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A NEW CLASS OF WEAKLY SYMMETRIC SPACES

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ABSTRACT. We prove that any simply connected Riemannian manifold which is equipped with a complete unit Killing vector field such that the reflections with respect to the flow lines of that field can be extended to global isometries, is a weakly symmetric space.

1. Introduction. Weakly symmetric spaces have been introduced by A. Selberg [**23**] in 1956. They may be characterized as connected Riemannian manifolds on which any two points can be interchanged by an isometry. Every Riemannian symmetric space is weakly symmetric, but the converse is not true. However, weakly symmetric spaces have many properties enjoyed by symmetric spaces. For example, in [**23**] it is proved that the algebra of all isometry-invariant differential operators on a weakly symmetric space is commutative and in [**1**], the authors show that all their geodesics are orbits of one-parameter groups of isometries of the manifold. Other geometric properties have been considered in [**2, 5, 6**].

At this moment many examples of nonsymmetric weakly symmetric spaces are known. We refer to [**3, 4, 7, 14, 15, 17, 23, 29**] for a detailed description. In particular, it has been proved in [**7**] that any simply connected φ-symmetric space is a weakly symmetric space. Such spaces have been introduced in [**24**] in the framework of contact geometry where they play a similar role as the Hermitian symmetric spaces in complex geometry. They provide examples of Riemannian manifolds which are equipped with a complete unit Killing vector field such that the reflections with respect to the flow lines can be extended to global isometries. Such Riemannian manifolds are called Killing-transversally symmetric spaces. Their local and global geometry as well

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as their classification has been studied in [**11, 12**] and in subsequent papers.

The proof of the weak symmetry of a simply connected φ-symmetric space is based on the fact that such a space is a certain circle or line bundle over a Hermitian symmetric space and on the existence of real forms, that is, fixed point sets of anti-holomorphic involutions, in Hermitian symmetric spaces with the same rank. The reflections of the bundle space in the lifted real forms provide the isometries interchanging two points.

The main purpose of this paper is to prove, using a similar procedure, that any simply connected Killing-transversally symmetric space is weakly symmetric. The proof will be given in Section 5. In Section 2 we start with a brief description of the needed material about the Killing-transversally symmetric spaces and in particular about the associated principal fiber bundle structure. In Section 3 we prove some useful results about immersions into manifolds equipped with a unit Killing vector field, and in Section 4 we relate this to the study of lifts of submanifolds of the base space to the total bundle space.

2. Preliminaries.

Let (M, g) be an n-dimensional, smooth, connected Riemannian manifold with $n \geq 2$. ∇ denotes the Levi Civita connection and R the associated Riemannian curvature tensor with the sign convention

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M.

Further, let ξ be a *unit* Killing vector field on (M, g) and $\mathfrak{F}_\xi$ the flow generated by it. It is a Riemannian flow which is called an *isometric* flow [**26**]. The leaves of this Riemannian foliation are geodesics and, moreover, a geodesic which is orthogonal to the flow field ξ at one of its points is orthogonal to it at all of its points. Such geodesics are called *transversal* (or *horizontal*) geodesics. Further, the foliation is locally a Riemannian submersion. So, let $m \in M$, and let $\mathcal{U}$ be a small open neighborhood of m such that ξ is regular on $\mathcal{U}$. Then the map $\pi : \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$ is submersion. Let g' denote the metric on $\mathcal{U}'$ defined by

$$g'(X', Y') = g(X'^*, Y'^*)$$

for $X', Y' \in \mathfrak{X}(\mathcal{U}')$ and where X'^*, Y'^* denote the horizontal lifts of X', Y' with respect to the $(n-1)$-dimensional horizontal distribution on $\mathcal{U}$ determined by the one-form η on M given by

$$\eta(U) = g(U, \xi),$$

$U \in \mathfrak{X}(M)$. Then the Levi Civita connection ∇' of g' is determined by

$$\nabla'_{X'} Y' = \pi_* \nabla_{X'^*} Y'^*,$$

$X', Y' \in \mathfrak{X}(\mathcal{U}')$.

Next, let A denote the integrability tensor of O'Neill [**19**], see also [**8, 22, 26**]. Then we have

$$\begin{aligned} A_U \xi &= \nabla_U \xi, & A_\xi U &= 0, \\ A_X Y &= (\nabla_X Y)^\nu = -A_Y X, & g(A_X Y, \xi) &= -g(A_X \xi, Y) \end{aligned}$$

for $U \in \mathfrak{X}(M)$ and where X, Y are horizontal vector fields. $\mathcal{V}$ denotes the vertical component. Note that for $U = \xi$ the first two formulas are consistent because ξ is a Killing vector field of constant length and, hence, its integral curves are geodesics. Further, put

$$HU = -A_U \xi$$

and define the $(0,2)$-tensor field h by

$$h(U, V) = g(HU, V),$$

$U, V \in \mathfrak{X}(M)$. (Note that H coincides with the (1,1)-tensor field A_ξ introduced by Kobayashi-Nomizu but we do not use this notation in order to avoid confusion with the integrability tensor A.) Since ξ is a Killing field, h is skew-symmetric. Then we obtain easily

$$A_X Y = h(X, Y)\xi = \frac{1}{2}\eta([X, Y])\xi$$

for all horizontal fields X, Y. This yields

$$h = -d\eta.$$

Further, ∇ and ∇' are related by

$$\nabla_{X'^*} Y'^* = (\nabla'_{X'} Y')^* + h(X'^*, Y'^*)\xi.$$

Now a straightforward computation yields

$$R(X, \xi, Y, \xi) = g(HX, HY) = -g(H^2 X, Y)$$

for horizontal vector fields X, Y. Here we use the notation $R(X, Y, Z, W) = g(R_{XY} Z, W)$. This formula implies that the ξ-*sectional curvature* $K(X, \xi)$ of the two-plane spanned by X and ξ is nonnegative for all horizontal X and, since $H\xi = 0$, $K(X, \xi) = 0$ for all horizontal X if and only if $h = 0$, or equivalently, $A = 0$ or ξ is parallel, that is, the horizontal distribution is integrable. Moreover, $K(X, \xi) > 0$ for each horizontal X if and only if H is of maximal rank $n - 1$. In this case n is necessarily odd and then η is a *contact* form on M. This leads to

Definition 2.1. $\mathfrak{F}_\xi$ is called a *contact flow* if η is a contact form, that is, if H is of maximal rank.

In what follows, we shall also need

Definition 2.2. $\mathfrak{F}_\xi$ is said to be *normal* if $R(X, Y, X, \xi) = 0$ for all horizontal vector fields X, Y.

Then we have that $\mathfrak{F}_\xi$ is normal if and only if

$$(\nabla_U H)V = g(HU, HV)\xi + \eta(V) H^2 U$$

for all $U, V \in \mathfrak{X}(M)$ and in this case the curvature tensor satisfies

$$R_{UV}\xi = \eta(V) H^2 U - \eta(U) H^2 V, \tag{2.1}$$

$$R_{U\xi} V = g(HU, HV)\xi + \eta(V) H^2 U. \tag{2.2}$$

Here it is worthwhile to note that a Sasakian manifold is a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that $K(X, \xi) = 1$ for all horizontal X, see [**9, 28**] for more details.

Next we recall the notion of a locally or globally Killing-transversally symmetric space. So let $\mathfrak{F}_\xi$ be an isometric flow on (M, g) as before.

Let $m \in M$ and denote by σ the geodesic flow line through m. A local diffeomorphism s_m of M defined in a neighborhood $\mathcal{U}$ of m is said to be a (*local*) *reflection* in σ if for every transversal geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathcal{U}$ and σ, we have

$$(s_m \circ \gamma)(s) = \gamma(-s)$$

for all s with $\gamma(\pm s) \in \mathcal{U}$, s being the arc length of γ. Now we consider

Definition 2.3. A *locally Killing-transversally symmetric space*, briefly a *locally* KTS-*space*, is a Riemannian manifold (M, g) equipped with an isometric flow $\mathfrak{F}_\xi$ such that the reflection s_m with respect to the flow line through m is an isometry for all $m \in M$.

These spaces may be characterized by [**11**]:

Proposition 2.1. $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space if and only if $\mathfrak{F}_\xi$ is normal and

$$(\nabla_X R)(X, Y, X, Y) = 0$$

for all horizontal X, Y.

Proposition 2.2. *Let $\mathfrak{F}_\xi$ be a normal flow on (M, g). Then $(M, g, \mathfrak{F}_\xi)$ is a locally* KTS-*space if and only if each base space $\mathcal{U}'$ of a local Riemannian submersion $\pi : \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$ is a locally symmetric space.*

So, according to the terminology used in [**27**], $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space if and only if $\mathfrak{F}_\xi$ is a normal transversally symmetric foliation.

Further, we have

Definition 2.4. Let (M, g) be a Riemannian manifold and ξ a complete unit Killing vector field on it. Then $(M, g, \mathfrak{F}_\xi)$ is said to be a (*globally*) *Killing-transversally symmetric space*, briefly, a KTS-*space*, if it is a locally KTS-space such that the local reflections s_m can be extended to global isometries for each $m \in M$.

For all these notions and for further properties and details, we refer to [**11, 12**]. We finish this section with some other important facts.

First we note that KTS-spaces are necessarily homogeneous spaces [**12**]. ξ is a regular vector field and the orbit space $M' = M/\xi$ admits a unique structure of differentiable manifold such that the natural projection $\pi : M \to M'$ is a submersion. Moreover, M is a principal fiber bundle over M' whose structural group is the one-parameter group of global isometries generated by ξ. Further, M' is a symmetric space and π intertwines the reflections s_m of M with the geodesic symmetries $s_{m'=\pi(m)}$ of M'. On M' we consider the tensor field of type (1,1) defined by

$$H'X' = \pi_* H X'^*$$

and the corresponding (0,2)-version h' given by

$$h'(X', Y') = g'(H'X', Y')$$

for all $X', Y' \in \mathfrak{X}(M')$. Then $h = \pi^*(h')$. Moreover, $\mathfrak{F}_\xi$ is normal if and only if $\nabla' H' = 0$. Further, when the ξ-sectional curvature is a nonvanishing constant $k = c^2$, then $H^2 = -kI$ and $(M, c^2 g, c^{-1}H, c^{-1}\xi, c\eta)$ is a Sasakian manifold which fibers over the Kähler manifold $(M', c^2 g', J = c^{-1}H')$ and this Sasakian manifold is a *φ-symmetric space* [**24**] if and only if $(M, g, \mathfrak{F}_\xi)$ is a KTS-space.

Next we note that *contact* KTS-spaces are always irreducible. Moreover, a reducible simply connected KTS-space is a Riemannian product of a Riemannian symmetric space and a contact KTS-space. For simply connected contact KTS-spaces the following result is proved in [**12**].

Proposition 2.3. *The base space (M', g') of a simply connected contact* KTS*-space $(M, g, \mathfrak{F}_\xi)$ is a (simply connected) Hermitian symmetric space. Moreover, we have*

(i) *if $M' = M'_0 \times M'_1 \times \cdots \times M'_r$ is its de Rham decomposition and $\mathcal{H}_i$, $i = 0, 1, \ldots, r$, are the smooth distributions on M obtained by the horizontal lifts of the tangent vectors to M'_i, then, for each $m \in M$, $\mathcal{H}(m) = \mathcal{H}_0(m) \oplus \mathcal{H}_1(m) \oplus \cdots \oplus \mathcal{H}_r(m)$ is an H-invariant orthogonal decomposition of the horizontal subspace $\mathcal{H}(m)$;*

(ii) *each sectional curvature $K(X_j, \xi)$, $X_j \in \mathcal{H}_j$, $j = 1, \ldots, r$, is a positive constant c_j^2;*

(iii) *the* $(1,1)$*-tensor field*

$$J = J_0 \times \frac{1}{c_1} H'_1 \times \cdots \times \frac{1}{c_r} H'_r,$$

is a Hermitian structure on (M', g') *where* J_0 *denotes the canonical almost complex structure on* $M'_0 = \mathbf{C}^p = E^{2p}(x_1, \ldots, x^{2p})$ *and* $H'_j = H' \circ p_j$, $j = 1, \ldots, r$, *where* $p_j : M' \to M'_j$ *denotes the projection of* M' *onto* M'_j;

(iv) $H'_0 = H' \circ p_0$ *on* $E^{2p}(x^1, \ldots, x^{2p})$ *is given by*

$$H' \frac{\partial}{\partial x^k} = \mu_k \frac{\partial}{\partial x^{p+k}}, \quad H' \frac{\partial}{\partial x^{p+k}} = -\mu_k \frac{\partial}{\partial x^k}$$

for certain positive real numbers $\mu_1, \ldots, \mu_p$.

3. Horizontal immersions. Let f be an immersion of an n-dimensional manifold M into an $\bar{n}$-dimensional Riemannian manifold $(\overline{M}, \bar{g})$ and denote by g the induced metric from $\bar{g}$. Then $f : (M, g) \to (\overline{M}, \bar{g})$ is an isometric immersion. Further, let $\overline{R}$ and R be the Riemannian curvature tensors of the Levi Civita connections $\bar{\nabla}$ and ∇ on $(\overline{M}, \bar{g})$ and (M, g), respectively. Moreover, we denote by α the second fundamental form of (M, f), by $\nabla^{\perp}$ the connection in the normal bundle $N(M)$ and by $R^{\perp}$ its curvature tensor. In what follows, and if the argument is local, we shall sometimes identify M with its image $f(M)$ under f to simplify the notation. Then we have the well-known Gauss and Weingarten formulas:

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \tag{3.1}$$

$$\bar{\nabla}_X U = -S_U X + \nabla^{\perp}_X U \tag{3.2}$$

where $X, Y \in \mathfrak{X}(M)$ and U is normal to M. S_U is the shape operator and it is related to α by $\bar{g}(\alpha(X, Y), U) = g(S_U X, Y)$.

We recall the following Gauss and Codazzi equations:

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + \bar{g}(\alpha(X, W), \alpha(Y, Z)) - \bar{g}(\alpha(X, Z), \alpha(Y, W)), \tag{3.3}$$

$$(\overline{R}_{XY} Z)^{\perp} = -(\tilde{\nabla}_X \alpha)(Y, Z) + (\bar{\nabla}_Y \alpha)(X, Z) \tag{3.4}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$, $U, V \in \mathfrak{X}(M)^\perp$. Here $\tilde{\nabla}\alpha$ is defined by

$$(\tilde{\nabla}_X \alpha)(Y, Z) = \nabla^\perp_X(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) \tag{3.5}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

f is said to be *totally geodesic* if $\alpha = 0$ and *parallel* if $\tilde{\nabla}\alpha = 0$. The normal vector field $\mu = (1/n)\mathrm{trace}\,\alpha$ is called the *mean curvature vector field* of f, and f is said to be *minimal* if $\mu = 0$. μ is said to be *parallel* if $\nabla^\perp \mu = 0$. Further, f is called a *totally umbilical immersion* if $\alpha(X, Y) = g(X, Y)\mu$ for all $X, Y \in \mathfrak{X}(M)$.

Now let $\mathfrak{F}_{\bar{\xi}}$ be the flow on $(\overline{M}, \bar{g})$ determined by a unit Killing vector field $\bar{\xi}$. An isometric immersion $f : (M, g) \to (\overline{M}, \bar{g})$ is said to be *tangent* if $\bar{\xi}$ is tangent to $f(M)$. In that case we shall denote by ξ the unit Killing vector field on (M, g) induced by $\bar{\xi}$, that is, $f_*\xi = \bar{\xi} \circ f$, and by η the one-form on M given by $\eta = f^*\bar{\eta}$. If $\bar{\xi}$ is normal to $f(M)$, that is, $f_* T_m M \subset \mathcal{H}(f(m))$ for all $m \in M$, then f is said to be *horizontal* (with respect to $\mathfrak{F}_{\bar{\xi}}$). f is said to be *anti-invariant* if $\overline{H} f_* T_m M \subset T^\perp_m M$ for all $m \in M$.

Many of the properties of horizontal submanifolds in Sasakian manifolds have analogs when we consider $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$. We give some examples which we shall need later on.

Proposition 3.1. *Let $f : (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ be a horizontal immersion. Then we have*

(i) *f is anti-invariant and, if $\mathfrak{F}_{\bar{\xi}}$ is a contact flow, $2n \le \bar{n} - 1$;*

(ii) *α is orthogonal to $\bar{\xi}$ and $S_{\bar{\xi}} = 0$;*

(iii) $\bar{\eta}((\tilde{\nabla}_X \alpha)(Y, Z)) = \bar{g}(\alpha(Y, Z), \overline{H} X)$;

(iv) $\bar{\eta}((\nabla^\perp_X \mu) = \bar{g}(\mu, \overline{H} X)$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proof. From (3.1) we get

$$\bar{\eta}(\alpha(X, Y)) = \bar{g}(\bar{\nabla}_X Y, \bar{\xi}) = \bar{g}(\overline{H} X, Y).$$

The first term is symmetric in X, Y and the last term is skew-symmetric in X, Y. Hence both terms vanish. So f is anti-invariant. Then (i) and

(ii) follow at once. Further, for arbitrary vector fields X, W on $\overline{M}$ we have

$$\bar{\eta}(\bar{\nabla}_X W) = X\bar{\eta}(W) + \bar{g}(\overline{H}X, W)$$

which, together with (3.5) and taking into account (ii), implies (iii). Finally, (iv) follows similarly. □

From (ii), (iii) and (iv) in this proposition, we get

Corollary 3.1. *Let $\mathfrak{F}_{\bar{\xi}}$ be a contact flow on a $(2d+1)$-dimensional Riemannian manifold $(\overline{M}, \bar{g})$, and let $f : (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ be a horizontal isometric immersion with* $\dim M = d$. *Then we have*

(i) *if the second fundamental form of f is parallel, then f is totally geodesic*;

(ii) *if the mean curvature vector of f is parallel, then f is minimal.*

Now we introduce

Definition 3.1. The second fundamental form α, respectively the mean curvature vector field μ, of a horizontal immersion $f : (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ is said to be *η-parallel* if $(\tilde{\nabla}_X \alpha)(Y, Z)$, respectively $\nabla^{\perp}_X \mu$, is vertical, that is, see Proposition 3.1 (iii), (iv),

$$(\tilde{\nabla}_X \alpha)(Y, Z) = \bar{g}(\alpha(Y, Z), \overline{H}X)\bar{\xi}, \tag{3.6}$$

respectively,

$$\nabla^{\perp}_X \mu = g(\mu, \overline{H}X)\bar{\xi}) \tag{3.7}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

In what follows we shall suppose that the orbit space $\overline{M}' = \overline{M}/\xi$ admits a (unique) structure of differentiable manifold such that the natural projection $\bar{\pi} : \overline{M} \to \overline{M}'$ is a submersion. As we have seen in Section 2, this is always so locally. For the global case, see [**13, 18**].

Proposition 3.2. *Let $f : (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ be a horizontal isometric immersion. Then the composition $f' = \bar{\pi} \circ f : (M, g) \to$*

$(\overline{M}, \bar{g}')$ is an isometric immersion satisfying

$$\overline{H}' f'_*(m)X \perp f'_* T_m M$$

for all $X \in T_m, M, m \in M$. In particular, if the ξ-sectional curvature on $\overline{M}$ is a nonvanishing constant $k = c^2$, then $f' : (M, g) \to (\overline{M}', \bar{g}', J = c^{-1}\overline{H}')$ is a totally real immersion.

Proof. Using the definition of $\bar{g}'$ and the fact that f is horizontal, it follows at once that f' is an isometric immersion. Further, since f is anti-invariant, we get

$$\bar{g}'(\overline{H}' f'_* X, f'_* Y) = \bar{g}'(\bar{\pi}_* \overline{H} f_* X, \bar{\pi}_* f_* Y) = \bar{g}(\overline{H} f_* X, f_* Y) = 0$$

for all $X, Y \in T_m M$ and all $m \in M$. This completes the proof. □

Now we derive some geometric relations between a horizontal isometric immersion f and its projection f'. We refer to [**21**] for the corresponding theory in the study of the canonical fibrations of Sasakian manifolds. First we need

Lemma 3.1. *Let $f : (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ be a horizontal isometric immersion. Let $X \in \mathfrak{X}(M)$ and denote by Y a vector field along f. Then we have*

$$\bar{\pi}_* \bar{\nabla}_X Y = \bar{\nabla}'_X \bar{\pi}_* Y - \bar{\eta}(Y) \overline{H}' X. \tag{3.8}$$

Proof. Since $\bar{\xi}$ is regular, any vector field Y along f is locally projectable, that is, for each $m \in M$, there exists a neighborhood $\mathcal{U}$ of $m \in M$ such that $\bar{\pi}_* Y_{|\mathcal{U}}$ is well-defined. Hence, $Z = Y - \bar{\eta}(Y)\bar{\xi} \circ f$ is a locally projectable horizontal vector field along f. Using the formulas given in Section 2, we then get

$$\bar{\nabla}'_X \bar{\pi}_* Y = \bar{\pi}_*(\bar{\nabla}_X Z) = \bar{\pi}_*(\bar{\nabla}_X Y) + \bar{\eta}(Y) \overline{H}' X,$$

and so (3.8) holds. □

Next let α', S', μ' and $\nabla'^{\perp}$ be the second fundamental form, the shape operator, the mean curvature vector field and the normal connection of f', respectively. Then we have

Proposition 3.3. *For a horizontal isometric immersion $f : (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$, we have*

(i) *α takes its values in the horizontal subbundle $\mathcal{H}$ and $\bar{\pi}_*\alpha(X, Y) = \alpha'(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$;*

(ii) *μ is horizontal and $\bar{\pi}_*\mu = \mu'$;*

(iii) *if u is normal for f, then $\bar{\pi}_*u$ is normal for f' and $S'_{\bar{\pi}_*u} = Su$.*

Proof. (i) follows from Proposition 3.1 (ii). Then (ii) and (iii) follow by using (3.8) and the Gauss formula (3.1). □

Corollary 3.2. *The horizontal isometric immersion f is totally geodesic, minimal or totally umbilical if and only if its projection f' has the corresponding properties.*

Corollary 3.3. *If U is a normal vector for f, then $\bar{\pi}_*U$ is normal for f' and $\nabla^{\perp}_X U$ satisfies*

$$\bar{\pi}_*\nabla^{\perp}_X U = \nabla'^{\perp}_X \bar{\pi}_*U - \bar{\eta}(U)\overline{H}'X$$

for tangent X.

Proof. The result follows easily from (3.8), Proposition 3.3 (iii) and the Weingarten formula (3.2). □

Corollary 3.4. *$\nabla^{\perp}\mu$ is determined by*

$$\bar{\eta}(\nabla^{\perp}_X\mu) = \bar{g}(\mu, \overline{H}X) \quad \text{and} \quad \bar{\pi}_*\nabla^{\perp}_X\mu = \nabla'^{\perp}_X\mu',$$

and, hence, μ is η-parallel if and only if μ' is parallel.

Proof. The result follows from Proposition 3.1 (iv), Proposition 3.3 (ii) and Corollary 3.3. □

Corollary 3.5. $\tilde{\nabla}_\alpha$ *is determined by*

$$\bar{\eta}((\tilde{\nabla}_X\alpha)(Y,Z)) = \bar{g}(\alpha(Y,Z), \overline{H}X),$$
$$\bar{\pi}_*((\tilde{\nabla}_X\alpha)(Y,Z)) = (\tilde{\nabla}'_X\alpha')(Y,Z).$$

So α is η-parallel if and only if α' is parallel.

Proof. The first formula is proved in Proposition 3.1. For the second one we use Corollary 3.3 and Proposition 3.3 (i) in (3.5). □

Corollary 3.6. *Let f be a horizontal immersion of (M,g) into $(\overline{M},\bar{g},\mathfrak{F}_{\bar{\xi}})$. If $(\overline{M},\bar{g},\mathfrak{F}_{\bar{\xi}})$ is a locally* KTS*-space and α is η-parallel, then (M,g) is locally symmetric.*

Proof. It follows from Corollary 3.5 that α' is parallel and Proposition 2.2 implies that M' is locally symmetric. The result then follows by applying the Gauss equation and (3.5). □

4. Horizontal lifts. Existence. Now we shall derive a criterion to characterize those immersions f' into the base space $\overline{M}' = \overline{M}/\bar{\xi}$ which are, at least locally, projections of horizontal immersions f.

Lemma 4.1. *Let $f:(M,g)\to(\overline{M},\bar{g},\mathfrak{F}_{\bar{\xi}})$ be a tangent anti-invariant isometric immersion. Then there exists a codimension one totally geodesic foliation $\mathfrak{F}$ of M such that the restriction $f|_{\mathcal{L}}:\mathcal{L}\to\overline{M}$ to each leaf $\mathcal{L}$ of $\mathfrak{F}$ is horizontal.*

Proof. The Gauss formula implies that $\bar{\nabla}_X\xi = -\bar{H}X$ is a normal vector field if and only if $\nabla_X\xi = 0$. Therefore, ξ is a parallel unit vector field of M and the horizontal distribution is integrable. Then the corresponding foliation satisfies the desired result. □

Theorem 4.1. *Let f' be an isometric immersion of (M,g) into the orbit space $(\overline{M}' = \overline{M}/\bar{\xi}, \bar{g}')$ of $(\overline{M},\bar{g},\mathfrak{F}_{\bar{\xi}})$. Then the following statements are equivalent:*

(i) *f' satisfies*

$$\overline{H}' f'_*(m)X \perp f'_* T_m M, \tag{4.1}$$

for all $X \in T_m M$ *and* $m \in M$;

(ii) f' *has locally horizontal lifts, that is, for every initial data* $(m, \bar{m}) \in M \times \overline{M}$ *with* $f'(m) = \bar{\pi}(\bar{m})$, *there exists a neighborhood* $\mathcal{U} = \mathcal{U}(m) \subset M$ *and a horizontal isometric immersion* $f : \mathcal{U} \to \overline{M}$ *with* $f(m) = \bar{m}$ *such that* $\bar{\pi} \circ f = f'_{|\mathcal{U}}$.

Proof. (ii) $\to$ (i) is the main content of Proposition 3.2.

To prove that (i) implies (ii), we consider the regular submanifold N of $M \times \overline{M}$ given by

$$N = \{(m, \bar{m}) \in M \times \overline{M} \mid f'(m) = \bar{\pi}(\bar{m})\}.$$

The projections $\pi : N \to M$ and $j : N \to \overline{M}$ are, respectively, a submersion and an immersion satisfying $f' \circ \pi = \bar{\pi} \circ j$. Equipping N with the Riemannian metric $j^*\bar{g}$, π becomes a Riemannian submersion. Since j maps every fiber $\pi^{-1}(m)$, $m \in M$, isometrically onto the fiber $\bar{\pi}^{-1}(f'(m))$, j is tangent. Moreover, using

$$\begin{aligned}\bar{g}(\overline{H} j_* X, j_* Y) &= \bar{g}'(\bar{\pi}_* \overline{H} j_* X, \bar{\pi}_* j_* Y) = \bar{g}'(\overline{H}' \bar{\pi}_* j_* X, \bar{\pi}_* j_* Y) \\ &= \bar{g}'(\overline{H}' f'_* \pi_* X, f'_* \pi_* Y) = 0\end{aligned}$$

it follows that j is also anti-invariant. Then, following Lemma 4.1, there exists a codimension one foliation $\mathfrak{F}$ of N such that $j|_{\mathcal{L}}$ is horizontal for each leaf $\mathcal{L}$ of $\mathfrak{F}$. Let $\mathcal{L}_{\tilde{m}}$ be the leaf containing $\tilde{m} = (m, \bar{m})$. Then $\psi = \pi|_{\mathcal{L}_{\tilde{m}}}$ is a local isometry into M, and hence there exists a neighborhood $\mathcal{V}$ and $\mathcal{U}$ of $\tilde{m}$ and m, respectively, such that ψ maps $\mathcal{V}$ isometrically onto $\mathcal{U}$. Now $f = j|_{\mathcal{L}_{\tilde{m}}} \circ \psi^{-1}_{|\mathcal{U}}$ is the desired horizontal immersion. □

We note that (4.1) is called the *integrability condition* for the immersion f', that is, the condition for the existence of a horizontal lift.

Corollary 4.1. *If the* $\bar{\xi}$*-sectional curvature of* $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ *is a nonvanishing constant* $k = c^2$, *then an isometric immersion* $f : M \to \overline{M}' = \overline{M}/\bar{\xi}$ *admits locally horizontal lifts if and only if it is totally real with respect to the almost Hermitian structure on* $\overline{M}'$ *induced by* $\overline{H}'$.

From this corollary and from [**12**, Theorem 3.1], we obtain

Corollary 4.2. *Let* $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ *be a contact* KTS-*space fibering over an irreducible Hermitian symmetric space* $(\overline{M}' = \overline{M}/\bar{\xi}, \bar{g}')$. *Then an isometric immersion* $f' : M \to \overline{M}'$ *admits locally horizontal lifts if and only if it is totally real.*

Now we prove

Corollary 4.3. *Let* $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ *be a simply connected* KTS-*space fibering over a Hermitian symmetric space* $(\overline{M}' = \overline{M}'_1 \times \cdots \times \overline{M}'_r, \bar{g}', J)$ *where each* $\overline{M}'_i$, $i = 1, \dots, r$, *is an irreducible Hermitian symmetric space, and let* f'_i *be totally real immersions of Riemannian manifolds* M_i *into* $\overline{M}'_i$. *Then the product immersion* $f' = f'_1 \times \cdots \times f'_r$ *of* $M = M_1 \times \cdots \times M_r$ *into* $\overline{M}'$ *admits locally horizontal lifts.*

Proof. Let $X \in T_m M$, $m \in M$. Then $f'_* X$ can be decomposed as $f'_* X = \sum_{i=1}^r f'_{i*} X_i$ where $X = \sum_{i=1}^r X_i$, $X_i \in T_m M_i$. From Proposition 2.3 it follows that there exist real numbers $c_1, \dots, c_r$ such that

$$\overline{H}' f'_* X = \sum_{i=1}^{r} c_i J f'_{i*} X_i.$$

Since each f'_i is a totally real immersion into $\overline{M}'_i$, it follows that f' satisfies the integrability condition. $\square$

Finally, we note that a global version of this theorem can be obtained for KTS-spaces $\overline{M}$ by applying [**20**, Theorem 5] taking into account that $\overline{M}$ is a principal bundle over $\overline{M}'$ whose structural group is the one-parameter group of global isometries generated by the flow [**12**].

Theorem 4.2. *Let* $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ *be a* KTS-*space fibering over* $(M' = \overline{M}/\bar{\xi}, \bar{g}')$, *and let* $f' : (M, g) \to (\overline{M}', \bar{g}')$ *be an isometric immersion satisfying the integrability condition* (4.1) *and* $(m, \bar{m}) \in M \times \overline{M}$ *some initial data with* $f'(m) = \bar{\pi}(\bar{m})$. *Then there exists a Riemannian* *manifold* $\tilde{M}$*, an isometric covering map* $\psi : \tilde{M} \to M$*, a horizontal isometric immersion* $\tilde{f} : \tilde{M} \to \overline{M}$*, and a point* $\tilde{m} \in \tilde{M}$ *such that*

$$\bar{\pi} \circ \tilde{f} = f' \circ \psi, \psi(\tilde{m}) = m, \tilde{f}(\tilde{m}) = \tilde{m}.$$

5. The weak symmetry of KTS-spaces. As mentioned in the introduction, a connected Riemannian manifold is said to be weakly symmetric if for any two points on it there exists an isometry interchanging them. Now we shall prove that this is the case for any simply connected KTS-space. To do this we first derive the following result which generalizes one given in [**4**] for reflections in submanifolds of (locally) φ-symmetric spaces.

Lemma 5.1. *Let* $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ *be a* $(2d+1)$*-dimensional contact locally* KTS*-space, and let* $f : (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ *be a totally geodesic horizontal isometric immersion with* $\dim M = d$*. Suppose that the following two conditions are satisfied*:

(i) $\overline{H}^2 Tf(M) \subset Tf(M)$;

(ii) $\overline{R}_{uv}u$ *is normal to* $f(M)$ *for all horizontal* u, v *normal to* $f(M)$*.*

Then the local reflections in $f(M)$ *are isometries.*

Proof. In what follows we shall again identify M with $f(M)$ to simplify the notation.

Let T be the tensor field of type (1,2) on $\overline{M}$ given by

$$T_U V = \bar{g}(\overline{H}U, V)\bar{\xi} + \bar{\eta}(U)\overline{H}V - \bar{\eta}(V)\overline{H}U$$

for $U, V \in \mathfrak{X}(\overline{M})$. It follows from [**11**] that T is a homogeneous structure satisfying $T_U U = 0$. Hence, $\bar{\nabla}_U T_U = 0$. Next, let $\overline{R}_U = \overline{R}(U, \cdot)U$ be the Jacobi operator of $\overline{M}$ with respect to U. Then the Jacobi operator $\overline{R}_U^{(n)} = (\bar{\nabla}^{(n)}_{U \cdots U}\overline{R})(U, \cdot)U$ of nth order is given by, see [**4**],

$$\bar{g}(\overline{R}_U^{(n)} V, W) = (-1)^n \sum_{\nu=0}^{n} \binom{n}{\nu} \bar{g}(\overline{R}_U T_U^{n-\nu}, T_U^{\nu} W).$$

If U is orthogonal to M, then the hypothesis implies

$$T_U(TM) \subset NM, \quad T_U(NM) \subset TM.$$

Further, since (i) is equivalent to $\overline{H}^2 NM \subset NM$, (2.1), (2.2) and (ii) yield

$$\overline{R}_U TM \subset TM, \quad \overline{R}_U NM \subset NM.$$

Considering these properties of T_U and R_U with the above expression for the Jacobi operator of order n, we get

$$0 = \bar{g}(\overline{R}_U^{(2k)} V, X), 0 = \bar{g}(\overline{R}_U^{(2k+1)} V, W), 0 = \bar{g}(\overline{R}_U^{(2k+1)} X, Y)$$

for all $k \in \mathbf{N}$ and all normal vectors U, V and tangent vectors X, Y. The required result then follows from [**10**, Theorem 1]. □

Now we prove the main result of this paper.

Theorem 5.1. *Any simply connected* KTS-*space is weakly symmetric.*

Proof. Let $(M, g, \mathfrak{F}_\xi)$ be a simply connected KTS-space. Then $M = M_1 \times M_2$ where M_1 is a contact KTS-space and M_2 is symmetric [**12**, Theorem 5.1]. So M is weakly symmetric if and only if M_1 is weakly symmetric [**6**]. Hence we may restrict to the case where $(M, g, \mathfrak{F}_\xi)$ is a simply connected contact KTS-space.

Let p and q be any two points in M and γ a geodesic connecting them. (This γ exists since M is homogeneous and, hence, complete. See [**12**] for details.) We will construct an isometry interchanging these two points. Denote by m the midpoint between p and q on γ, and let v be the unit tangent vector to γ at m. We assume that v is different from ξ. As is easily seen, the case $v = \xi$ may be proved by a similar construction and by putting $\pi_* v = 0$ in the proof below. From Proposition 2.3 it follows that the orbit space $M' = M/\xi$ is a Hermitian symmetric space $M' = M'_0 \times M'_1 \times \cdots \times M'_r$ where $M'_0 = \mathbf{C}^p = E^{2p}(x^1, \dots, x^{2p})$, $p \geq 0$, and where M'_i, $i = 1, \dots, r$ is an irreducible, simply connected Hermitian symmetric space. In each M'_i there exist connected, complete, totally real, totally geodesic submanifolds P_i with $\dim_{\mathbf{R}} P_i = \dim_{\mathbf{C}} M_i'$ and $\operatorname{rank} P_i = \operatorname{rank} M_i'$, see [**7, 16, 25**]. We can take P_i such that $(\pi(m))_i \in P_i$ and with $(\pi_* v)_i$ tangent to P_i, see [**7**]. Each P_i is a reflective submanifold of M_i', that is, the reflection of M_i' in P_i is a well-defined global isometry of M_i'. Hence there exists a connected totally geodesic submanifold Q_i of M_i' with $(\pi(m))_i \in Q_i$ and such that $T_{(\pi(m))_i} Q_i$ coincides with the normal space of P_i at $(\pi(m))_i$ [**10**, Theorem 3].

For $i = 0$, put

$$(\pi_* v)_0 = \sum_{k=1}^{p} \left(v^k \frac{\partial}{\partial x^k} + v^{p+k} \frac{\partial}{\partial x^{p+k}} \right).$$

For simplicity we suppose that the first q vectors, $q \leq p$,

$$v_l = v^l \frac{\partial}{\partial x^l} + v^{p+l} \frac{\partial}{\partial x^{p+l}}, \quad l = 1, \ldots, q,$$

are the nonzero ones. (Any other case may be treated in a similar way.) Here v_l satisfies

$$H'^2 v_l = -\mu_l^2 v_l. \tag{5.1}$$

Hence,

$$\left\{ v_1, \ldots, v_q, \frac{\partial}{\partial x^{q+1}}, \cdots, \frac{\partial}{\partial x^p}, H' v_1, \ldots, H' v_q, \frac{\partial}{\partial x^{p+q+1}}, \cdots, \frac{\partial}{\partial x^{2p}} \right\}$$

forms an orthogonal basis of $T_{(\pi(m))_0} E^{2p}$. We consider the p-plane Q_0 through $(\pi(m))_0$ parallel to $\{w_1', \ldots, w_p'\}$ where $w_1' = H' v_1, \ldots, w_q' = H' v_q$, $w_{q+1}' = (\partial / \partial x^{p+q+1}), \ldots, w_p' = (\partial / \partial x^{2p})$.

Now $Q = Q_0 \times Q_1 \times \cdots \times Q_r$ is a connected, complete totally geodesic submanifold with $\pi(m) \in Q$ and $\pi_* v$ perpendicular to $T_{\pi(m)} Q$. Moreover, since Q_0 satisfies the integrability condition (4.1) and each Q_i, $i = 1, \ldots, r$, is totally real, we get by Corollary 4.3 that Q admits locally a horizontal lift. Let $\tilde{Q}$ be a horizontal lift of Q with $m \in \tilde{Q}$. Then v is perpendicular to $T_m \tilde{Q}$ and, following Corollary 3.2, $\tilde{Q}$ is totally geodesic.

Let u be a tangent vector of $\tilde{Q}$. Then u can be written as

$$u = u_0'^* + \sum_{i=1}^{r} u_i'^* = \sum_{k=1}^{p} u_0'^k w_k'^* + \sum_{i=1}^{r} u_i'^*$$

where $u'_j \in TQ_j$, $j = 0, 1, \dots, r$. Then, using (5.1) and Proposition 2.3, we have

$$\overline{H}^2 u = -\left\{ \sum_{k=1}^{p} \mu_k^2 u_0'^k w_k'^* + \sum_{i=1}^{r} c_i^2 u_i'^* \right\}$$

and so $\overline{H}^2 T\tilde{Q} \subset T\tilde{Q}$.

Further, each Q_i is reflective and so, at each point of Q_i, there exists a totally real, totally geodesic submanifold of M'_i containing this point and tangent to the normal space of Q_i at this point. Taking products we obtain totally geodesic submanifolds at each point of Q tangent to the normal space of Q at this point. Again, all these perpendicular totally geodesic submanifolds have local horizontal lifts which are totally geodesic in M. Then the Codazzi equation (3.4) yields that $R_{uv}u$ is perpendicular to $\tilde{Q}$ whenever u, v are horizontal vectors normal to $\tilde{Q}$. It follows from Lemma 5.1 that the local reflections with respect to $\tilde{Q}$ are isometries. As (M, g) is real analytic, complete (since it is homogeneous), connected and simply connected, these local isometric reflections can be extended to a global isometry which reverses geodesics perpendicular to $\tilde{Q}$. Since v is perpendicular to $\tilde{Q}$, this global reflection interchanges p and q and, consequently, the theorem is proved. □

Remark 5.1. It follows from [**11**] that a simply connected naturally reductive Riemannian manifold of dimension not greater than five is symmetric or the product of a symmetric space and an irreducible KTS-space or an irreducible KTS-space, and conversely. From this and from Theorem 5.1 we get that *any simply connected naturally reductive Riemannian manifold* (M, g) *with* $\dim M \leq 5$ *is weakly symmetric.* This result was proved in a different way for $\dim M \leq 4$ in [**6**] and for $\dim M = 5$ in [**15**].

REFERENCES

1. J. Berndt, O. Kowalski and L. Vanhecke, *Geodesics in weakly symmetric spaces*, Ann. Global Anal. Geom. **15** (1997), 153–156.

2. J. Berndt, F. Prüfer and L. Vanhecke, *Symmetric-like Riemannian manifolds and geodesic symmetries*, Proc. Roy. Soc. Edinburgh **125** (1995), 265–282.

3. J. Berndt, F. Ricci and L. Vanhecke, *Weakly symmetric groups of Heisenberg type*, Differential Geom. Appl. **8** (1998), 275–284.

4. J. Berndt, Ph. Tondeur and L. Vanhecke, *Examples of weakly symmetric spaces in contact geometry*, Boll. Un. Mat. Ital. **11-B** (1997), suppl. fasc. 2, 1–10.

5. J. Berndt, F. Tricerri and L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Math. **1598** (1995).

6. J. Berndt and L. Vanhecke, *Geometry of weakly symmetric spaces*, J. Math. Soc. Japan **48** (1996), 745–760.

7. ———, *φ-symmetric spaces and weak symmetry*, Boll. Un. Mat. Ital., to appear.

8. A.L. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb. (3) **10** (1987).

9. D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. **509** (1976).

10. B.Y. Chen and L. Vanhecke, *Isometric, holomorphic and symplectic reflections*, Geom. Dedicata **29** (1989), 259–277.

11. J.C. González-Dávila, M.C. González-Dávila and L. Vanhecke, *Reflections and isometric flows*, Kyungpook Math. J. **35** (1995), 113–144.

12. ———, *Classification of Killing-transversally symmetric spaces*, Tsukuba J. Math. **20** (1996), 321–347.

13. ———, *Invariant submanifolds in flow geometry*, J. Austral. Math. Soc. **62** (1997), 290–314.

14. J.C. González-Dávila and L. Vanhecke, *New examples of weakly symmetric spaces*, Monatsh. Math. **125** (1998), 309–314.

15. O. Kowalski and R.A. Marinosci, *Weakly symmetric spaces in dimension five*, J. Geom. **58** (1997), 123–131.

16. D.S.P. Leung, *Reflective submanifolds*, III. *Congruency of isometric reflective submanifolds and corrigenda to the classification of reflective submanifolds*, J. Differential Geom. **14** (1979), 167–177.

17. S. Nagai, *Weakly symmetric spaces in complex and quaternionic space forms*, Arch. Math. **65** (1995), 342–351.

18. K. Ogiue, *On fiberings of almost contact manifolds*, Kōdai Math. Sem. Rep. **17** (1965), 53–62.

19. B. O'Neill, *The fundamental equation of a submersion*, Michigan Math. J. **13** (1966), 459–469.

20. H. Reckziegel, *Horizontal lifts of isometric immersions into the bundle space of a pseudo-Riemannian submersion*, in *Global differential geometry and global analysis* (D. Ferus, R.B. Gardner, S. Helgason and U. Simon, eds.), Lecture Notes in Math. **1156** (1976), 264–279.

21. ———, *A correspondence between horizontal submanifolds of Sasakian manifolds and totally real submanifolds of Kählerian manifolds*, in *Topics in differential geometry*, I, II (J. Szenthe and L. Tamássy, eds.), North-Holland, Amsterdam, 1988, 1063–1081.

22. B.L. Reinhart, *Differential geometry of foliations*, Ergeb. Math. Grenzgeb. (3) **99** (1983).

23. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.

24. T. Takahashi, *Sasakian φ-symmetric spaces*, Tôhoku Math. J. **29** (1977), 91–113.

25. M. Takeuchi, *Stability of certain minimal submanifolds of compact Hermitian symmetric spaces*, Tôhoku Math. J. **36** (1984), 293–314.

26. Ph. Tondeur, *Foliations on Riemannian manifolds*, Universitext, Springer-Verlag, Berlin, 1988.

27. Ph. Tondeur and L. Vanhecke, *Transversally symmetric Riemannian foliations*, Tôhoku Math. J. **42** (1990), 307–317.

28. K. Yano and M. Kon, *Structures on manifolds*, World Scientific, Singapore, 1984.

29. W. Ziller, *Weakly symmetric spaces*, in *Topics in Geometry*: *In memory of Joseph D'Atri*, (S. Gindikin, ed.), Progr. Nonlinear Differential Equations Appl. **20** Birkhäuser, Boston, 1996, 355–368.

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