

MATRICAL RANGES OF QUADRATIC OPERATORS

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ABSTRACT. We show that if T is a quadratic operator on a Hilbert space, then (1) the numerical range of T is an (open or closed) elliptical disc (or its degenerate form) and (2) for every $n \geq 1$, the n th matricial range of T consists of $n \times n$ matrices whose numerical ranges are contained in the closure of the numerical range of T .

For a bounded linear operator T on a complex Hilbert space H its *numerical range* $W(T)$ is by definition the set $\{\langle Tx, x \rangle : x \in H \text{ and } \|x\| = 1\}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in H . As is well known, to determine the numerical range of a general operator is a very difficult task. Toeplitz, in the earliest paper on this subject [10], did this for operators on a two-dimensional space: their numerical ranges are (closed) elliptical discs. One purpose of this paper is to show that an analogous result holds for quadratic operators on a Hilbert space. Recall that T is *quadratic* if it satisfies $T^2 + \lambda_1 T + \lambda_2 I = 0$ for some scalars λ_1 and λ_2 . In contrast to the finite-dimensional case, numerical ranges of operators on an infinite-dimensional space are in general not closed. We will determine for quadratic operators when their numerical ranges are.

In the literature, there are miscellaneous generalizations of the numerical range. The one to the matricial range seems to be most natural and useful. Specifically, for every $n \geq 1$, the n th *matricial range* $W^n(T)$ of an operator T on H consists of $n \times n$ matrices of the form $\phi(T)$, where ϕ is a unital completely positive linear map from $\mathcal{B}(H)$, the C^* -algebra of all operators on H , to M_n , the C^* -algebra of $n \times n$ matrices. This was first introduced by Arveson [2]. As was shown by him, they provide complete unitary invariants for certain compact operators. Note

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that $W^1(T)$, the first matricial range of T , coincides with $\overline{W(T)}$. Another result we obtain in this paper is a complete determination of all matricial ranges of any quadratic operator.

In Section 1 below we start by giving the canonical form for quadratic operators. This will be used for the description of their numerical ranges in Section 2 and their matricial ranges in Section 3.

For properties of completely positive maps and related dilation theorems, the reader can consult Paulsen's monograph [9].

1. Canonical form. There are two frequently seen subclasses of quadratic operators: *square-zero operators* ($T^2 = 0$) and *idempotents* ($T^2 = T$). In fact, every quadratic operator is an affine function of one of these two types of operators. For if T is quadratic, say, $T^2 + \lambda_1 T + \lambda_2 I = 0$ for scalars λ_1 and λ_2 , then T satisfies $(T - aI)(T - bI) = 0$, where $a, b = (-\lambda_1 \pm (\lambda_1^2 - 4\lambda_2)^{1/2})/2$. If $a = b$, then $T - aI$ is square-zero; otherwise, $(b - a)^{-1}(T - aI)$ is idempotent. Hence, T is an affine function of one or the other depending on whether its spectrum consists of one or two points.

The next theorem gives a canonical form for quadratic operators. The special case for idempotents on a finite-dimensional space was treated before in [4].

Theorem 1.1. *Let T be an operator on H . Then T is quadratic if and only if it is unitarily equivalent to an operator of the form*

$$aI \oplus bI \oplus \begin{bmatrix} aI & A \\ 0 & bI \end{bmatrix} \quad \text{on } H_1 \oplus H_2 \oplus (H_3 \oplus H_3),$$

where a and b are scalars and A is positive definite, that is, $\langle Ax, x \rangle > 0$ for all nonzero vectors x in H_3 . In this representation, a, b and the dimensions of H_1, H_2 ($H_1 \oplus H_2$ if $a = b$) and H_3 are unique while A is unique up to unitary equivalence.

Proof. If T is of the above form, then obviously $(T - aI)(T - bI) = 0$, which shows that T is quadratic. For the rest of the proof, we consider only quadratic operator T with spectrum consisting of two distinct points a and b . The case for $a = b$ can be treated analogously. Since

$(b - a)^{-1}(T - aI)$ is idempotent, it is clear that to prove the converse we need only consider such operators.

If T is idempotent on H , then $T = 0 \oplus T'$ on $H = (\ker T \cap \ker T^*) \oplus H'$, where H' is the orthogonal complement of $\ker T \cap \ker T^*$ in H . We may represent T' as

$$\begin{bmatrix} 0 & B \\ 0 & C \end{bmatrix}$$

on $H' = \ker T' \oplus \overline{\text{ran } T'^*}$. Since T' is also idempotent, it is easily deduced that $C = I$. Hence, T' has the finer representation

$$\begin{bmatrix} 0 & 0 & D \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

with respect to the decomposition $H' = \ker T' \oplus \ker B \oplus (\overline{\text{ran } T'^*} \ominus \ker B)$ or the representation

$$I \oplus \begin{bmatrix} 0 & D \\ 0 & I \end{bmatrix}$$

on $\ker B \oplus (\ker T' \oplus (\overline{\text{ran } T'^*} \ominus \ker B))$. From the above construction, it is easily seen that D is injective and has dense range. If $D = UA$ is the polar decomposition of D , where U is unitary and A is positive definite, then

$$\begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & D \\ 0 & I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}.$$

This shows that T' is unitarily equivalent to

$$I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$$

and hence has the asserted representation.

For the uniqueness (for idempotents), if T has the form

$$0 \oplus I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} \quad \text{on } H_1 \oplus H_2 \oplus (H_3 \oplus H_3),$$

then $H_1 = \ker T \cap \ker T^*$ and $H_2 = \ker(T-I) \cap \ker(T-I)^*$. Hence their dimensions are uniquely determined by T . To prove the uniqueness of A , assume that

$$\begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & A' \\ 0 & I \end{bmatrix}$$

are unitarily equivalent, where A and A' are positive definite, with the intertwining unitary operator

$$U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}.$$

Carrying out the matrix multiplications in

$$\begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A' \\ 0 & I \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix},$$

we obtain $U_3 = 0$ and $U_1A + U_2 = A'U_4$. Since U^* is the inverse of U , we also have

$$\begin{bmatrix} U_1^* & U_3^* \\ U_2^* & U_4^* \end{bmatrix} \begin{bmatrix} 0 & A' \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} \begin{bmatrix} U_1^* & U_3^* \\ U_2^* & U_4^* \end{bmatrix},$$

from which $U_2^* = 0$ or $U_2 = 0$ follows. Hence, both U_1 and U_4 are unitary and $U_1A = A'U_4$. We have

$$A^2 = (AU_1^*)(U_1A) = (U_4^*A')(A'U_4) = U_4^*A'^2U_4.$$

From this we infer that A and A' are unitarily equivalent, completing the proof. \square

2. Numerical range. The main result of this section is the following theorem specifying the numerical range of a quadratic operator.

Theorem 2.1. *Let T be a quadratic operator with $\sigma(T) = \{a, b\}$.*

(1) *If $a = b$, then $W(T)$ is either the singleton $\{a\}$ or the (open or closed) circular disc with center a and radius $\|T - aI\|/2$.*

(2) *If $a \neq b$, then $W(T)$ is either the closed line segment connecting a and b or the (open or closed) elliptical disc with foci at a and b , major axis $\|T - aI\|$ and minor axis $(\|T - aI\|^2 - |a - b|^2)^{1/2}$.*

(3) *The following conditions are equivalent for T :*

- (i) *$W(T)$ is closed;*
- (ii) *T attains its norm;*
- (iii) *T attains its numerical radius.*

The *numerical radius* $w(T)$ of an operator T is the quantity $\sup\{|\lambda| : \lambda \in W(T)\}$. T is said to *attain its norm*, respectively *attain its numerical radius*, if there is a unit vector x such that $\|Tx\| = \|T\|$, respectively $|\langle Tx, x \rangle| = w(T)$. For general properties of the numerical range and numerical radius, the reader can consult [6, Chapter 22].

We prove Theorem 2.1 via the following lemma.

Lemma 2.2. *If*

$$T = \begin{bmatrix} aI & A \\ 0 & bI \end{bmatrix} \quad \text{on } H = H_1 \oplus H_2,$$

then

- (1) $\|T - aI\|^2 = |a - b|^2 + \|A\|^2$, *and*
- (2) *T attains its norm if and only if A does.*

Proof. Let x be a unit vector in H . We decompose x as $\alpha y \oplus \beta z$, where y and z are unit vectors in H_1 and H_2 , respectively, and α and β are scalars satisfying $|\alpha|^2 + |\beta|^2 = 1$. Then

$$\begin{aligned} \|Tx\|^2 &= \|(a\alpha y + \beta Az) \oplus b\beta z\|^2 \\ &= |a|^2|\alpha|^2 + 2\operatorname{Re}(\bar{a}\bar{\alpha}\beta\langle Ax, y \rangle) + |\beta|^2\|Az\|^2 + |b|^2|\beta|^2 \\ (*) \quad &\geq \left\| \begin{bmatrix} a & \langle Az, y \rangle \\ 0 & b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|^2. \end{aligned}$$

From this, we deduce that

$$\|T\| \geq \sup \left\{ \left\| \begin{bmatrix} a & \langle Az, y \rangle \\ 0 & b \end{bmatrix} \right\| : y \in H_1, z \in H_2 \quad \text{and} \quad \|y\| = \|z\| = 1 \right\}.$$

Since, for fixed a and b , the norm of the 2×2 matrix

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

is an increasing function of $|c|$ and $\|A\| = \sup\{|\langle Az, y \rangle| : y \in H_1, z \in H_2 \text{ and } \|y\| = \|z\| = 1\}$, we obtain

$$\|T\| \geq \left\| \begin{bmatrix} a & \|A\| \\ 0 & b \end{bmatrix} \right\|.$$

On the other hand, from (*), we also have

$$\begin{aligned} \|Tx\|^2 &\leq |a|^2|\alpha|^2 + 2|a||\alpha||\beta|\|A\| + |\beta|^2\|A\|^2 + |b|^2|\beta|^2 \\ (**) \quad &= \left\| \begin{bmatrix} a & \|A\| \\ 0 & b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta e^{i\theta} \end{bmatrix} \right\|^2, \end{aligned}$$

where θ is a suitable real number with $\bar{a}\bar{\alpha}\beta\|A\|e^{i\theta} \geq 0$. Therefore,

$$\|T\| \leq \left\| \begin{bmatrix} a & \|A\| \\ 0 & b \end{bmatrix} \right\|.$$

This proves that

$$(***) \quad \|T\| = \left\| \begin{bmatrix} a & \|A\| \\ 0 & b \end{bmatrix} \right\|.$$

(1) Applying the above formula to $T - aI$ results in

$$\|T - aI\|^2 = \left\| \begin{bmatrix} 0 & \|A\| \\ 0 & b - a \end{bmatrix} \right\|^2 = |a - b|^2 + \|A\|^2.$$

(2) If T attains its norm, say, $\|T\| = \|Tx\|$ for the unit vector x , then from (**) and (***) we deduce that $\|Tx\|^2 = |a|^2|\alpha|^2 + 2|a||\alpha||\beta|\|A\| + |\beta|^2\|A\|^2 + |b|^2|\beta|^2$. This, together with (*), implies that $\|A\| = \|Az\|$ if $\beta \neq 0$. On the other hand, if $\beta = 0$, then $\|T\| = \|Tx\| = \|a\alpha y\| = |a|$. But, for any unit vector y in H_1 , we also have

$$|a| = \|T^*\| \geq \|T^*(y \oplus 0)\| = (|a|^2 + \|A^*y\|^2)^{1/2},$$

from which it follows that $A = 0$. Hence, in either case, A attains its norm.

Conversely, if A attains its norm, say, $\|A\| = \|Az\|$ for the unit vector z in H_2 , then $y \equiv Az/\|Az\|$ is such that $\langle Az, y \rangle = \|A\|$. Let α and β be scalars satisfying $|\alpha|^2 + |\beta|^2 = 1$ and

$$\left\| \begin{bmatrix} a & \|A\| \\ 0 & b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\| = \left\| \begin{bmatrix} a & \|A\| \\ 0 & b \end{bmatrix} \right\|.$$

Let $x = \alpha y \oplus \beta z$. From (*) and (**), we have

$$\|Tx\|^2 \geq \left\| \begin{bmatrix} a & \langle Az, y \rangle \\ 0 & b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} a & \|A\| \\ 0 & b \end{bmatrix} \right\|^2 = \|T\|^2.$$

It follows that $\|Tx\| = \|T\|$, that is, T attains its norm. \square

Proof of Theorem 2.1. By Theorem 1.1 it suffices to prove the assertions for the operator T in Lemma 2.2. We start by showing

$$W(T) = \cup W \left(\begin{bmatrix} a & \langle Az, y \rangle \\ 0 & b \end{bmatrix} \right),$$

where the union is taken over all unit vectors y and z in H_1 and H_2 , respectively. Let x be a unit vector in H . As before, we decompose x as $\alpha y \oplus \beta z$, where y and z are unit vectors in H_1 and H_2 , respectively, and α and β are scalars satisfying $|\alpha|^2 + |\beta|^2 = 1$. Then

$$\begin{aligned} \langle Tx, x \rangle &= \left\langle \begin{bmatrix} aI & A \\ 0 & bI \end{bmatrix} \begin{bmatrix} \alpha y \\ \beta z \end{bmatrix}, \begin{bmatrix} \alpha y \\ \beta z \end{bmatrix} \right\rangle \\ (*) \quad &= a|\alpha|^2 + \beta \bar{\alpha} \langle Az, y \rangle + b|\beta|^2 \\ &= \left\langle \begin{bmatrix} a & \langle Az, y \rangle \\ 0 & b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle. \end{aligned}$$

This proves the inclusion

$$W(T) \subseteq \cup W \left(\begin{bmatrix} a & \langle Az, y \rangle \\ 0 & b \end{bmatrix} \right).$$

The converse containment follows by reversing the above arguments. Since the numerical range of the 2×2 matrix

$$\begin{bmatrix} a & \langle Az, y \rangle \\ 0 & b \end{bmatrix}$$

is the closed elliptical disc (or its degenerate form) with foci at a and b , major axis $(|a - b|^2 + |\langle Az, y \rangle|^2)^{1/2}$ and minor axis $|\langle Az, y \rangle|$, and $\|A\| = \sup\{|\langle Az, y \rangle| : y \in H_1, z \in H_2 \text{ and } \|y\| = \|z\| = 1\}$, the assertions in (1) and (2) on the shape of $W(T)$ follow easily from Lemma 2.2 (1).

For the proof of (3), note that it is easily seen from above that $W(T)$ is closed if and only if $\|A\| = |\langle Az, y \rangle|$ for some unit vectors y and z in H_1 and H_2 , respectively. This latter condition is equivalent to A being norm-attaining. The equivalence of (i) and (ii) in (3) follows by Lemma 2.2 (2). That (i) and (iii) are equivalent since the function $\lambda \mapsto |\lambda|$ attains a maximum value on each compact subset of the plane, but not on any open ellipse. This completes the proof. \square

An interesting consequence of Theorem 2.1 is that, if a quadratic operator T is such that $T - \lambda I$ attains its norm for some scalar λ , then it does so for all scalars.

Note that every rank-one operator is quadratic. Indeed, if T has rank one, then, letting $K = \text{ran } T \vee \text{ran } T^*$, we can represent T as $T_1 \oplus 0$ on $K \oplus K^\perp$. Since T_1 is acting on a space of dimension at most two, it is easily seen that T is annihilated by a quadratic polynomial. There is another representation for rank-one operators: if x and y are nonzero vectors in $\text{ran } T$ and $\text{ran } T^*$, respectively, then $T = x \otimes y$ in the sense that $Tz = \langle z, y \rangle x$ for any vector z . The following result is an immediate consequence of Theorem 2.1; the assertion on the numerical radius appeared before in [5, Theorem 1].

Corollary 2.3. *If $T = x \otimes y$ is a rank-one operator, then $W(T)$ is the closed elliptical disc with foci 0 and $\langle x, y \rangle$, major axis $\|x\| \cdot \|y\|$ and minor axis $(\|x\|^2 \cdot \|y\|^2 - |\langle x, y \rangle|^2)^{1/2}$, and $w(T)$ is equal to $(\|x\| \cdot \|y\| + |\langle x, y \rangle|)/2$.*

3. Matricial range. In this section we only consider operators on a separable space. The next theorem determines the matricial ranges of a quadratic operator in terms of its numerical range.

Theorem 3.1. *If T is a quadratic operator, then, for every $n \geq 1$, $W^n(T)$ consists of all $n \times n$ matrices A with $W(A) \subseteq \overline{W(T)}$.*

For its proof, we need some preparation. Recall that the operator A on H is said to *dilate* to operator B on K if there exists an isometry V from H to K such that $A = V^*BV$. Alternatively, this is the same as requiring B being unitarily equivalent to some 2×2 operator matrix $\begin{bmatrix} A & * \\ * & * \end{bmatrix}$ in which A appears in its upper left corner.

Our first result is well known, which establishes the equivalence of dilation and infinite C^* -convex combination.

Lemma 3.2. *Let T and T_n be operators on H and H_n , $n \geq 1$, respectively. Then T dilates to $\sum_n \oplus T_n$ if and only if there are operators X_n from H to H_n such that $\sum_n X_n^*X_n = I$ and $\sum_n X_n^*T_nX_n = T$ in the strong operator topology (SOT).*

Proof. If T dilates to $\sum_n \oplus T_n$, say, $T = V^*(\sum_n \oplus T_n)V$ for some isometry V from H to $\sum_n \oplus H_n$, then letting $X_n = P_nV$, where P_n denotes the (orthogonal) projection from $\sum_n \oplus H_n$ onto H_n , we obtain $T = \sum_n X_n^*T_nX_n$ in SOT. On the other hand, $V^*V = I$ implies that $\sum_n X_n^*X_n = I$ in SOT.

For the converse, let X_n , $n \geq 1$, be the operators satisfying the asserted conditions. It is easy to see that the operator $V = [X_1X_2\cdots]^t$ is an isometry from H to $\sum_n \oplus H_n$ and satisfies $T = V^*(\sum_n \oplus T_n)V$. This shows that T dilates to $\sum_n \oplus T_n$, completing the proof. \square

If T_j and X_j , $1 \leq j \leq n$, are operators on H with $\sum_{j=1}^n X_j^*X_j = I$, then the operator $\sum_{j=1}^n X_j^*T_jX_j$ is called a C^* -convex combination of T_j 's. By the preceding lemma, T is a C^* -convex combination of T_1, \dots, T_n if and only if T dilates to $T_1 \oplus \cdots \oplus T_n$. A subset of $\mathcal{B}(H)$ is C^* -convex if it is closed under the operation of C^* -convex combination. For any subset \mathcal{S} of $\mathcal{B}(H)$, the smallest C^* -convex subset of $\mathcal{B}(H)$ which contains \mathcal{S} is called the C^* -convex hull of \mathcal{S} .

The next lemma gives alternative descriptions of the matricial range in terms of the notion of dilation.

Lemma 3.3. *If T is an operator on H , then, for every $n \geq 1$, $W^n(T)$ is equal to the closure of the set of $n \times n$ matrices which can be*

dilated to $\underbrace{T \oplus \cdots \oplus T}_m$ for some m , $1 \leq m < \sqrt{3n}$.

Proof. If H is finite-dimensional, then the assertion is an easy consequence of [8, Lemma 3.1] and Lemma 3.2. Hence, from now on, we assume that H is infinite-dimensional. By [3, Theorem 3.5], $W^n(T)$ is equal to the closure of the C^* -convex hull of $W_s^n(T)$, the n th spatial matricial range of T (recall that $W_s^n(T)$ consists of $n \times n$ matrices which can be dilated to T). Hence, every operator in $W^n(T)$ can be approximated by C^* -convex combinations of a finite number of operators each of which can be dilated to T . From Lemma 3.2, each such combination can be dilated to $\underbrace{T \oplus \cdots \oplus T}_m$ for some $m \geq 1$. This

proves that $W^n(T)$ is contained in the closure of $n \times n$ matrices dilatable to direct sums of copies of T . That m can be chosen to be less than $\sqrt{3n}$ is a consequence of [8, Lemma 3.1].

To prove the reverse containment, assume that A is an $n \times n$ matrix which can be dilated to $T \oplus T \oplus \cdots$, say $A = V^*(T \oplus T \oplus \cdots)V$, where V is an isometry from \mathcal{C}^n to $H \oplus H \oplus \cdots$. It is easily seen that the map $\Phi : \mathcal{B}(H) \rightarrow M_n$ defined by $\Phi(X) = V^*(X \oplus X \oplus \cdots)V$ for X in $\mathcal{B}(H)$ is unital, linear, completely positive and satisfies $\Phi(T) = A$. This shows that A is in $W^n(T)$. Since $W^n(T)$ is closed, it contains the closure of all such A 's. \square

The next lemma is the genesis of our Theorem 3.1.

Lemma 3.4. *An operator A is dilatable to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \cdots$ if and only if $w(A) \leq (1/2)$.*

Proof. Since $w(A) \leq (1/2)$ is equivalent to the condition that $1 + 2\operatorname{Re}(zA) \geq 0$ for all z , $|z| = 1$, the assertion follows from [2, Theorem 1.3.1]. \square

A more concrete proof of the preceding lemma can be given based on Ando's result in [1]. Indeed, if $w(A) \leq (1/2)$, then it was shown in [1, Theorem 1] that $2A = (1 + B)^{1/2}C(1 - B)^{1/2}$ for some Hermitian

contraction B and contraction C . Then the expression

$$A = \left[\begin{pmatrix} \frac{1}{2}(1+B) \\ \frac{1}{2}(1-B) \end{pmatrix}^{1/2} \right] \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \left[\begin{pmatrix} ((1+B)/2)^{1/2} \\ ((1-B)/2)^{1/2} \end{pmatrix} \right]$$

with $\left[\begin{pmatrix} ((1+B)/2)^{1/2} & \\ & ((1-B)/2)^{1/2} \end{pmatrix} \right]^t$ isometry shows that A dilates to $\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$. Since C is a contraction, it dilates to the unitary operator

$$U = \begin{bmatrix} C & (1 - CC^*)^{1/2} \\ (1 - C^*C)^{1/2} & -C^* \end{bmatrix}.$$

Hence, A dilates to $\begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}$. Since the latter is unitarily equivalent to $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \dots$, we have that A dilates to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \dots$ as asserted. This argument is shown to us by M.-D. Choi.

Now we generalize this lemma successively up to its full force.

Lemma 3.5. *Let T be a quadratic operator with closed numerical range. Then an operator A is dilatable to $T \oplus T \oplus \dots$ if and only if $W(A)$ is contained in $W(T)$.*

Proof. One direction is trivial. To prove the other, assume that $W(A) \subseteq W(T)$. If T is a scalar operator aI , then $W(T)$ and $W(A)$ are both the singleton $\{a\}$. Hence, A is also equal to aI and thus dilatable to $T \oplus T \oplus \dots$. On the other hand, if $T = aI \oplus bI$ with $a \neq b$, then, letting $S = (b - a)^{-1}(T - aI)$ and $B = (b - a)^{-1}(A - aI)$, we have $W(B) \subseteq W(S) = [0, 1]$. Thus, B is a positive contraction and S is an orthogonal projection. Since the projection

$$\begin{bmatrix} B & (B - B^2)^{1/2} \\ (B - B^2)^{1/2} & 1 - B \end{bmatrix}$$

is a dilation of B , the latter can be dilated to $S \oplus S \oplus \dots$, and hence A can be dilated to $T \oplus T \oplus \dots$ as required. For the rest of the proof, we may assume that

$$T = aI \oplus bI \oplus \begin{bmatrix} aI & A' \\ 0 & bI \end{bmatrix},$$

where A' is positive definite and the last summand is not missing. Since $W(T)$ is closed, from the results in Section 2 we have that A' attains its norm and hence $c \equiv \|A'\|$ is its eigenvalue. Therefore, the 2×2 matrix

$$R = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

is a direct summand of

$$\begin{bmatrix} aI & A' \\ 0 & bI \end{bmatrix}$$

and has the same numerical range as the latter. To complete the proof, we need to show that $W(A) \subseteq W(R)$ implies that A is dilatable to $R \oplus R \oplus \dots$.

Indeed, the above implication is unaffected by adding a scalar operator to R or by multiplying it by a nonzero scalar. This reduces the proof to the case $R = \begin{bmatrix} 1 & d \\ 0 & -1 \end{bmatrix}$. Consider the function f defined by

$$f(\lambda) = (4 + |d|^2)^{-1/2} \operatorname{Re} \lambda + i|d|^{-1} \operatorname{Im} \lambda$$

for $\lambda \in \mathcal{C}$. We have

$$W(f(A)) \subseteq W(f(R)) = \{\lambda : |\lambda| \leq (1/2)\} = W\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

Lemma 3.4 implies that $f(A)$ dilates to

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \dots$$

and hence A dilates to $R \oplus R \oplus \dots$. \square

The special case of the preceding lemma when T is a 2×2 matrix was obtained earlier by M.-D. Choi, C.-K. Li and N.-K. Tsing and communicated to the second author by Choi. The idea of using the function f to reduce the consideration of general 2×2 matrices to the Jordan block $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has been exploited by C.-K. Li in his recent simple proof for the numerical range of 2×2 matrices [7].

We next drop the restriction of the closedness of the numerical range in the preceding lemma. This we achieve through the next

Lemma 3.6. *Let T be a quadratic operator. If A is an operator with $W(A) \subseteq W(T)$ and $\text{dist}(W(A), \partial W(T)) > 0$, then A dilates to $T \oplus T \oplus \dots$.*

Proof. As in the proof of Lemma 3.5, we may assume that

$$T = aI \oplus bI \oplus \begin{bmatrix} aI & A' \\ 0 & bI \end{bmatrix},$$

where A' is positive definite and the last summand is not missing. By our assumption and the proof of Theorem 2.1, we can find a number c , $0 < c < \|A'\|$, in $W(A')$ with the property that the numerical range of

$$S = aI \oplus bI \oplus \begin{bmatrix} aI & cI \\ 0 & bI \end{bmatrix}$$

contains that of A . Now $W(S)$ is closed and hence Lemma 3.5 is applicable. We obtain that A dilates to $S \oplus S \oplus \dots$. But, from the choice of c , it is easily seen that the 2×2 matrix

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

dilates to

$$\begin{bmatrix} aI & A' \\ 0 & bI \end{bmatrix}.$$

Hence S dilates to $T \oplus T \oplus \dots$. Combining these, we obtain that A dilates to $T \oplus T \oplus \dots$. \square

We are now ready for

Proof of Theorem 3.1. By Lemma 3.3, we can approximate any A in $W^n(T)$ by a sequence of $n \times n$ matrices $\{A_j\}$, each of which is dilatable to $T \oplus T \oplus \dots$. Since $W(A_j) \subseteq W(T)$ for every j and since the map which takes any operator to the closure of its numerical range is continuous, cf. [6, Problem 220], we infer that $W(A) \subseteq \overline{W(T)}$.

Conversely, assume that $W(A) \subseteq \overline{W(T)}$ and

$$T = aI \oplus bI \oplus \begin{bmatrix} aI & A' \\ 0 & bI \end{bmatrix},$$

where A' is positive definite and the last summand is not missing. For each $j \geq 2$, let $A_j = (1 - (1/j))(A - ((a+b)/2)I) + ((a+b)/2)I$. Then A_j is an $n \times n$ matrix with $W(A_j) \subseteq W(T)$ and $\text{dist}(W(A_j), \partial W(T)) > 0$. Lemma 3.6 implies that A_j can be dilated to $T \oplus T \oplus \cdots$. Since $A_j \rightarrow A$ in norm, we deduce from Lemma 3.3 that A is in $W^n(T)$. \square

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