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EMBEDDING DERIVATIVES OF $\mathcal{M}$-HARMONIC FUNCTIONS INTO L^p SPACES

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ABSTRACT. A characterization is given of those Borel measures μ on B, the unit ball in C^n, such that differentiation of order m maps the $\mathcal{M}$-harmonic Hardy space $\mathcal{H}^p$ boundedly into $L^q(\mu)$, $0 < q < p < +\infty$.

1. Introduction. Let B denote the unit ball in C^n, $n \geq 1$, and m the $2n$-dimensional Lebesgue measure on B normalized so that $m(B) = 1$, while σ is the normalized surface measure on its boundary S. We set $d\tau(z) = (1 - |z|^2)^{-1-n}\, dm(z)$. For the most part, we will follow the notation and terminology of Rudin [**10**]. If $\alpha > 0$ and $\xi \in S$, the corresponding Koranyi approach region is defined by

$$D_\alpha(\xi) = \{z = r\eta \in B : |1 - \langle \eta, \xi\rangle| < \alpha(1 - r)\},$$

those regions are equivalent to the standard approach regions $\{z \in B : |1 - \langle z, \xi\rangle| < 2^{-1}\beta(1 - |z|^2),\ \beta > 1\}$. For any function u on B we define a scale of maximal functions by

$$M_\alpha u(\xi) = \sup\{|u(z)| : z \in D_\alpha(\xi)\}.$$

Let $\tilde{\Delta}$ be the invariant Laplacian on B. That is,

$$(\tilde{\Delta}u)(z) = \frac{1}{n+1}\Delta(u \circ \phi_z)(0), \quad u \in C^2(B),$$

where Δ is the ordinary Laplacian and ϕ_z the standard involutive automorphism of B taking 0 to z, see [**10**]. A function u defined on B is $\mathcal{M}$-harmonic, $u \in \mathcal{M}$, if $\tilde{\Delta}u = 0$.

For $0 < p < \infty$, $\mathcal{M}$-harmonic Hardy space $\mathcal{H}^p$ is defined to be the space of all functions $u \in \mathcal{M}$ such that $M_\alpha u \in L^p(\sigma)$ for some $\alpha > 0$, $\|u\|_p = \|M_\alpha u\|_p$. This definition is independent of α and the

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corresponding norms are all equivalent. The space $\mathcal{H}^p \cap H(B)$ is the usual Hardy space and it will be denoted by H^p.

For $0 < \varepsilon < 1$, we define nonisotropic balls, that is, the balls in the Bergman metric on B, by $E_\varepsilon(z) = \{w \in B : |\phi_z(w)| < \varepsilon\} = \phi_z(B_\varepsilon(0))$. For $\xi \in S$ and $0 < \delta \le 2$, set $Q_\delta(\xi) = \{\eta \in S : |1 - \langle \eta, \xi \rangle| < \delta\}$. Note that $\sigma(Q_\delta(\xi)) \sim \delta^n$. Also, for $w \in B$ and $\alpha > 0$, define $S_\alpha(w) = \{\xi \in S : w \in D_\alpha(\xi)\}$. For a measurable function g on S, we define its maximal function with respect to nonisotropic balls Q_δ:

$$M^* g(\xi) = \sup_\delta \frac{1}{\sigma(Q_\delta(\xi))} \int_{Q_\delta(\xi)} |g| \, d\sigma.$$

We say that a sequence (z_k) in B is ε-separated if the nonisotropic balls $E_\varepsilon(z_k)$ are mutually disjoint.

All our definitions are independent of the aperture α; the omission of a subscript α implies that $\alpha = 1$. If μ is a positive Borel measure on B and if $0 < r < \infty$, we define

$$A_{r,\mu} u(\xi) = \left(\int_{D(\xi)} |u|^r \, d\mu \right)^{1/r};$$

we also set $A_{\infty,\mu} u(\xi) = \mu - \operatorname{sup\,ess} \{|u(z)| : z \in D(\xi)\}$. If $d\mu = d\tau$, we write simply $A_r u(\xi)$. Now we define tent spaces $T_r^s(\mu)$, $0 < r \le \infty$, $0 < s < \infty$, see [**2**] and [**9**], as the set of all (equivalence classes of) measurable functions u on B satisfying $\|u\|_{r,s} = \|A_{r,\mu}\|_{L^s(\sigma)} < \infty$. If (z_k) is a sequence of points in B and if $\mu = \sum_k \delta_{z_k}$, where δ_z denotes the unit mass measure at z, then $T_r^s(z_k)$ stands for $T_r^s(\mu)$. In that case elements u in $T_r^s(z_k)$ are in fact sequences $b_k = u(z_k)$.

For $u \in C^1(B)$, we set $\nabla u = (\partial u/\partial z_1, \dots, \partial u/\partial z_n, \partial u/\partial \bar{z}_1, \dots, \partial u/\partial \bar{z}_n)$, $z_k = x_{2k-1} + i x_{2k}$, $k = 1, \dots, n$. More generally, for $m \ge 1$, we define, using multiindex notation, $\nabla^m u = (\partial^\alpha \bar{\partial}^\beta u)_{|\alpha|+|\beta|=m}$ and

$$|\nabla^m u(z)|^2 = \sum_{|\alpha|+|\beta|=m} |\partial^\alpha \bar{\partial}^\beta u(z)|^2.$$

We introduce the area integrals by

$$S_{m,\alpha}^2 u(\xi) = \int_{D_\alpha(\xi)} |\nabla^m u(z)|^2 (1 - |z|)^{2m-n-1} \, dm(z), \qquad \xi \in S, \quad \alpha > 0, \quad m \ge 1.$$

Let us consider the following problem.

Find a necessary and sufficient condition on a positive Borel measure μ on B such that $\nabla^m u \in L^q(\mu)$ whenever $u \in \mathcal{H}^p$.

A standard application of the closed graph theorem tells us that the above is equivalent to the existence of a constant $C < \infty$ such that $\|\nabla^m u\|_{L^q(\mu)} \le C\|u\|_p$.

The case $0 < p \le q < \infty$ was settled in a series of papers [**6, 5** and **4**]. In this paper we solve the question in the remaining case $0 < q < p < \infty$. We note here that the same problem was treated in the R^n-setting, a complete solution was obtained in papers by Luecking and Shirokov [**9, 8, 11** and **12**]. The main result of this paper is the following theorem, which appeared as a conjecture in [**6**].

Theorem 1. *Let μ be a positive measure on B, let $m \ge 1$ be an integer, and let $0 < \varepsilon < 1$. Assume that $0 < q < p < \infty$. Set*

$$g_\varepsilon(z) = \frac{\mu(E_\varepsilon(z))}{(1-|z|)^{n+mq}}.$$

A) *If $2 \le q$, then the following conditions are equivalent:*

1) *$\nabla^m u \in L^q(\mu)$ for every $u \in \mathcal{H}^p$.*

2) *There is a constant $C < \infty$ such that $\|\nabla^m u\|_{L^q(\mu)} \le C\|u\|_p$ for all $u \in \mathcal{H}^p$.*

3) *$A_\infty g_\varepsilon \in L^{p/(p-q)}(\sigma)$.*

4) *If (z_k) is an ε-separated sequence in B, then*

$$\int_S \left(\sup_{z_k \in D(\xi)} \frac{\mu(E_\varepsilon(z_k))}{(1-|z_k|)^{n+mq}} \right)^{p/(p-q)} d\sigma(\xi) < \infty.$$

B) *If $0 < q < 2$, then the following conditions are equivalent:*

1) *$\nabla^m u \in L^q(\mu)$ for every $u \in \mathcal{H}^p$.*

2) *There is a constant $C < \infty$ such that $\|\nabla^m u\|_{L^q(\mu)} \le C\|u\|_p$ for all $u \in \mathcal{H}^p$.*

3) *$A_{2/(2-q)} g_\varepsilon \in L^{p/(p-q)}(\sigma)$.*

4) *if* (z_k) *is an* ε*-separated sequence in* B*, then*

$$\Big[\sum_{z_k\in D(\xi)}\Big(\frac{\mu(E_\varepsilon(z_k))}{(1-|z_k|)^{n+mq}}\Big)^{2/(2-q)}\Big]^{(2-q)/2}\in L^{p/(p-q)}(d\sigma(\xi)).$$

Constants will be denoted by C which may indicate a different constant from one occurrence to the next.

This paper is organized as follows. In Section 2 some preliminaries and auxiliary results are collected. In Section 3 we prove the main theorem.

2. Preliminaries. We collect here some results for future reference.

Lemma 1 [**7**]. *Let* $m\geq l$ *be a nonnegative integer,* $0<p<\infty$ *and* $0<\varepsilon<1$*. Then there is a constant* $C=C(m,n,l,p,\varepsilon)$ *such that, for every* $\mathcal{M}$*-harmonic function* u *we have*

$$|\nabla^m u(w)|^p\leq C(1-|w|)^{(l-m)p}\int_{E_\varepsilon(w)}|\nabla^l u(z)|^p\,d\tau(z),\quad w\in B.$$

For the next lemma, see [**6**] and [**3**].

Lemma 2. *For* $u\in\mathcal{M}$*, the following are equivalent*:

(i) $u\in\mathcal{H}^p$.

(ii) *For some* $m\geq 1$*,* $S_{m,\alpha}u\in L^p(d\sigma)$*.*

(iii) *For all* $m\geq 1$*,* $S_{m,\alpha}u\in L^p(d\sigma)$*.*

Note that the norms $\|u\|_p$ and $\|S_{m,\alpha}u\|_{L^p(\sigma)}$ are equivalent on $\mathcal{H}^p$.

Lemma 3. *Let* $\lambda>1$ *and* $1<p\leq+\infty$*. Set, for* $g\in L^1(\sigma)$*,*

$$H_\lambda g(z)=\int_S\Big(\frac{1-|z|}{|1-\langle z,\eta\rangle|}\Big)^{\lambda n}(1-|z|)^{-n}g(\eta)\,d\sigma(\eta).$$

There is a constant $C = C(\lambda, n)$ such that, for every $\xi \in S$,

$$(A_\infty H_\lambda g)(\xi) \leq C(M^* g)(\xi). \tag{1}$$

In particular, $\|A_\infty H_\lambda g\|)_{L^p(\sigma)} \leq C_{\lambda,n,p}\|g\|_{L^p(\sigma)}$.

Proof. Let us prove (1). Fix $g \in L^1(\sigma)$, $\xi \in S$ and $z \in D(\xi)$, $|z| = r$. Set $S_0 = \{\eta \in S : |1 - \langle \eta, \xi\rangle| < 1 - r$ and $S_k = \{\eta \in S : 2^{k-1}(1-r) \leq |1-\langle \eta, \xi\rangle| < 2^k(1-r)\}$ for $k \geq 1$. Since $|1-\langle z, \eta\rangle| \geq 1-|\langle z, \eta\rangle| \geq 1-|z|$ and since $|1 - \langle z, \eta\rangle| \geq C|1 - \langle \xi, \eta\rangle|$, see [**10**, Lemma 5.4.3], we have

$$\begin{aligned}
|H_\lambda g(z)| &\leq (1-r)^{-n} \int_{S_0} |g(\eta)|\, d\sigma(\eta) \\
&\qquad + C \sum_k \int_{S_k} \frac{(1-r)^{\lambda n - n}|g(\eta)|}{|1 - \langle \xi, \eta\rangle|^{\lambda n}}\, d\sigma(\eta) \\
&\leq C(M^* g)(\xi) \\
&\qquad + \sum_k \int_{Q_{2^k(1-r)}(\xi)} \frac{(1-r)^{\lambda n - n}|g(\eta)|}{[2^{k-1}(1-r)]^{\lambda n}}\, d\sigma(\eta) \\
&\leq \left[C + C2^{\lambda n} \sum_k 2^{n(1-\lambda)k}\right] M^* g(\xi).
\end{aligned}$$

Hence, $A_\infty H_\lambda g(\xi) \leq C(M^* g)(\xi)$. The L^p estimates follow from the corresponding estimates for the maximal function $M^* g$.

If ν is a positive Borel measure on B and if $F \geq 0$ is measurable on B, then for some constants $C_1 = C_1(n, \alpha)$ and $C_2 = C_2(n, \alpha)$:

$$\begin{aligned}
C_1 \int_S \int_{D_\alpha(\xi)} \frac{F(w) d\nu(w)}{(1 - |w|)^n}\, d\sigma(\xi) &\leq \int_B F(w)\, d\nu(w) \\
&\leq C_2 \int_S \int_{D_\alpha(\xi)} \frac{F(w) d\nu(w)}{(1 - |w|)^n}\, d\sigma(\xi).
\end{aligned} \tag{2}$$

To prove this, one applies Fubini's theorem to $F(w)\chi_A(w, \xi)\, d\nu(w)\, d\sigma(\xi)$, $A = \{(w, \xi) \in B \times S : w \in D_\alpha(\xi)\}$ and uses an estimate $\sigma(S_\alpha(w)) \sim (1 - |w|)^n$. □

The first application of this simple observation is in the following lemma.

Lemma 4. *Let $0 < s < \infty$, $\lambda > \max(1, 1/s)$. Then there are constants $C_1 = C_1(s,n,\lambda,\alpha)$ and $C_2 = C_2(s,n,\lambda,\alpha)$ such that*

$$C_1 \int_S (\nu(D_\alpha(\xi)))^s \, d\sigma(\xi) \le \int_S \left(\int_B \left(\frac{1-|z|}{|1-\langle z,\xi\rangle|} \right)^{\lambda n} d\nu(z) \right)^s d\sigma(\xi) \le C_2 \int_S (\nu(D_\alpha(\xi)))^s \, d\sigma(\xi)$$

for every positive measure $d\nu$ on B.

Proof. The first estimate follows from the fact that $1 - |z| > \alpha^{-1}|1-\langle z,\xi\rangle|$ for $z \in D_\alpha(\xi)$. The second estimate in case $0 < s < 1$ was stated in [**6**]. However, the condition $\lambda s > 1$ is missing there and the proof contains a gap which we are going to fill in below.

In that paper the problem is reduced to proving an inequality

$$(3) \qquad \int_S \left(\int_B \left(\frac{1-|z|}{|1-\langle z,\xi\rangle|} \right)^{\lambda n} g(z)^r \, d\tau(z) \right)^{1/r} d\sigma(\xi) \le C\|g\|_{T^1_r}$$

where $rs = 1$ and $g(z) \ge 0$ is a $\beta - T^1_r$ atom, $\beta > \alpha$, so that g is supported in $\hat{Q}_\beta$ and

$$\int_{\hat{Q}_\beta} |g(z)|^r (1-|z|)^{-1} \, dm(z) \le \sigma(Q)^{1-r}.$$

Note that, with the above normalization, we have to estimate the lefthand side of (3) by a constant. One can assume $Q = Q(e_1, \delta)$ and then split the outer integral into two parts: the integral I_1 over $|1-\xi_1| > 2\delta$ and the integral I_0 over $|1-\xi_1| \le 2\delta$.

The estimation of I_0 is carried out in [**6**], we now give a correct estimation of I_1. Since

$$\left(\frac{1-|z|}{|1-\langle z,\xi\rangle|} \right)^{\lambda n} (1-|z|)^{-n-1} \le \frac{C}{1-|z|} \frac{(1-|z|)^{\lambda n - n}}{|1-\xi_1|^{\lambda n}}, \qquad z \in \hat{Q}, \quad |1-\xi_1| > 2\delta,$$

we have, using [**10**, formula 1.4.5],

$$I_1 \le C \biggl(\int_{\hat{Q}_\beta} \frac{g(z)^r}{1-|z|} \, dm(z) \biggr)^{1/r} \delta^{(\lambda n - n)/r} \int_{|1-\xi_1|>2\delta} \frac{d\sigma(\xi)}{|1-\xi_1|^{\lambda n/r}} \le C\delta^{(1-r)n/r} \delta^{(\lambda n - n)/r} J_{\delta,r},$$

where

$$J_{\delta,r} = \int_{U_\delta} (1-\rho^2)^{n-2} |1-\rho e^{i\theta}|^{-\lambda n/r} \rho d\, \rho d\theta,$$

and $U_\delta = \{w \in C : |w| < 1, |1-w| > 2\delta\}$. Using inequality $|1-\rho| \le |1-\rho e^{i\theta}|$, $\rho < 1$, and then working in polar coordinates $w = 1 + ue^{it}$, we get

$$\begin{aligned} J_{\delta,r} &\le C \int_{U_\delta} |1-\rho e^{i\theta}|^{-\lambda n/r+n-2} \rho d\, \rho d\theta \\ &\le C \int_{2\delta}^{2} u^{-\lambda n/r+n-1}\, du \\ &\le C\delta^{n(1-\lambda/r)}. \end{aligned}$$

The last inequality is based on the assumption $\lambda s > 1$. Combining these estimates we get $I_1 \le C$, as desired.

We now assume $s \ge 1$ and use a duality argument. Let s' be the conjugate exponent and choose a nonnegative $\psi \in L^{s'}(\sigma)$. If we set

$$\psi_\lambda(z) = \int_S \left(\frac{1-|z|}{|1-\langle z,\xi\rangle|}\right)^{\lambda n} \psi(\xi)\, d\sigma(\xi),$$

then, using (2) and the previous lemma, we get

$$\begin{aligned} J &= \int_S \psi(\xi) \left(\int_B \left(\frac{1-|z|}{|1-\langle z,\xi\rangle|}\right)^{\lambda n} d\nu(z) \right) d\sigma(\xi) \\ &= \int_B \psi_\lambda(z)\, d\nu(z) \\ &\le C \int_S \int_{D_\alpha(\xi)} \frac{\psi_\lambda(z)}{(1-|z|)^n}\, d\nu(z)\, d\sigma(\xi) \\ &\le C \int_S A_\infty\left(\frac{\psi_\lambda(z)}{(1-|z|)^n}\right) \nu(D_\alpha(\xi))\, d\sigma(\xi) \\ &\le C \|A_\infty H_\lambda \psi\|_{L^{s'}(\sigma)} \|\nu(D_\alpha(\xi))\|_{L^s(d\sigma(\xi))} \\ &\le C\|\psi\|_{s'} \|\nu(D_\alpha(\xi))\|_s, \end{aligned}$$

where we used that $\psi_\lambda(z)(1-|z|)^{-n} = H_\lambda \psi(z)$. □

Lemma 5. *Let* $0 < p < \infty$, $t > 0$ *and* $\lambda = n+1+t$. *Assume* $\lambda p > 2$, *let* (z_k) *be an* ε*-separated sequence, and define*

$$S_\lambda(b_k)(z) = \sum_k b_k \left(\frac{1-|z_k|}{1-\langle z, z_k\rangle} \right)^\lambda$$

for $(b_k) \in T_2^p$. *Then* $S_\lambda : T_2^p(z_k) \to H^p$ *is a bounded operator.*

This lemma was proved in [**6**] for $0 < p < 2$. However, the same proof extends to the full range since we have at our disposal the previous lemma, valid for $0 < s < \infty$. See [**6**] for details.

We conclude this section with a duality result for tent spaces.

Lemma 6. *Let* $1 \le r < +\infty$, $1 < s < +\infty$, *and let* r' *and* s' *be the corresponding conjugate exponents. Assume* $d\nu$ *is a positive Borel measure on* B. *Then the dual of* $T_r^s(\nu)$ *is* $T_{r'}^{s'}(\nu)$, *with respect to a pairing*

$$\langle u, v\rangle = \int_B u(z)v(z)(1-|z|)^n \, d\nu(z).$$

The dual of $T_r^1(\nu)$ was described in [**6**].

Proof. If $u \in T_r^s(\nu)$ and $v \in T_{r'}^{s'}(\nu)$, then, using (2),

$$|\langle u, v\rangle| \le C \int_S \int_{D(\xi)} |uv| \, d\nu \, d\sigma(\xi) \le C\|u\|_{r,s}\|v\|_{r',s'},$$

which shows that $T_{r'}^{s'}(\nu)$ is contained in $T_r^s(\nu)^*$.

Now assume that $\Lambda \in T_r^s(\nu)^*$. We follow a method used in [**9**]. The space $T_r^s(\nu)$ embeds isometrically into a Banach space $L^rL^s = L^rL^s(d\nu(z)d\sigma(\xi))$ consisting of all measurable functions $F(z,\xi)$ such that

$$\|F\|_{L^rL^s}^s = \int_S \left(\int_B |F(z,\xi)|^r \, d\nu(z) \right)^{s/r} d\sigma(\xi) < +\infty$$

by a mapping $u(z) \mapsto u(z)\chi_{D(\xi)}(z)$. Since the dual of L^rL^s is $L^{r'}L^{s'}$, see [**1**], the Hahn-Banach theorem implies that there is a function

$g \in L^{r'}L^{s'}$ such that $\Lambda u = \int_S \int_{D(\xi)} g(z,\xi)u(z)\,d\nu(z)\,d\sigma(\xi)$. Using Fubini's theorem, we conclude that $\Lambda u = \int_B u(z)Pg(z)(1-|z|)^n\,d\nu(z)$, where

$$Pg(z) = (1-|z|)^n \int_{z\in D(\xi)} g(z,\xi)\,d\sigma(\xi).$$

Therefore, it remains to prove that P maps $L^{r'}L^{s'}$ boundedly into $T^{s'}_{r'}(\nu)$.

Assume first that $r' = +\infty$; then $h(\zeta) = \nu - \operatorname{supp\,ess}\{|g(z,\zeta)| : z \in B\}$ belongs to $L^{s'}(d\sigma)$. If $z \in D(\zeta)$, then $\xi \in Q_{4(1-|z|)}(\zeta)$ whenever $z \in D(\xi)$. Therefore, for $z \in D(\zeta)$,

$$|Pg(z)| \le (1-|z|)^{-n} \int_{Q_{4(1-|s|)}(\zeta)} |g(z,\xi)|\,d\sigma(\xi) \le CM^*h(\zeta),$$

which implies that $Pg \in T^{s'}_{r'}(\nu)$.

Next assume $r' = s'$. Using (2) we see that the space $T^{s'}_{r'}(\nu)$ is characterized by the condition $\int_B |u(z)|^{r'}(1-|z|)^n\,d\nu(z) < +\infty$. Hence, by Holder's inequality and the estimate $\sigma(S(z)) \sim (1-|z|)^n$, we have

$$\begin{aligned}\int_B |Pg(z)|^{r'}(1-|z|)^n\,d\nu(z) &= \int_B \left|(1-|z|)^n \int_{z\in D(\xi)} g(z,\xi)\,d\sigma(\xi)\right|^{r'} \\ &\le C\int_B \int_{z\in D(\xi)} |g(z,\xi)|\,d\sigma(\xi)\,d\nu(z) \\ &\le C\|g\|_{L^{r'}L^{s'}}.\end{aligned}$$

Interpolation gives the result for $s' \le r' \le +\infty$. Using a duality argument and self-adjointness of $\chi_{D(\xi)}P : L^rL^s \to L^rL^s$, one extends this to the case $1 \le r' \le s' < \infty$ and therefore finishes the proof. □

3. Proof of the theorem. We have already noted that 1) is equivalent to 2). Let us prove that 3) implies 2). Using Lemma 1, we have, with $E = E_{1/2}$,

$$\begin{aligned} I &= \int_B |\nabla^m u|^q \, d\mu \\ &\le C \int_B \int_{E(z)} |\nabla^m u|^q \, d\tau(w) \, d\mu(z) \\ &= C \int_B \mu(E(w)) |\nabla^m u(w)|^q (1-|w|^2)^{-n-1} \, dw \\ &\le C \int_B |(1-|w|)^m \nabla^m u(w)|^q g(w) (1-|w|)^{-1} \, dw. \end{aligned}$$

We consider first case A). Using (2), we obtain

$$\begin{aligned} I &\le C \int_S \int_{D(\xi)} |(1-|w|)^m \nabla^m u(w)|^q \\ &\qquad\qquad \cdot g(w)(1-|w|)^{-n-1} \, dw \, d\sigma(\xi) \\ &\le C \int_S (A_\infty g)(\xi) S_m^2(\xi) \\ &\qquad\qquad \cdot \sup_{D(\xi)} |(1-|w|)^m \nabla^m u(w)|^{q-2} \, d\sigma(\xi). \end{aligned}$$

But it is easy to derive from Lemma 1 that $\sup_{D(\xi)} |(1-|w|)^m \nabla^m u(w)| \le C \sup_{D_{\beta(\xi)}} |u| = \phi(\xi)$ whenever $\beta > 1$. Hence, $I \le C \int_S A_\infty g S_m^2 \phi^{q-2} \, d\sigma$, and that suffices because $A_\infty g \in L^{p/(p-q)}$, $\phi \in L^p$ and $S_m \in L^p$ by Lemma 2.

Now we turn to case B). By the assumption $g(w) \in T_{2/(2-q)}^{p/(p-q)}(\tau)$ and by Lemma 2, $(1-|w|)^m |\nabla^m u(w)|^q \in T_{2/q}^{p/q}(\tau)$. The result follows from (4) and duality, Lemma 6.

Proposition 1. *Let $0 < q < p < \infty$, and let μ be a positive Borel measure on B such that $\|\nabla^m u\|_{L^q(\mu)} \le C\|u\|_p$, $u \in \mathcal{H}^p$. Assume that (z_k) is an ε-separated sequence in B. Then*

$$d_k = \frac{\mu(E_\varepsilon(z_k))}{(1-|z_k|)^{qm+n}}$$

belongs to the dual of $T_{2/q}^{p/q}(z_k)$, with respect to duality

$$\langle (d_k), (c_k) \rangle = \sum_k d_k c_k (1-|z_k|)^n.$$

We can assume that $|z_k|$ is bounded away from zero: $|z_k| \geq \delta > 0$.

Proof. We choose $(b_k) \in T_2^p(z_k)$ and set $u = S_\lambda(b_k)$ where $\lambda > \max(n+1, 2/p)$. By Lemma 5 we have

$$\begin{aligned}\|(b_k)\|^q_{T_2^p(z_k)} &\geq C\|u\|_p^q \\ &\geq C\|\nabla^m u\|^q_{L^q(\mu)} \\ &\geq C \sum_{|\beta|=m} \|\partial^\beta S_\lambda(b_k)\|^q_{L^q(\mu)}.\end{aligned}$$

Now we fix β, $|\beta| = m$ and replace b_k by $b_k r_k(t)$ where $r_k(t)$, $0 \leq t \leq 1$ are Rademacher's functions. Note that this does not change the norm in $T_2^p(z_k)$. Integration in t, Fubini's theorem and the Khinchine inequality give

$$\begin{aligned}&\|\partial^\beta S_\lambda(b_k)\|^q_{L^q(\mu)} \\ &\qquad \geq C \int_B \int_0^1 \left| \sum_k b_k z_k^\beta \delta_k r_k(t) \frac{(1-|z_k|)^\lambda}{(1-\langle z, z_k\rangle)^{\lambda+m}} \right|^q dt\, d\mu(z) \\ &\qquad \geq C \int_B \left(\sum_k \left| b_k z_k^\beta \frac{(1-|z_k|)^\lambda}{(1-\langle z, z_k\rangle)^{\lambda+m}} \right|^2 \right)^{q/2} d\mu(z) \\ &\qquad \geq C \sum_k \int_{E_\varepsilon(z_k)} \left| b_k z_k^\beta \frac{(1-|z_k|)^\lambda}{(1-\langle z, z_k\rangle|^{\lambda+m}} \right|^q d\mu(z) \\ &\qquad \geq C \sum_k |z_k^\beta|^q |b_k|^q \frac{\mu(E_\varepsilon(z_k))}{(1-|z_k|)^{qm+n}} (1-|z_k|)^n.\end{aligned}$$

The last inequality follows from an elementary fact that $1-|z| \sim 1-|z_k|$ for $z \in E_\varepsilon(z_k)$. Therefore,

$$\|(b_k)\|^q_{T_2^p(z_k)} \geq C \sum_{|\beta|=m} \sum_k |z_k^\beta|^q |b_k|^q \frac{\mu(E_\varepsilon(z_k))}{(1-|z_k|)^{qm+n}} (1-|z_k|)^n.$$

Since $\sum_{|\beta|=m} |z_k^\beta|^q \geq C_{n,m,\delta,q}$ uniformly over k, we can set $c_k = |b_k|^q$ to obtain

$$\sum_k c_k \frac{\mu(E_\varepsilon(z_k))}{(1-|z_k|)^{qm+n}} (1-|z_k|)^n \leq C\|(c_k)\|_{T^{p/q}_{2/q}(z_k)}$$

for all positive $(c_k) \in T^{p/q}_{2/q}(z_k)$, $c_k \geq 0$, and then remove positivity restriction.

This proposition, along with Lemma 6, gives implication 2) $\Rightarrow$ 4) in case B). The next proposition gives the same implication in case A). $\square$

Proposition 2. *Let (z_k) be an ε-separated sequence, $0 < r < 1 < s < \infty$. Then the dual of $T^s_r(z_k)$ can be identified with $T^{s'}_\infty(z_k)$, where s' is the conjugate exponent to s, in the following manner: for $L \in T^s_r(z_k)^*$ there is a unique $(b_k) \in T^{s'}_\infty(z_k)$ such that*

$$Lu = \sum_k u(z_k) b_k (1 - |z_k|)^n, \quad u \in T^s_r(z_k),$$

and $\|(b_k)\| \sim \|L\|$. Conversely, every $(b_k) \in T^{s'}_\infty(z_k)$ induces a continuous linear functional on $T^s_r(z_k)$ by the above formula.

Proof. Let $(b_k) \in T^{s'}_\infty(z_k)$. Since $\sigma(S(z_k)) \sim (1 - |z_k|)^n$ and since $\sum_k a_k \leq (\sum_k a_k^r)^{1/r}$ for $0 < r < 1$, we have

$$\begin{aligned}
|Lu| &\leq C \sum_k |u(z_k) b_k| \int_{\{\xi : z_k \in D(\xi)\}} d\sigma(\xi) \\
&= C \int_S \sum_{z_k \in D(\xi)} |u(z_k) b_k| \, d\sigma(\xi) \\
&\leq C \int_S \left(\sup_{z_k \in D(\xi)} |b_k| \right) \left(\sum_{z_k \in D(\xi)} |u(z_k)|^r \right)^{1/r} d\sigma(\xi) \\
&\leq C \|(b_k)\| \cdot \|u\|_{r,s}.
\end{aligned}$$

We now turn to the harder part. Let us fix a continuous linear functional L. Let $e_k(z_j) = \delta_{j,k}$, where $\delta_{j,k}$ is the Kronecker's delta symbol, and let F be the vector space of all finite linear combinations of e_ks. Set $b_k = L(e_k)(1 - |z_k|)^{-n}$ and $v = (b_k)$. Then $Lu = \langle u, (b_k) \rangle = \langle u, v \rangle$ whenever $u \in F$. Once we prove that $\|(b_k)\|_{\infty, s'} \leq C < +\infty$, the proof is finished. It suffices to prove this estimate under an additional assumption that $b_k = 0$ for $k \geq k_0$, but with C independent of k_0. Pick $\alpha > 0$ and $0 < \delta < 1$. Set $Z_j = \{z_k : \delta^{j+1} < 1 - |z_k| \leq \delta^j\}$, $j \geq 1$. If

$z_k \in Z_j$, we say that the rank of z_k is j. Note that $b_k = 0$ if $z_k \in Z_j$ for $j \geq j_0$.

We can choose $\alpha_1 > 0$ and $\delta < 1$, depending on ε and n only, such that for each $\xi \in S$ and every j there is at most one $z_k \in Z_j$ such that $z_k \in D_{\alpha_1}(\xi)$. Set $B(k, \xi, \alpha, \delta) = \{z \in D_\alpha(\xi) : \delta^{k+1} < 1 - |z| \leq \delta^k\}$. If two different points z_j and z_l belong to $B(k, \xi, \alpha, \delta)$, then $\operatorname{diam}(B(k, \xi, \alpha, \delta)) \geq \operatorname{diam}(E_\varepsilon(z_j)) = \operatorname{diam}(E_\varepsilon(0)) = C(\varepsilon, n)$, diameters are taken with respect to the Bergman metric. Hence it suffices to prove that we can make the diameter of $B(k, \xi, \alpha, \delta)$ arbitrarily small, uniformly over k and ξ. Choose $0 \leq r < 0$, set $w_0 = r\xi$ and $Y(r, \xi, \alpha) = \{r\eta : \eta \in S, r\eta \in D_\alpha(\xi)\}$. If $w = r\eta \in Y(r, \xi, \alpha)$, then, since $|1 - \langle w, w_0 \rangle| = |1 - r^2\langle \xi, \eta \rangle| \leq 1 - r^2 + r^2|1 - \langle \xi, \eta \rangle| \leq (1 - r)(1 + r + \alpha r^2)$,

$$\begin{aligned} 1 - |\phi_{w_0}(w)|^2 &= \frac{(1 - |w_0|^2)(1 - |w|^2)}{|1 - \langle w_0, w \rangle|^2} \\ &\geq \frac{(1 + r)^2}{(1 + r + \alpha r^2)} \geq (1 - \alpha)^2. \end{aligned}$$

Therefore, $|\phi_{w_0}(w)| < \sqrt{2\alpha}$ and hence the diameter of $Y(r, \xi, \alpha)$ tends to zero, uniformly over r and $\xi \in S$, as $\alpha \to 0$. Since the Bergman distance from $r_1 S$ to $r_2 S$ tends to zero as $r_2/r_1 \to 1$, we see that $\operatorname{diam} B(k, \xi, \alpha, \delta) \to 0$ as $\alpha \to 0$.

From now on we fix such α_1 and δ. Set $\alpha_0 = \delta\alpha_1/4$. Then $S_{\alpha_0}(z_k) \subset S_{\alpha_1}(z_l)$ whenever $S_{\alpha_0}(z_k) \cap S_{\alpha_0}(z_l) \neq \varnothing$, $z_k \in Z_{j_1}$, $z_l \in Z_{j_2}$, $j_1 > j_2$ (absorbing property); also $S_{\alpha_1}(z_k) \cap S_{\alpha_1}(z_l) = \varnothing$ if z_k and z_l are distinct points in Z_j.

If $u = (a_k)$, we define a sequence $u_{j,0}$ of functions on $S : u_{j,0}(\xi) = a_k$ if $\xi \in S_{\alpha_0}(z_k)$ for some (and unique) $z_k \in Z_j$; otherwise set $u_{j,0}(\xi) = 0$. Similarly we define a sequence $u_{j,1}$ using aperture α_1.

Set $c_k = |b_k|^{s'-1}$ and $w = (c_k)$. We are going to define inductively a sequence $\tilde{w} = (\tilde{c}_k)$ as a function on the set of all z_ks. If $z_k \in Z_1$, set $\tilde{c}_k = c_k$. Now assume that $\tilde{c}_k$ has been defined for all $z_k \in Z_i$, $i \leq j$, and choose $z_k \in Z_{j+1}$. Let us say that a point $z_k \in Z_i$, $i \leq j$, is selected if $\tilde{c}_k \neq 0$. If $S_{\alpha_0}(z_k) \cap S_{\alpha_0}(z_l) = \varnothing$ for every selected z_l, set $\tilde{c}_k = c_k$. If $S_{\alpha_0}(z_k) \cap S_{\alpha_0}(z_l) \neq \varnothing$ for some selected z_l, then choose z_l of highest rank and set $\tilde{c}_k = c_k$ if $c_k > 2c_l$ (in that case we say that

z_k dominates z_l) and $\tilde{c}_k = 0$ otherwise. Since $Z_j = \varnothing$ for $j \geq j_0$, this process terminates. Using absorbing property one easily shows that

$$\sup_j w_{j,0}(\xi) \leq 2 \sup_j \tilde{w}_{j,1}(\xi), \quad \xi \in S. \tag{4}$$

Now choose $\xi \in S$ and assume that $\tilde{w}_{j,0}(\xi) \neq 0$ for some j. Let $j_l > j_{l-1} > \cdots > j_1$ be those indices j for which $\tilde{w}_{j,0}(\xi) \neq 0$. Then $\xi \in S_\alpha(z_{k_i})$ for unique $z_{k_i} \in Z_{j_i}$, $1 \leq i \leq l$, and $z_{k_{i+1}}$ dominates z_{k_i}. Hence $\tilde{w}_{j_l,1}(\xi) \geq 2^{l-i} \tilde{w}_{j_i,0}(\xi)$ which gives $\sum_j \tilde{w}_{j,0}(\xi)^r = \sum_{i=1}^l \tilde{w}_{j_i,0}(\xi)^r \leq \tilde{w}_{j_l,1}(\xi) \sum_{i=1}^l 2^{r(i-l)}$. Hence, there is a constant C_r such that

$$\Big(\sum_j \tilde{w}_{j,0}(\xi)^r \Big)^{1/r} \leq C_r \sup_j \tilde{w}_{j,1}(\xi). \tag{5}$$

For each k choose a unimodular λ_k so that $\lambda_k b_k = |b_k|$ and set $u = (\lambda_k \tilde{c}_k)$. From (4) we get

$$\sum_j \tilde{w}_{j,1} v_{j,1} = \sum_j \tilde{w}^s_{j,1} \geq (\sup_j \tilde{w}^s_{j,0})^s \geq 2^{-s} \sup_j v^{s'}_{j,0}. \tag{6}$$

Note that $(1-|z_k|)^n = C_k \sigma(S_{\alpha_0}(z_k))$ where $0 < M_1 \leq C_k \leq M_2 < +\infty$. Therefore we get

$$Lu = \int_S \sum_{z_k \in D_{\alpha_0}(\xi)} u(z_k) b_k C_k \, d\sigma(\xi),$$

whenever $u \in F$. Then, using (5), (6) and the continuity of L, we obtain

$$\begin{aligned}
2^{-s} \int_S \sup_j v_{j,0}(\xi)^{s'} \, d\sigma(\xi) &\leq \int_S \sum_j \tilde{w}_{j,0}(\xi) v_{j,0}(\xi) \, d\sigma(\xi) \\
&\leq C \Big[\int_S \Big(\sum_j \tilde{w}_{j,0}(\xi)^r \Big)^{s/r} d\sigma(\xi) \Big]^{1/s} \\
&\leq C \cdot C_r \Big(\int_S (\sup_j \tilde{w}_{j,1}(\xi))^s \, d\sigma(\xi) \Big)^{1/s} \\
&\leq C \Big(\int_S (\sup_j v_{j,1}(\xi))^{s'} \, d\sigma(\xi) \Big)^{1/s}.
\end{aligned}$$

Since different apertures give equivalent norms, this implies

$$2^{-s}\|v\|_{\infty,s'}^{s'} \le C\|v\|_{\infty,s'}^{s'/s}$$

and that suffices.

To finish the proof of the theorem we have to show that 4) implies 3). Let us choose an ε-separated sequence z_k in B such that the nonisotropic balls $E_\delta(z_k)$ cover B for some $\delta < 1$. There is a $\delta_1 \in (\delta, 1)$ such that $E_\delta(w) \subset E_{\delta_1}(z)$ whenever $w \in E_\delta(z)$. If $q < 2$, we use

$$\begin{aligned}\int_{D(\xi)} g(z)^{2/(2-q)}\,d\tau(z) &\le \sum_{z_k\in D(\xi)} \int_{E_\delta(z_k)} g(z)^{2/(2-q)}\,d\tau(z)\\ &\le C \sum_{z_k\in D(\xi)} \sup_{E_\delta(z_k)} g(z)^{2/(2-q)}\\ &\le C \sum_{z_k\in D(\xi)} \left(\frac{\mu(E_{\delta_1}(z_k))}{(1-|z|)^{mq+n}}\right)^{2/(2-q)}.\end{aligned}$$

The case $q \ge 2$ is even simpler, and we leave it to the reader. □

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