^{q-r+1}).$$

Switching the sums and introducing $q^* = q+m$, this term is seen to equal the last term in (3.19). Similarly one proves (3.18) when $s \in \{1,\dots,k_1\}\cup\{k_2+1,\dots,k_3\}$ using Lemmas 3.5 and 3.6. We have verified (3.11).

Assume $\alpha_* + \varepsilon > \operatorname{trace}(L)/\lambda_*$. Now define

$$x(t,v) = H_v(t) + t^{\alpha_*+\varepsilon} f(t,v)$$

and observe

$$\begin{aligned}\det D_2x(t,0) &= \det\left\{\exp\left(L\frac{1}{\lambda_*}\ln t\right) + t^{\alpha_*+\varepsilon}D_2f(t,0)\right\}\\ &= \det\,\exp\left(L\frac{1}{\lambda_*}\ln t\right) + t^{\alpha_*+\varepsilon}h(t)\\ &= \exp\left(\operatorname{trace}\left(L\frac{1}{\lambda_*}\ln t\right)\right) + t^{\alpha_*+\varepsilon}h(t)\\ &= \exp\left(\operatorname{trace}(L)\frac{1}{\lambda_*}\ln t\right) + t^{\alpha_*+\varepsilon}h(t)\\ &\neq 0,\end{aligned}$$

for sufficiently small t. Let a denote such a small t. Then there exists an open neighborhood U around the origin in $\mathbf{R}^n$ such that

$$v \in U \longmapsto x_a(v)$$

is a diffeomorphism onto its open image V. Since $y(t) \to 0$ as $t \to -\infty$, there exists a $t_0 > 0$ such that

$$y(t_0 \in V)$$

hence

$$y(t_0) = x(a,v) = x_v(\exp(\lambda_*, b))$$

for some $v \in U$. Thus

$$y(t) = x_v(\exp(\lambda_*(t - t_0 + b)) = x_v(k \exp(\lambda_* t)), \quad k > 0.$$

Finally

$$\begin{aligned} y\left(\frac{1}{\lambda_*} \ln t\right) &= x_v(kt) \\ &= H_v(kt) + t^{\alpha_* + \varepsilon} k^{\alpha_* + \varepsilon} f(kt) \\ &= H_w(t) + t^{\alpha_* + \varepsilon} k^{\alpha_* + \varepsilon} f(kt), \end{aligned}$$

and the theorem follows.

Corollary 3.7. $\eta^{\mathbf{s}} = \beta^{\mathbf{s}}$, $q = 0, \dots, np_*$, $\mathbf{pa} \le \alpha_*$, $|\mathbf{r}| \le p$.

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