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SOME IDENTITIES CONNECTING PARTITION FUNCTIONS TO OTHER NUMBER THEORETIC FUNCTIONS

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Introduction. Let A be a subset of N, the set of all natural numbers. If n belongs to N, let $p_A(n)$ denote the number of partitions of n into parts belonging to A. Let $\sigma_A(n)$ denote the sum of the divisors of n that belong to A. In particular, if $A = N$, then $p_A(n) = p(n)$, the unrestricted partition function, and $\sigma_A(n) = \sigma(n)$; if A is the set of odd natural numbers, then $p_A(n) = q(n)$, the number of partitions of n into odd parts, and $\sigma_A(n) = \sigma^0(n)$, our notation for the sum of the odd divisors of n. Let $q_0(n)$ denote the number of partitions of n into distinct odd parts. Let $E(n) = n(3n-1)/2$. The integers $E(\pm n)$, where $n \geq 0$, are know as the pentagonal numbers. Let $T(n) = n(n+1)/2$. The integers $T(n)$, where $n \geq 0$, are known as the triangular numbers. Consider the following general theorem:

Theorem X. *Let $f : A \to N$ be a function such that both*

$$F_A(x) = \prod_{n \in A} (1 - x^n)^{-f(n)/n} = 1 + \sum_{n=1}^{\infty} p_{A,f}(n) x^n$$

and

$$G_A(x) = \sum_{n \in A} \frac{f(n)}{n} x^n$$

converge absolutely and represent analytic functions in the unit disk: $|x| < 1$. Let $p_{A,f}(0) = 1$ and $f_A(k) = \sum \{f(d) : d \mid k, d \in A\}$. Then

$$np_{A,f}(n) = \sum_{k=1}^{n} p_{A,f}(n-k) f_A(k). \tag{1}$$

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Theorem X is Theorem 14.8 in [**1**]; its proof is obtained by logarithmic differentiation of the generating function $F_A(x)$. Theorem X is a master identity that can be used to derive many partition identities. For example, if we let $f(n) = n$, then $p_{A,f}(n) = p_A(n)$ and $f_A(k) = \sigma_A(k)$. We thus obtain

$$np_A(n) = \sum_{k=1}^{n} p_A(n-k)\sigma_A(k). \tag{1.1}$$

If we let $A = N$, we obtain

$$np(n) = \sum_{k=1}^{n} p(n-k)\sigma(k). \tag{1.2}$$

In [**4**], Erdos used (1.2) in the equivalent form

$$np(n) = \sum_{m=1}^{n} \sum_{k=1}^{[n/m]} mp(n-km) \tag{1.3}$$

to prove by elementary means that $p(n) \sim (C/n)\exp(2\pi(n/6)^{1/2})$. (This important estimate had been proven earlier in more precise form, namely with $C = 48^{-1/2}$, by Hardy and Ramanujan. See [**5**].

In [**6**], Kraetzel used (1.3) to prove the bound

$$p(n) \le 5^{n/4}$$

with equality only when $n = 4$. If we let $A = N$ and $f(n) = -n$, then (by Euler's well-known pentagonal number formula) we have $f_A(k) = -\sigma(k)$ and

$$P_{A,f}(n) = \begin{cases} (-1)^m & \text{if } n = E(\pm m) \\ 0 & \text{otherwise.} \end{cases}$$

This yields

$$\sigma(n) + \sum_{k \ge 1} (-1)^k \{\sigma(n - E(k)) + \sigma(n - E(-k))\} = \begin{cases} n(-1)^{m-1} & \text{if } n = E(\pm m) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In this note, in Theorems 1 and 2a below, we use (1) to derive identities pertaining to $q(n)$ and $q_0(n)$ that are analogous to (1.2). In Theorem 2b and Corollary 1, we obtain explicit formulas for $\sigma^0(n)$ and $\sigma(n)$ in terms of $q(n)$ and $q_0(n)$. (As for the sum of the even divisors of n, which we denote $\sigma^E(n)$, it is easily seen that, if $n = 2^k m$ where $k \geq 0$ and m is odd, so that $\sigma^0(n) = \sigma(m)$, then $\sigma^E(n) = (2^{k+1} - 2)\sigma^0(n)$.)

In addition to the results mentioned above, we prove Theorem Y, which is Exercise 14.10 in [**1**]. We use Theorem Y to prove several additional identities. The first two, which involve $\mu(n)$ and $\phi(n)$, respectively, are not new; the third yields a new formula for $p(n)$ in terms of $\sigma(n)$.

2. Preliminaries. Let $|x| < 1$. Then

$$\prod_{n=1}^{\infty}(1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n)x^n \tag{2.1}$$

$$\prod_{n=1}^{\infty}(1 - x^{2n-1})^{-1} = \sum_{n=0}^{\infty} q(n)x^n \tag{2.2}$$

$$\prod_{n=1}^{\infty}(1 + x^{2n-1}) = \sum_{n=0}^{\infty} q_0(n)x^n \tag{2.3}$$

$$\prod_{n=1}^{\infty}(1 - x^n) = 1 + \sum_{n=1}^{\infty}(-1)^n\{x^{E(n)} + x^{E(-n)}\} \qquad \text{(Euler)} \tag{2.4}$$

$$\prod_{n=1}^{\infty}(1 - x^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n + 1)x^{T(n)} \qquad \text{(Jacobi).} \tag{2.5}$$

Remarks. Equations (2.1) through (2.5) may be found in [**1**]. (See Table 14.1 and Theorems 14.3 and 14.7.)

3. The main theorems.

Theorem 1.

$$nq(n) = \sum_{k=1}^{n} q(n-k)\sigma^0(k).$$

Proof. Let A be the set of all odd natural numbers. Then $p_A(n) = q(n), \sigma_A(k) = \sigma^0(k)$, and the conclusion follows from (1.1). □

Theorem 2.

(a) $$nq_0(n) = \sum_{k=1}^{n}(-1)^{k-1} q_0(n-k)\sigma^0(k)$$

(b) $$\sigma^0(n) = \sum_{k=1}^{n}(-1)^{k-1} k q_0(k) q(n-k).$$

Proof of (a). Let A be the set of all odd natural numbers. Write (2.3) in the form:

$$\sum_{n=0}^{\infty} q_0(n)x^n = \prod_{n\in A}(1+x^n).$$

Replace x by $-x$ to obtain

$$\sum_{n=0}^{\infty}(-1)^n q_0(n) = \prod_{n\in A}(1-x^n).$$

Let $f(n) = -n$, so that $f_A(k) = -\sigma^0(k)$. Now Theorem X applies, with $p_{A,f}(n) = (-1)^n q_0(n)$, so that

(*) $$n(-1)^n q_0(n) = \sum_{k=1}^{n}(-1)^{n-k} q_0(n-k)(-\sigma^0(k)).$$

Upon simplifying, we obtain (a). □

Proof of (b). $(*)$ implies that

$$\sum_{k=1}^{n}(-1)^{n-k}q_0(n-k)\sigma^0(k) = n(-1)^{n-1}q_0(n).$$

If $|x| < 1$, then we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n}(-1)^{n-k}q_0(n-k)\sigma^0(k)\right)x^n = \sum_{n=0}^{\infty} n(-1)^{n-1}q_0(n)x^n,$$

that is,

$$\left(\sum_{n=0}^{\infty}(-1)^n q_0(n)x^n\right)\left(\sum_{n=0}^{\infty}\sigma^0(n)x^n\right) = \sum_{n=0}^{\infty} n(-1)^{n-1}q_0(n)x^n.$$

Now (2.3) implies

$$\prod_{n=1}^{\infty}(1-x^{2n-1})\left(\sum_{n=0}^{\infty}\sigma^0(n)x^n\right) = \sum_{n=0}^{\infty} n(-1)^{n-1}q_0(n)x^n$$

so that

$$\sum_{n=0}^{\infty}\sigma^0(n)x^n = \left(\prod_{n=1}^{\infty}(1-x^{2n-1})^{-1}\right)\left(\sum_{n=0}^{\infty} n(-1)^{n-1}q_0(n)x^n\right).$$

Now (2.2) implies

$$\sum_{n=0}^{\infty}\sigma^0(n)x^n = \left(\sum_{n=0}^{\infty}q(n)x^n\right)\left(\sum_{n=0}^{\infty} n(-1)^{n-1}q_0(n)x^n\right).$$

Therefore

$$\sum_{n=1}^{\infty}\sigma^0(n)x^n = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} k(-1)^{k-1}q_0(k)q(n-k)\right)x^n.$$

The conclusion now follows by equating coefficients of like powers of x.
□

Remark. Note that, if n is odd, then $\sigma^0(n) = \sigma(n)$. More generally, we have

Corollary 1. *If $n = 2^j m$, where $j \geq 0$ and m is odd, then*

$$\sigma(n) = (2^{j+1} - 1)\sum_{k=1}^{m}(-1)^{k-1}kq_0(k)q(m-k).$$

Proof. By hypothesis, $(2^j, m) = 1$. Therefore

$$\sigma(n) = \sigma(2^j)\sigma(m) = (2^{j+1} - 1)\sum_{k=1}^{m}(-1)^{k-1}kq_0(k)q(m-k)$$

by Theorem 2, part (b). □

Theorem 3.

$$\sum_{k\geq 0}(-1)^k(2k+1)\sigma(n - T(k)) = \begin{cases} (-1)^{m-1}(2m+1)n/3 & \text{if } n = T(m) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write the conclusion of Theorem X as

$$(**) \qquad np_{A,f}(n) = \sum_{k=0}^{n-1} p_{A,f}(k)f_A(n-k).$$

Let $A = N$, $f(n) = -3n$, so $f_A(k) = -3\sigma(k)$. Write (2.5) as

$$\prod_{n=1}^{\infty}(1 - x^n)^3 = \sum_{n=0}^{\infty} p_{A,f}(n)x^n$$

where

$$p_{A,f}(n) = \begin{cases} (-1)^k(2k+1) & \text{if } n = T(k) \\ 0 & \text{otherwise.} \end{cases}$$

Applying $(**)$ and dividing by -3, we obtain the conclusion. □

Remark. Theorem 3 is not a new result; it appears on page 127 of [**7**]. The proof suggested by the distinguished authors involves the logarithmic differentiation of (2.5), as well as use of the identities:

$$\sum_{n=1}^{\infty} \sigma(n) n^{-s} = \zeta(s)\zeta(s-1)$$

and

$$\sum_{n=1}^{\infty} \sigma(n) x^n = \sum_{n=1}^{\infty} n x^n/(1-x^n).$$

(See [**7**, pp. 121, 127 and 318].) Our proof is thus somewhat simpler.

Theorem 4.

$$\sigma(n) = np(n) + \sum_{k\geq 1} (-1)^k \{(n-E(k))p(n-E(k)) + (n-E(-k))p(n-E(-k))\}.$$

Proof. If $|x| < 1$, then (1.2) implies

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p(k)\sigma(n-k) \right) x^n = \sum_{n=0}^{\infty} np(n) x^n,$$

that is,

$$\left(\sum_{n=0}^{\infty} p(n) x^n \right) \left(\sum_{n=0}^{\infty} \sigma(n) x^n \right) = \sum_{n=0}^{\infty} np(n) x^n.$$

Now (2.1) implies

$$\prod_{n=1}^{\infty} (1-x^n)^{-1} \left(\sum_{n=1}^{\infty} \sigma(n) x^n \right) = \sum_{n=0}^{\infty} np(n) x^n,$$

so that

$$\sum_{n=1}^{\infty} \sigma(n) x^n = \left(\prod_{n=1}^{\infty} (1-x^n) \right) \left(\sum_{n=0}^{\infty} np(n) x^n \right).$$

Now (2.4) implies

$$\sum_{n=1}^{\infty} \sigma(n)x^n = \Big(1 + \sum_{k\geq 1}(-1)^k(x^{E(k)} + x^{E(-k)})\Big)\Big(\sum_{n=0}^{\infty} np(n)x^n\Big).$$

The conclusion now follows from matching coefficients of like powers of x. □

Theorem Y (using the notation of Theorem X). *Let*

$$F_A(x) = \prod_{n\in A}(1-x^n)^{-f(n)/n},$$
$$H_A(x) = \sum_{k=1}^{\infty} f_A(k)x^k.$$

Then $F_A(x) = \exp\{\int_0^x (H_A(t)/t)\,dt\}$.

Proof. From the proof of Theorem X, see [**1**, p. 322], we have

$$xF_A'(x) = F_A(x)H_A(x).$$

Thus

$$F_A'(x)/F_A(x) = H_A(x)/x.$$

Hence,

$$\int_0^x (F_A'(t)/F(t))\,dt = \int_0^x (H_A(t)/t)\,dt,$$

that is,

$$\operatorname{Log} F_A(x) = \int_0^x (H_A(t)/t)\,dt,$$

from which the conclusion follows.

Theorem 5.

$$\prod_{n=1}^{\infty}(1-x^n)^{\mu(n)/n} = e^{-x} \quad \text{if } |x| < 1.$$

Proof. Let $A = N$, $f(n) = -\mu(n)$. Now

$$f_A(k) = -\sum_{d|K} \mu(d) = \begin{cases} -1 & \text{if } k = 1, \\ 0 & \text{if } k \neq 1. \end{cases}$$

Thus $H_A(x) = -x$. The conclusion now follows from Theorem Y. □

Theorem 6.

$$\prod_{n=1}^{\infty} (1 - x^n)^{\phi(n)/n} = e^{-x/(1-x)} \quad \text{if } |x| < 1.$$

Proof. Let $A = N$, $f(n) = -\phi(n)$. Now $f_A(k) = -\sum_{d|k} \phi(d) = -k$. Thus

$$\begin{aligned} H_A(x) &= -\sum_{k=1}^{\infty} kx^k = -x\left(\sum_{k=1}^{\infty} kx^{k-1}\right) \\ &= -x\left(\sum_{k=1}^{\infty} x^k\right)' = -x((1-x)^{-1})' \\ &= -x(1-x)^{-2}. \end{aligned}$$

Now

$$\begin{aligned} \int_0^x (H_A(t)/t)\, dt &= \int_0^x -(1-t)^{-2}\, dt \\ &= -(1-t)^{-1}|_0^x = \frac{-x}{1-x}. \end{aligned}$$

The conclusion now follows from Theorem Y. □

Remark. Theorem 5 was posed as a problem in the American Mathematics Monthly in 1943, see [**2**]; a solution was given by Buck [**3**]. Theorems 5 and 6 appear as Exercise 72.1 in [**7**, p. 126]. The suggested solution in [**7**] involves use of the identities

$$\sum_{n=1}^{\infty} \mu(n) n^{-s} = 1/\zeta(s),$$
$$\sum_{n=1}^{\infty} \varnothing(n) n^{-s} = \zeta(s-1)/\zeta(s).$$

We conclude with two additional applications of Theorem Y.

Theorem 7.

$$p(n) = \sum_{k=1}^{n} \frac{1}{k!} \sum_{i_1+i_2+\cdots+i_k=n} \prod_{j=1}^{k} \sigma(i_j)/i_j.$$

Proof. Let $f(n) = n$, $A = N$. Now

$$F_A(x) = \prod_{n=1}^{\infty} (1-x^n)^{-1} = \sum_{n=0}^{\infty} p(n)x^n$$

by (2.1); also $f_A(k) = \sum_{d|k} d = \sigma(k)$, so $H_A(x) = \sum_{k=1}^{\infty} \sigma(k)x^k$.

$$\int_0^x H_A(t)/t\,dt = \int_0^x \sum_{k=1}^{\infty} \sigma(k)t^{k-1} = \sum_{k=1}^{\infty} \frac{\sigma(k)}{k} x^k.$$

Thus

$$\sum_{n=0}^{\infty} p(n)x^n = \exp\left\{ \sum_{k=1}^{\infty} \frac{\sigma(k)}{k} x^k \right\} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\sum_{j=1}^{k} \frac{\sigma(j)}{j} x^j \right).$$

The conclusion now follows from matching coefficients of like powers of x. □

Theorem 8.

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \sum_{i_1+i_2+\cdots+i_k=n} \prod_{j=1}^{k} \sigma(i_j)/i_j = \begin{cases} (-1)^{k-1} & \textit{if } n = E(\pm m) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $f(n) = -n$, $A = N$, use (4) and Theorem Y. □

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