ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 3, Fall 1998

TORAL ARRANGEMENTS AND HYPERPLANE ARRANGEMENTS

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ABSTRACT. We consider pairs, $(T, \mathcal{A})$, where T is a torus and $\mathcal{A}$ is a finite set of characters of T. Then $d\mathcal{A} = \{\ker(d\chi) \mid \chi \in \mathcal{A}\}$ is a finite set of hyperplanes in the Lie algebra of T. Let $\mathcal{O}_T$ be the coordinate ring of T, and $\mathcal{O}_{T,e}$ the local ring of the identity in T. In analogy with hyperplane arrangements, put $y = \prod_i (\chi_i - 1)$, and consider the set, $D(\mathcal{A})$, of derivations, θ, of $\mathcal{O}_T$ that satisfy $\theta(y) \in y\mathcal{O}_T$. The main results are that the localization of $D(\mathcal{A})$ at the identity of T is a free $\mathcal{O}_{T,e}$-module if and only if $d\mathcal{A}$ is a free hyperplane arrangement, and that if this is the case, then the exponents of $d\mathcal{A}$ can be recovered from $\mathcal{A}$.

1. Introduction. Let k be an algebraically closed field of characteristic zero. In analogy with the definition of a hyperplane arrangement, we will define a toral arrangement to be a pair, $(T, \mathcal{A})$, where T is a torus defined over k and $\mathcal{A} = \{\chi_1, \dots, \chi_s\}$ is a finite set of characters of T. Let $\mathfrak{t}$ be the Lie algebra of T. Then if χ is a character of T, its derivative d_χ is a linear functional on $\mathfrak{t}$. Let $d\mathcal{A}$ be the multiset of hyperplanes $\{\ker(d\chi_1), \dots, \ker(d\chi_s)\}$. Then the distinct hyperplanes in $d\mathcal{A}$ form a central hyperplane arrangement in $\mathfrak{t}$. In this way, a central hyperplane arrangement is canonically associated with each toral arrangement.

Define $y = \prod_{i=1}^s (\chi_i - 1)$ in $\mathcal{O}_T$ and $y' = \prod_{i=1}^s d\chi_i$ in $\mathcal{O}_\mathfrak{t}$. Let $\mathrm{Der}_k(\mathcal{O}_T)$ and $\mathrm{Der}_k(\mathcal{O}_t)$ be the modules of k-linear derivations of $\mathcal{O}_T$ and $\mathcal{O}_\mathfrak{t}$, respectively. Then $D(d\mathcal{A})$ is defined to be $\{\theta \in \mathrm{Der}_k(\mathcal{O}_t) \mid \theta(y') \in y'\mathcal{O}_\mathfrak{t}\}$, and $d\mathcal{A}$ is said to be free if $D(d\mathcal{A})$ is a free $\mathcal{O}_\mathfrak{t}$-module. Terao [**7**, Theorem 2.5] has shown, using the homogeneity of y', that $D(d\mathcal{A})$ is a free $\mathcal{O}_\mathfrak{t}$-module if and only if its localization at $\mathfrak{m}_0$ is a free $\mathcal{O}_{\mathfrak{t},0}$-module. Notice that we have modified the standard definitions slightly, see [**6**], since the linear functions $\{d_{\chi_j} \mid 1 \le j \le s\}$ are not necessarily distinct. It will be shown in Section 2, after Corollary 2.5, that our definition of $D(d\mathcal{A})$ gives the same $\mathcal{O}_\mathfrak{t}$-module as the

Received by the editors on November 7, 1994, and in revised form on December 11, 1996.

module of derivations of $\mathcal{O}_\mathfrak{t}$ usually associated with the arrangement of distinct hyperplanes in $d\mathcal{A}$. For a toral arrangement, we define $D(\mathcal{A}) = \{\theta \in \mathrm{Der}_k(\mathcal{O}_T) \mid \theta(y) \in y\mathcal{O}_T\}$, but we will say that $(T, \mathcal{A})$, or just $\mathcal{A}$, is free if $D(\mathcal{A})_e$ is a free $\mathcal{O}_{T,e}$-module, where e is the identity in T.

Our main results are that $D(\mathcal{A})_e$ is a free $\mathcal{O}_{T,e}$-module if and only if $D(d\mathcal{A})$ is a free $\mathcal{O}_t$-module and that if this is the case, then the exponents of $(\mathfrak{t}, d\mathcal{A})$ can be recovered from $D(\mathcal{A})_e$. In particular, $(T, \mathcal{A})$ is a free toral arrangement if and only if $(\mathfrak{t}, d\mathcal{A})$ is a free hyperplane arrangement.

Since $\mathfrak{t}$ depends only on a neighborhood of the identity in T, it is plausible that $d\mathcal{A}$ only gives information near e. We will give an example in Section 4 which shows that this is indeed the case. Thus, it can happen that $D(\mathcal{A})_e$ is free, but $D(\mathcal{A})$ is not.

Our motivation for considering toral arrangements is a result of Lehrer and Shoji [**4**] in which the exponents of certain restricted arrangements are shown to arise in the decomposition of Springer representations of Weyl groups. In that setup, G is a complex reductive Lie group, L is a Levi factor of a parabolic subgroup of G, T_0 is a maximal torus in L and T is the center of L. The characters involved are the restrictions to T of the roots of (G, T_0). If u is a regular unipotent element in L and $\mathfrak{B}_u$ is the variety of Borel subgroups of G that contain u, then T acts on $\mathfrak{B}_u$, and the Weyl group, W, acts on the cohomology of $\mathfrak{B}_u$. It can be shown that the set $\{\mathrm{Lie}\,(T_x) \mid x \in \mathfrak{B}_u\}$, where T_x is the stabilizer of x, is precisely the lattice of subspaces of the corresponding hyperplane arrangement $(\mathfrak{t}, d\mathcal{A})$. Lehrer and Shoji show that the reflection representation of W appears in $H^*(\mathfrak{B}_u)$ in degrees given by the exponents of $d\mathcal{A}$. The results in this paper show that the exponents of $d\mathcal{A}$, and hence the multiplicity of the reflection representation in the $H^*(\mathfrak{B}_u)$, can be computed in the coordinate ring of T without passing to Lie algebras.

More generally, suppose T is any torus and V is a finite dimensional, rational T-module. Let X be a T-stable subvariety of V or a T-stable subvariety of the projective space of V. For $x \in X$, the stabilizer of x in T is then an intersection of kernels of characters of T. It can happen that there are characters $\chi_1, \dots, \chi_s$ of T so that $\{T_x \mid x \in X\} = \{\cap_{j\in J} \ker(\chi_j) \mid J \subseteq \{1, \dots, s\}\}$. This happens, for example, if $X = V$

or $X = \mathbf{P}(V)$. If $\{T_x \mid x \in X\} = \{\cap_{j\in J} \ker(\chi_j) \mid J \subseteq \{1, \dots, s\}\}$, then there is a hyperplane arrangement canonically associated with the action of T on X and it might be hoped that this hyperplane arrangement controls in some way the structure of X. In order to realize such a hope, the first step is to understand the relationships between toral arrangements and their associated hyperplane arrangements.

Our approach to toral arrangements is based on an explicit isomorphism constructed by Atiyah and Hirzebruch [**1**, Section 4.3] from the completed representation ring of T to the completion of the ring of regular functions on $\mathfrak{t}$ and on an analysis of the completions of $D(\mathcal{A})$ and $D(d\mathcal{A})$. More generally, if k is a commutative ring, A is a commutative k-algebra, and y is in A, we can consider the A-module $D(A, y) = \{\theta \in \mathrm{Der}_k(A) \mid \theta(y) \in yA\}$. In Section 2 we prove some general results about these modules as well as consider how derivations behave relative to completion. In Section 3 we discuss the factorization of regular functions on T of the form $\chi - 1$ where χ is a character of T. Finally, in Section 4 we apply the results of Sections 2 and 3 to toral arrangements and prove the main results mentioned above.

2. Derivations and completion. In this section we will start with a commutative, Noetherian ring, k, a finitely generated, commutative k-algebra, A, and an ideal, $\mathfrak{m}$, of A. We will add hypotheses on k, A and $\mathfrak{m}$ as necessary.

Let $\Omega_{A/k}$ be the module of Kähler differentials of A over k, and let $d_{A/k}$ be the natural A-linear map from A to $\Omega_{A/k}$. We will denote the A-module of k-linear derivations of A by $\mathrm{Der}_k(A)$. Then $\mathrm{Der}_k(A)$ can be naturally identified with $\mathrm{Hom}_A(\Omega_{A/k}, A)$. If θ is in $\mathrm{Der}_k(A)$, we will often identify θ with the unique A-linear map $\Omega_{A/k} \to A$ it determines. With this convention, if y is in A, then $\theta d_{A/k}(y) = \theta(y)$.

All A-modules will be considered with the $\mathfrak{m}$-adic topology. If M is an A-module, then $\widehat{M}$ will denote the completion of M. Recall that if M is finitely generated, then $\widehat{M}$ is naturally isomorphic to $\hat{A} \otimes_A M$, that $\hat{A}$ is a flat A-module, and that the $\hat{\mathfrak{m}}$-adic topology is the same as the $\mathfrak{m}$-adic topology for any $\hat{A}$-module [**2**, Chapter 10].

We first consider how derivations behave with respect to completion.

Lemma 2.1. *If M is an A-module and $\theta \in \mathrm{Der}_k(A, M)$, then θ is continuous.*

Proof. Given $f_1, \dots, f_m$ in $\mathfrak{m}$, $\theta(f_1 \cdots f_m) = \sum_{i=1}^m f_1 \cdots \theta(f_i) \cdots f_m \in \mathfrak{m}^{m-1}M$. Thus, $\theta(\mathfrak{m}^m) \subseteq \mathfrak{m}^{m-1}M$ and so θ is continuous. □

Lemma 2.2. *Suppose that X is an $\hat{A}$-module and $\theta : A \to X$ is a k-linear derivation. Then θ extends to a k-derivation $\tilde{\theta} : \hat{A} \to X$.*

Proof. Since $\Omega_{A/k}$ is a finitely generated A-module, so is the image of θ. Let Y be the $\hat{A}$-submodule of X generated by the image of θ, so Y is a finitely generated, hence complete, $\hat{A}$-module. Suppose $\hat{a} \in \hat{A}$ and $\{a_n\}_{n=1}^\infty$ is a Cauchy sequence in A converging to $\hat{a}$. Then by Lemma 2.1, $\{\theta(a_n)\}_{n=1}^\infty$ is a Cauchy sequence in Y. Since $\theta(0) = 0$, if $\{b_n\}_{n=1}^\infty$ is another Cauchy sequence in A converging to $\hat{a}$, then $\{\theta(a_n)\}_{n=1}^\infty$ and $\{\theta(b_n)\}_{n=1}^\infty$ both converge to the same element of Y. Thus we can define $\tilde{\theta}(\hat{a}) = \lim_{n\to\infty} \theta(a_n)$. Since addition and multiplication are continuous, it follows that $\tilde{\theta}$ is a derivation. Finally, extending the range of $\tilde{\theta}$ to all of X gives the result. □

Let $\widetilde{\mathrm{Der}}_k(A)$ denote the $\hat{A}$ submodule of $\mathrm{Der}_k(\hat{A})$ generated by the extensions, $\tilde{\theta}$, for $\theta \in \mathrm{Der}_k(A)$.

Proposition 2.3. *The $\hat{A}$ module, $\mathrm{Der}_A(\hat{A})$, of A-linear derivations of $\hat{A}$ is a complement to $\widetilde{\mathrm{Der}}_k(A)$ in $\mathrm{Der}_k(\hat{A})$. Thus, $\mathrm{Der}_k(\hat{A}) \cong \widetilde{\mathrm{Der}}_k(A) \oplus \mathrm{Der}_A(\hat{A})$.*

Proof. It follows from [**5**, Theorem 57], using Lemma 2.2, that there is a split exact sequence of $\hat{A}$-modules,

$$0 \longrightarrow \hat{A} \otimes_A \Omega_{A/k} \xrightarrow{v} \Omega_{\hat{A}/k} \longrightarrow \Omega_{\hat{A}/A} \longrightarrow 0,$$

where $v(\hat{a} \otimes d_{A/k}b) = \hat{a}d_{\hat{A}/k}b$. Since $\hat{A}$ is a flat A-algebra, $\mathrm{Hom}_{\hat{A}}(\hat{A} \otimes_A \Omega_{A/k}, \hat{A}) \cong \hat{A} \otimes_A \mathrm{Der}_k(A)$ and so we can identify the exact sequence of dual $\hat{A}$-modules with

$$0 \longrightarrow \mathrm{Der}_A(\hat{A}) \to \mathrm{Der}_k(\hat{A}) \xrightarrow{v^*} \hat{A} \otimes_A \mathrm{Der}_k(A) \longrightarrow 0, \tag{2.3a}$$

where v^* is induced by v. It is straightforward to check that if θ is in $\mathrm{Der}_k(A)$, then $v^*(\tilde{\theta}) = 1 \otimes \theta$, so $1 \otimes \theta \mapsto \tilde{\theta}$ determines a splitting of the sequence (2.3a) whose image in $\mathrm{Der}_k(\hat{A})$ is $\widetilde{\mathrm{Der}_k}(A)$. It follows that $\mathrm{Der}_k(\hat{A}) \cong \widetilde{\mathrm{Der}_k}(A) \oplus \mathrm{Der}_A(\hat{A})$ as claimed. □

As in the introduction, if y is in A, define $D(A, y) = \{\theta \in \mathrm{Der}_k(A) \mid \theta(y) \in yA\}$. Then $D(A, y)$ is an A-submodule of $\mathrm{Der}_k(A)$. In the next proposition we collect several simple properties of $D(A, y)$.

Proposition 2.4. *With the preceding notation*

(a) *If u is a unit in A, then $D(A, u) = \mathrm{Der}_k(A)$ and $D(A, yu) = D(A, y)$.*

(b) *If S is a multiplicatively closed subset of A, then the natural isomorphism given by the quotient rule from $S^{-1}\mathrm{Der}_k(A)$ to $\mathrm{Der}_k(S^{-1}A)$ restricts to an isomorphism from $S^{-1}D(A, y)$ to $D(S^{-1}A, y)$.*

(c) *If A is an integral domain, y_1 and y_2 are elements in A, and $y_1y_2A = y_1A \cap y_2A$, then $D(A, y_1y_2) = D(A, y_1) \cap D(A, y_2)$.*

(d) *If k is a field with characteristic zero, A is an integral domain and n is a positive integer, then $D(A, y^n) = D(A, y)$.*

Proof. The proof of (a) is straightforward and will be omitted.

To prove (b), suppose $\theta \in \mathrm{Der}_k(A)$. Let θ' denote the extension of θ to a derivation of $S^{-1}A$. Clearly, if $\theta \in D(A, y)$, then $\theta' \in D(S^{-1}A, y)$. Hence, $S^{-1}D(A, y) \subseteq D(S^{-1}A, y)$. Conversely, suppose $\eta \in D(S^{-1}A, y)$. Then $\eta = \sum_{i=1}^{n}(a_i/b_i)\theta_i'$ where the a_i's are in A, the b_i's are in S, and the θ_i's are in $\mathrm{Der}_k(A)$. Suppose $\eta(y) = (c/d)y$ where c is in A and d is in S. Put $\bar{b} = \prod_{i=1}^{n} b_i$. Then $\sum_{i=1}^{n} \bar{b}d(a_i/b_i)\theta_i(y) = \bar{b}cy$, so $\sum_{i=1}^{n} \bar{b}d(a_i/b_i)\theta_i \in D(A, y)$ and $\eta = (1/\bar{b}d) \sum_{i=1}^{n} \bar{b}d(a_i/b_i)\theta_i \in S^{-1}D(A, y)$.

Now suppose that A is an integral domain and $\theta \in D(A, y_1y_2)$. Then

$$\theta(y_1y_2) = y_1\theta(y_2) + y_2\theta(y_1) \in y_1y_2A = y_1A \cap y_2A.$$

Thus $y_1\theta(y_2) + y_2\theta(y_1) \in y_1A$, so $y_2\theta(y_1) \in y_1A$. It follows that $y_2\theta(y_1) \in y_1A \cap y_2A = y_1y_2A$. Since A is an integral domain, we must have $\theta(y_1) \in y_1A$. Similarly, $\theta(y_2) \in y_2A$. Thus $D(A, y_1, y_2) \subseteq$

$D(A, y_1) \cap D(A, y_2)$. An easy computation shows that the reverse containment always holds. This completes the proof of (c).

Finally, suppose that k is a field with characteristic zero and A is an integral domain. If $\theta \in \mathrm{Der}_k(A)$, then $\theta(y^n) = ny^{n-1}\theta(y)$, so $\theta(y^n) \in y^n A$ if and only if $\theta(y) \in yA$. Thus, $D(A, y^n) = D(A, y)$. This completes the proof of the proposition. □

Corollary 2.5. *If $y_1, \dots, y_n$ are elements in A with $y_1 \cdots y_n A = y_1 A \cap \cdots \cap y_n A$ and A is an integral domain, then $D(A, y_1 \cdots y_n) = D(A, y_1) \cap \cdots \cap D(A, y_n)$.*

Proof. This results follows easily by induction from Proposition 2.4(c) since, with the given hypotheses on $y_1, \dots, y_n$, we have $y_1 \cdots y_n A = y_1 A \cap y_2 \cdots y_n A$. We will omit the details. □

Notice that it follows from Corollary 2.5 and Proposition 2.4(d) that, with the notation in the introduction, if y_1' is the product of the distinct factors of y', then $D(\mathcal{O}_{\mathfrak{t}}, y') = D(\mathcal{O}_{\mathfrak{t}}, y_1')$. Thus our definition of $D(d\mathcal{A})$ agrees with the usual one.

For the rest of this section, we will assume that A is an integral domain, that $\Omega_{A/k}$ is a free A-module, and that finitely generated projective A-modules are free. These conditions all hold, for example, if A is the coordinate ring of a torus or an affine space, or if A is a regular local ring. Let l be the rank of $\mathrm{Der}_k(A)$.

We next consider the $\hat{A}$ submodule of $\mathrm{Der}_k(\hat{A})$ generated by $\{\tilde{\theta} \mid \theta \in D(A, y)\}$. Denote this submodule by $\tilde{D}(A, y)$. Since $\hat{A}$ is a flat A-algebra, we can consider $\hat{A} \otimes_A D(A, y)$ as a submodule of $\hat{A} \otimes_A \mathrm{Der}_k(A)$. It follows that $\tilde{D}(A, y) \cong \hat{A} \otimes_A D(A, y)$. Also, A is an integral domain, so we can consider A as a subring of $\hat{A}$. In particular, $D(\hat{A}, y)$ is defined for y in A. If $\theta \in D(A, y)$, then clearly $\tilde{\theta}(y) = \theta(y) \in yA \subseteq y\hat{A}$, so $\tilde{D}(A, y) \subseteq D(\hat{A}, y)$. Moreover, any A-linear derivation of $\hat{A}$ is obviously in $D(\hat{A}, y)$ and so $\tilde{D}(A, y) + \mathrm{Der}_A(\hat{A}) \subseteq D(\hat{A}, y)$. Our next goal is to show that, under certain circumstances, $\tilde{D}(A, y) = D(\hat{A}, y) \cap \widetilde{\mathrm{Der}_k}(A)$. It then follows easily that we have a direct sum decomposition, analogous to that of Proposition 2.3, $D(\hat{A}, y) \cong \tilde{D}(A, y) \oplus \mathrm{Der}_A(\hat{A})$.

If $\omega \in \Omega_{A/k}$, then evaluation at ω is an A-valued function on its dual, $\mathrm{Der}_k(A)$. We will denote this function also by ω. Hence, if θ is in $\mathrm{Der}_k(A)$, then $\omega(\theta) = \theta(\omega)$ and $d_{A/k}y(\theta) = \theta(y)$ for y in A. Because $\Omega_{A/k}$ and $\mathrm{Der}_k(A)$ are dual A-modules, there is a natural pairing, given by evaluation, from $\Omega_{A/k} \times \mathrm{Der}_k(A)$ to A. This pairing will be denoted by $\langle \cdot, \cdot \rangle$.

Proposition 2.6. *Assume A is an integral domain, $\Omega_{A/k}$ is a free A-module, and finitely generated projective A-modules are free. Suppose $y \in A$ and $d_{A/k}y(\mathrm{Der}_k(A)) = A$. Then $D(A, y)$ is a free A-module.*

Proof. If $d_{A/k}y(\mathrm{Der}_k(A)) = A$, then we can find $\theta \in \mathrm{Der}_k(A)$ so that $d_{A/k}y(\theta) = 1$. Because $\mathrm{Der}_k(A)$ is a free A-module, we can identify θ with the A-linear functional on $\Omega_{A/k}$ defined by $\omega \mapsto \omega(\theta)$. Then θ is a linear functional on $\Omega_{A/k}$ which takes the value 1 on $d_{A/k}y$, and using our assumption that finitely generated projective A-modules are free, it follows that there is a basis of $\Omega_{A/k}$ containing $d_{A/k}y$. Notice that, conversely, if $d_{A/k}y$ is contained in a basis, then $d_{A/k}y(\mathrm{Der}_k(A)) = A$.

Since $\Omega_{A/k}$ has a basis containing $d_{A/k}y$, we can choose dual bases $\{\omega_1, \dots, \omega_l\}$ and $\{D_1, \dots, D_l\}$ of $\Omega_{A/k}$ and $\mathrm{Der}_k(A)$, respectively, with $d_{A/k}y = \omega_1$. Thus, $\langle \omega_i, D_j \rangle = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta.

Define $\theta_1 = yD_1$ and, for $i > 1$, define $\theta_i = D_i$. Then $\theta_1(y) = yD_1(y) \in yA$ and, for $i > 1$, $\theta_i(y) = \langle d_{A/k}y, \theta_i \rangle = \langle \omega_1, \theta_i \rangle = 0$. Thus, $\{\theta_1, \dots, \theta_l\}$ is contained in $D(A, y)$. We will show that $\{\theta_1, \dots, \theta_l\}$ is a basis of $D(A, y)$. Linear independence follows from the linear independence of $\{D_1, \dots, D_l\}$ and the assumption that A is an integral domain. So suppose θ is in $D(A, y)$ and $\theta(y) = ya$ where $a \in A$. Then

$$\begin{aligned}\theta &= \langle d_{A/k}y, \theta \rangle D_1 + \sum_{i=2}^{l} \langle \omega_i, \theta \rangle D_i \\ &= \theta(y) D_1 + \sum_{i=2}^{l} \langle \omega_i, \theta \rangle \theta_i = a\theta_1 + \sum_{i=2}^{l} \langle \omega_i, \theta \rangle \theta_i.\end{aligned}$$

Thus $\{\theta_1, \dots, \theta_l\}$ spans $D(A, y)$. This completes the proof of the proposition. □

Corollary 2.7. *With the assumptions of Proposition* 2.6, $\tilde{D}(A,y) = D(\hat{A},y) \cap \widetilde{\mathrm{Der}}_k(A)$ *and* $D(\hat{A},y) \cong \tilde{D}(A,y) \oplus \mathrm{Der}_A(\hat{A})$.

Proof. For $1 \leq i \leq l$, let D_i, ω_i and θ_i be as in the proof of Proposition 2.6, and let $\tilde{D}_i$ and $\tilde{\theta}_i$ be the extensions of D_i and θ_i to derivations of $\hat{A}$, respectively. Then, since $\tilde{D}(A,y) \cong \hat{A} \otimes_A D(A,y)$, $\{\tilde{\theta}_1, \dots, \tilde{\theta}_l\}$ is a basis of $\tilde{D}(A,y)$. Also, it is easily seen that $\tilde{\theta}_1 = y\tilde{D}_1$.

Using the natural isomorphism, $\widetilde{\mathrm{Der}}_k(A) \cong \mathrm{Hom}_{\hat{A}}(\hat{A} \otimes_A \Omega_{A/k}, \hat{A})$, we may consider the $\tilde{D}_i$'s as $\hat{A}$-linear functionals on $\hat{A} \otimes_A \Omega_{A/k}$. With this identification, it is easily checked that $\{\tilde{D}_1, \dots, \tilde{D}_l\}$ is the basis dual to the basis $\{1 \otimes \omega_1, \dots, 1 \otimes \omega_l\}$ of $\hat{A} \otimes_A \Omega_{A/k}$.

Suppose $\hat{\theta}$ is in $\mathrm{Der}_k(\hat{A})$, and recall the map v from the proof of Proposition 2.3. Then $\hat{\theta}(v(1 \otimes d_{A/k}y)) = \hat{\theta}(d_{\hat{A}/k}y) = \hat{\theta}(y)$. Now suppose $\hat{\theta}$ is in $D(\hat{A},y) \cap \widetilde{\mathrm{Der}}_k(A)$, say $\hat{\theta} = \sum_{i=1}^{l} \hat{a}_i \tilde{D}_i$ and $\hat{\theta}(y) = y\hat{b}$ where the $\hat{a}_i$'s and $\hat{b}$ are in $\hat{A}$. Then $\hat{a}_1 = \langle 1 \otimes d_{A/k}y, \hat{\theta} \rangle = \hat{\theta}(y) = y\hat{b}$, so $\hat{\theta} = \hat{b}\tilde{\theta}_1 + \sum_{i=2}^{l} \hat{a}_i \tilde{\theta}_i$. Thus, $\{\tilde{\theta}_1, \dots, \tilde{\theta}_l\}$ spans $D(\hat{A},y) \cap \widetilde{\mathrm{Der}}_k(A)$, and so $D(\hat{A},y) \cap \widetilde{\mathrm{Der}}_k(A) \subseteq \tilde{D}(A,y)$. Conversely, $\tilde{D}(A,y) \subseteq D(\hat{A},y) \cap \widetilde{\mathrm{Der}}_k(A)$. Therefore, $\tilde{D}(A,y) = D(\hat{A},y) \cap \widetilde{\mathrm{Der}}_k(A)$.

Clearly, $\tilde{D}(A,y) \cap \mathrm{Der}_A(\hat{A}) = 0$. If $\hat{\theta}$ is in $D(\hat{A},y)$, then by Proposition 2.3, $\hat{\theta} = \tilde{\eta} + \psi$ where $\tilde{\eta}$ is in $\widetilde{\mathrm{Der}}_k(A)$ and ψ is in $\mathrm{Der}_A(\hat{A})$. Then $\hat{\theta} - \psi = \tilde{\eta}$ is in $D(\hat{A},y) \cap \widetilde{\mathrm{Der}}_k(A) = \tilde{D}(A,y)$. It follows that $D(\hat{A},y) = \tilde{D}(A,y) + \mathrm{Der}_A(\hat{A})$, and so $D(\hat{A},y) \cong \tilde{D}(A,y) \oplus \mathrm{Der}_A(\hat{A})$. This completes the proof of the corollary. □

We can now state the main result of this section.

Theorem 2.8. *Assume A is an integral domain, $\Omega_{A/k}$ is a free A-module, and finitely generated projective A-module are free. Suppose $y_1, \dots, y_n$ are elements of A satisfying*:

(a) $y_1 \cdots y_n A = y_1 A \cap \cdots \cap y_n A$ *and*

(b) $d_{A/k} y_i(\mathrm{Der}_k(A)) = A$ *for* $1 \leq i \leq n$.

Let $y = y_1 \cdots y_n$. *Then* $\tilde{D}(A,y) \cong D(\hat{A},y) \cap \widetilde{\mathrm{Der}}_k(A)$ *and* $D(\hat{A},y) \cong \tilde{D}(A,y) \oplus \mathrm{Der}_A(\hat{A})$.

Proof. First notice that, since $\hat{A}$ is a flat A-algebra, $\hat{A}y = \hat{A}(Ay) = \hat{A}(\cap_{i=1}^{n} y_i A) = \cap_{i=1}^{n}(\hat{A}Ay_i) = \cap_{i=1}^{n}\hat{A}y_i$. It follows from Corollary 2.5, applied to A and $\hat{A}$, that $D(A,y) = \cap_{i=1}^{n} D(A,y_i)$ and $D(\hat{A},y) = \cap_{i=1}^{n} D(\hat{A},y_i)$. Thus, again using that $\hat{A}$ is flat as well as Corollary 2.7, we see

$$\begin{aligned}
\hat{A}\otimes_A D(A,y) &= \hat{A}\otimes_A \left(\bigcap_{i=1}^{n} D(A,y_i)\right) \\
&= \bigcap_{i=1}^{n}(\hat{A}\otimes_A D(A,y_i)) \\
&\cong \bigcap_{i=1}^{n}(D(\hat{A},y_i)\cap \widetilde{\mathrm{Der}}_k(A)) \\
&= \left(\bigcap_{i=1}^{n} D(\hat{A},y_i)\right)\cap \widetilde{\mathrm{Der}}_k(A) \\
&= D(\hat{A},y)\cap \widetilde{\mathrm{Der}}_k(A).
\end{aligned}$$

The argument in the proof of Corollary 2.7 shows that $D(\hat{A},y) \cong \tilde{D}(A,y)\oplus \mathrm{Der}_A(\hat{A})$. This completes the proof of the theorem. □

For the rest of this section, we will assume that k is a field, A is a finitely generated, regular, local k-algebra which is an integral domain, $\mathfrak{m}$ is the maximal ideal in A and A contains a field isomorphic to its residue field. Choose $x_1, \dots, x_l$ in $\mathfrak{m}$ so that $\{x_1+\mathfrak{m}^2, \dots, x_l+\mathfrak{m}^2\}$ is a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. It is shown by Hartshorne [**3**, Proposition II.8.7] that $x+\mathfrak{m}^2 \mapsto dx\otimes 1$ defines an isomorphism of k-vector spaces from $\mathfrak{m}/\mathfrak{m}^2$ to $\Omega_{A/k}\otimes_A k$, so it follows from Nakayama's lemma that $\{dx_1, \dots, dx_l\}$ is an A-basis of $\Omega_{A/k}$. Let $\{D_1, \dots, D_l\}$ be the dual basis of $\mathrm{Der}_k(A)$. Thus $D_i(x_j) = \delta_{i,j}$ for $1 \le i,j \le l$.

If M is an A-module, $m \in M$ and $p \in \mathbf{N}$, define $\nu(m) = p$ if and only if $m \in \mathfrak{m}^p M\backslash \mathfrak{m}^{p+1}M$ for some nonnegative integer, p, and define $\nu(m) = \infty$ if m is in $\mathfrak{m}^p M$ for all $p \ge 0$. Notice that, if $\hat{\theta} \in \mathrm{Der}_k(\hat{A})$, then $\nu(\hat{\theta}) = p$ if and only if $\hat{\theta} \in \mathfrak{m}^p\mathrm{Der}_k(\hat{A})\backslash \mathfrak{m}^{p+1}\mathrm{Der}_k(\hat{A})$ because $\hat{\mathfrak{m}}\hat{A} = \mathfrak{m}\hat{A}$.

Lemma 2.9. *If $\theta \in \operatorname{Der}_k(A)$, then $\nu(\theta) \geq p$ if and only if $\theta(\mathfrak{m}) \subseteq \mathfrak{m}^p$. Similarly, if $\hat{\theta} \in \operatorname{Der}_k(A)$, then $\nu(\hat{\theta}) \geq p$ if and only if $\hat{\theta}(\hat{\mathfrak{m}}) \subseteq \hat{\mathfrak{m}}^p$.*

Proof. Clearly, if $f \in \mathfrak{m}^p$ and $\theta \in \operatorname{Der}_k(A)$, then $f\theta(\mathfrak{m}) \subseteq \mathfrak{m}^p$. Hence, if $\nu(\theta) \geq p$, then $\theta(\mathfrak{m}) \subseteq \mathfrak{m}^p$. Conversely, suppose that $\theta(\mathfrak{m}) \subseteq \mathfrak{m}^p$. Then $\theta = \sum_{i=1}^{l} \langle dx_i, \theta \rangle D_i$, and $\langle dx_i, \theta \rangle = \theta(x_i) \in \mathfrak{m}^p$ for all i, so $\nu(\theta) \geq p$. The same argument applies to $\operatorname{Der}_k(\hat{A})$. This proves the lemma. $\square$

Proposition 2.10. *Suppose $\theta \in \operatorname{Der}_k(A)$ and $\tilde{\theta}$ is the extension of θ to a derivation of $\hat{A}$. Then $\nu(\theta) = \nu(\tilde{\theta})$.*

Proof. It suffices to show that $\nu(\theta) \geq p$ if and only if $\nu(\tilde{\theta}) \geq p$ for all $p \geq 0$. Using Lemma 2.9, it suffices to show that $\theta(\mathfrak{m}) \subseteq \mathfrak{m}^p$ if and only if $\tilde{\theta}(\hat{\mathfrak{m}}) \subseteq \hat{\mathfrak{m}}^p$.

Suppose $\tilde{\theta}(\hat{\mathfrak{m}}) \subseteq \hat{\mathfrak{m}}^p$. Then $\theta(\mathfrak{m}) \subseteq \tilde{\theta}(\hat{\mathfrak{m}}) \cap A \subseteq \hat{\mathfrak{m}}^p \cap A = \mathfrak{m}^p$, where the last equality follows from the fact that $\hat{A}$ is faithfully flat. Conversely, suppose $\theta(\mathfrak{m}) \subseteq \mathfrak{m}^p$ and $\hat{a} \in \hat{\mathfrak{m}}$. Notice that our assumptions on A and $\mathfrak{m}$ imply that the topology on A is Hausdorff. Choose a Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ contained in $\mathfrak{m}$ and converging to $\hat{a}$. Then the sequence $\{\theta(a_n)\}_{n=1}^{\infty}$ converges to $\tilde{\theta}(\hat{a})$, each $\theta(a_n)$ is in $\mathfrak{m}^p \subseteq \hat{\mathfrak{m}}^p$, and $\hat{\mathfrak{m}}^p \cong \hat{\mathfrak{m}}^p$ is complete, so $\tilde{\theta}(\hat{a}) \in \hat{\mathfrak{m}}^p$. This completes the proof of the proposition. $\square$

Finally, notice that, with the preceding notation, $\hat{A} \cong k[[x_1, \dots, x_n]]$, and for $f \in k[[x_1, \dots, x_n]]$, $\nu(f)$ is the degree of the leading form of f. Moreover, if $\theta = \sum_{i=1}^{l} f_i D_i \in \operatorname{Der}_k(A)$, then $\nu(\theta)$ is the minimum of $\{\nu(f_1), \dots, \nu(f_l)\}$.

3. Factorization in $\mathcal{O}_T$. In this section k is an algebraically closed field and $T \cong \mathbf{G}_m^l$ is a k-torus. Recall that the k-algebra of regular functions on T is $\mathcal{O}_T$ and the identity in T by e. Notice that $\mathcal{O}_T$ is a unique factorization domain. Our goal in this section is to describe the factorization into irreducibles of a regular function of the form $\chi - 1$ where χ is a character of T. If χ is a character of T, then $\ker^0(\chi)$ will denote the identity component of the kernel of χ. If f is any function in $\mathcal{O}_T$, $Z(f)$ will denote the zero set of f in T, and df will denote the derivative of f, a k-valued function on $\mathrm{Lie}\,(T)$.

Proposition 3.1. *Suppose χ_1 and χ_2 are characters of T. Then $\ker^0(\chi_1) = \ker^0(\chi_2)$ if and only if $\ker(d\chi_1) = \ker(d\chi_2)$.*

Proof. First notice that, for any character χ, any integer n, and any $t \in T$, $\chi(t) = 1$ implies $\chi(t)^n = 1$, so $\ker(\chi) \subseteq \ker(\chi^n)$. In particular, $\ker^0(\chi) = \ker^0(\chi^n)$ for all n.

Now assume that $\ker^0(\chi_1) = \ker^0(\chi_2)$. Then $\mathcal{O}_T$ is a unique factorization domain, so there is an irreducible regular function, $f \in \mathcal{O}_T$, so that $\ker^0(\chi_1) = \ker^0(\chi_2) = Z(f)$. Hence, f divides $\chi_1 - 1$ and $\chi_2 - 1$. Say $\chi_1 - 1 = g_1 f$ and $\chi_2 - 1 = g_2 f$. Taking derivatives, we have $d\chi_1 = d(\chi_1 - 1) = d(g_1 f) = g_1(e)\, df + f(e)\, dg_1$. But $f(e) = 0$, so $d\chi_1 = g_1(e)\, df$. Similarly, $d\chi_2 = g_2(e)\, df$. We may assume that χ_1 and χ_2 are not the trivial characters, so $d\chi_1$ and $d\chi_2$ are not identically zero. It follows that $g_1(e) \neq 0$ and $g_2(e) \neq 0$, so $\ker(d\chi_1) = \ker(df) = \ker(d\chi_2)$.

Conversely, suppose $\ker(d\chi_1) = \ker(d\chi_2)$. Let $\{z_1, \dots, z_l\}$ be a basis of the character group of T. Then $\{z_1, \dots, z_l\}$ are global coordinates on T and $\{dz_1, \dots, dz_l\}$ is a basis of the dual space of $\mathfrak{t}$. There are integers $n_1, \dots, n_l$ and $m_1, \dots, m_l$ so that $\chi_1 = z_1^{m_1} \cdots z_l^{m_l}$ and $\chi_2 = z_1^{n_1} \cdots z_l^{n_l}$. But then $d\chi_1 = \sum_{i=1}^{l} m_i dz_i$ and $d\chi_2 = \sum_{i=1}^{l} n_i dz_i$. Since $\ker(d\chi_1) = \ker(d\chi_2)$, it follows that there is a rational number, p/q, so that $n_i = (p/q) m_i$ for all i. Hence $pm_i = qn_i$ for all i, so $\chi_1^p = \chi_2^q$. Thus $\ker^0(\chi_1) = \ker^0(\chi_1^p) = \ker^0(\chi_2^q) = \ker^0(\chi_2)$. This completes the proof of Proposition 3.1. □

Proposition 3.2. *If χ_1 and χ_2 are characters of T with $\ker d\chi_1 \neq \ker d\chi_2$, then $\chi_1 - 1$ and $\chi_2 - 1$ are relatively prime in $\mathcal{O}_T$.*

Proof. Suppose $f \in \mathcal{O}_T$, f is irreducible and f divides both $\chi_1 - 1$ and $\chi_2 - 1$. Then $Z(f) \subseteq \ker(\chi_1) \cap \ker(\chi_2)$. But $Z(f)$, $\ker(\chi_1)$ and $\ker(\chi_2)$ are all hypersurfaces and $Z(f)$ is connected, which by Proposition 3.1 contradicts the assumption that $\ker d\chi_1 \neq \ker d\chi_2$. Therefore, no such f can exist and so $\chi_1 - 1$ and $\chi_2 - 1$ are relatively prime. □

Proposition 3.3. *Suppose χ is a character of T. Then there is a character χ_0 and an integer n so that $\chi = \chi_0^n$ and $\ker^0(\chi) = \ker(\chi_0)$.*

Proof. Put $S = \ker^0(\chi)$, and let $X(T), X(S)$ denote the character groups of T and S, respectively. Recall that restriction from T to S defines a split surjection from the character group of T to the character group of S. Thus, we can choose a basis, $\{\chi_1, \ldots, \chi_l\}$, of $X(T)$ so that, if $\chi' \in X(T)$, then $S \subseteq \ker(\chi')$ if and only if $\chi' = \chi_l^n$ for some n, and the span of $\{\chi_1, \ldots, \chi_{l-1}\}$ is isomorphic, via restriction, to $X(S)$. Now $S = \cap \ker(\chi')$ where the intersection is over all $\chi' \in X(T)$ with $S \subseteq \ker(\chi')$, so $S = \cap_{n\in\mathbf{Z}} \ker(\chi_l^n)$. As noted in the proof of Proposition 3.1, $\ker(\chi_l) \subseteq \ker(\chi_l^n)$ for all n, so $S = \ker(\chi_l)$. Moreover, $S \subseteq \ker(\chi)$, so $\chi = \chi_l^n$ for some n. Taking $\chi_0 = \chi_l$ gives the result. $\square$

Now suppose χ is a character of T and $\chi = \chi_0^n$ where χ_0 has a connected kernel. Let μ_n be the group of nth roots of unity in k. Then $\chi - 1 = \chi_0^n - 1 = \prod_{\zeta\in\mu_n}(\chi_0 - \zeta)$. We can find a one-parameter subgroup, λ, of T so that T is the direct product of the image of λ and the kernel of χ_0, and replacing λ by the natural map from $\mathbf{G}_m/\ker(\lambda)$ to T, if necessary, we may assume that λ is injective. It follows that $\chi_0 \circ \lambda$ is the identity character of $\mathbf{G}_m$. For $\zeta \in \mathbf{G}_m$ set $t_\zeta = \lambda(\zeta)$. Then, if $t \in T$, t can be expressed uniquely as $t = t_\zeta t'$ with $t' \in \ker(\chi_0)$ and so $\chi(t) = \chi_0(t_\zeta)^n \chi_0(t')^n = \chi_0\lambda(\zeta)^n = \zeta^n$. Hence, $t = t_\zeta t' \in \ker(\chi)$ if and only if $\zeta \in \mu_n$, so $\ker(\chi) = \coprod_{\zeta\in\mu_n} t_\zeta \ker(\chi_0)$.

It is easily seen that, if ζ_1 and ζ_2 are in $\mathbf{G}_m$ and $t' \in \ker(\chi_0)$, then $(\chi_0 - \zeta_1)(t_{\zeta_2}t') = \zeta_2 - \zeta_1$, so $t_{\zeta_2}t' \in Z(\chi_0 - \zeta_1)$ if and only if $\zeta_1 = \zeta_2$. It follows that $Z(\chi_0 - \zeta) = t_\zeta \ker(\chi_0)$ for all $\zeta \in \mathbf{G}_m$. Now, for $\zeta \in \mathbf{G}_m$, $Z(\chi_0 - \zeta) = t_\zeta \ker(\chi_0)$ is an irreducible hypersurface and $\mathcal{O}_T$ is a unique factorization domain, so $\chi_0 - \zeta$ is irreducible, and we have proved the following proposition:

Proposition 3.4. *Suppose χ is a character of T and $\chi = \chi_0^n$ where χ_0 has a connected kernel. Then $\chi - 1 = \prod_{\zeta\in\mu_n}(\chi_0 - \zeta)$ is the factorization of $\chi - 1$ into irreducible factors.*

4. Freeness and toral arrangements. In this section k is an algebraically closed field of characteristic zero and $(T, \mathcal{A})$ is a fixed toral arrangement. Say $\mathcal{A} = \{\chi_1, \dots, \chi_s\}$, so $d\mathcal{A} = \{\ker d\chi_1, \dots, \ker d\chi_s\}$. Define $y = \prod_{i=1}^s (\chi_i - 1)$ and $y' = \prod_{i=1}^s d\chi_i$. Then $D(\mathcal{A}) = D(\mathcal{O}_T, y)$ and $D(d\mathcal{A}) = D(\mathcal{O}_\mathfrak{t}, y')$. Recall that $\mathcal{A}$ is free if $D(\mathcal{A})_e$ is a free $\mathcal{O}_{T,e}$-module and $d\mathcal{A}$ is free if $D(d\mathcal{A})$ is a free $\mathcal{O}_t$-module. Using Proposition 2.4(b), we see that $\mathcal{A}$ is free if and only if $D(\mathcal{O}_{T,e}, y)$ is a free $\mathcal{O}_{T,e}$-module.

We can now state the main theorem of this paper.

Theorem 4.1. *Suppose $(T, \mathcal{A})$ is a toral arrangement. Then $\mathcal{A}$ is free if and only if the associated central hyperplane arrangement, $d\mathcal{A}$, is free.*

Proof. In order to apply the results of Section 2, we need two lemmas.

Lemma 4.2. *Suppose χ is a nontrivial character of T. Then $d_{\mathcal{O}_T/k}(\chi - 1)(\mathrm{Der}_k(\mathcal{O}_T)) = \mathcal{O}_T$.*

Proof. Choose a basis of $X(T)$, say $\{z_1, \dots, z_l\}$. Then $\chi = z_1^{n_1} \cdots z_l^{n_l}$ for some integers $n_1, \dots, n_l$, $D_i = z_i(\partial/\partial z_i)$ is a derivation of $\mathcal{O}_T$, and $\{D_1, \dots, D_l\}$ is a basis of $\mathrm{Der}_k(\mathcal{O}_T)$. Since χ is not the trivial character, we can choose i so that $n_i \neq 0$. Then $d_{\mathcal{O}_T/k}(\chi - 1)(D_i) = D_i(\chi - 1) = n_i\chi$ is a unit in $\mathcal{O}_T$. Thus, the image of $d_{\mathcal{O}_T/k}(\chi - 1)$ contains a unit. Now $d_{\mathcal{O}_T/k}(\chi - 1)$ is $\mathcal{O}_T$-linear, so it follows that $d_{\mathcal{O}_T/k}(\chi - 1)$ is surjective. $\square$

We next discuss the isomorphism from $\widehat{\mathcal{O}}_T$ to $\widehat{\mathcal{O}}_\mathfrak{t}$. Suppose χ is a character of T, so $d\chi$ is a linear functional on $\mathfrak{t}$. Let $e^{d\chi}$ denote the power series $\sum_{n=0}^\infty (d\chi)^n/n!$ in $\widehat{\mathcal{O}}_\mathfrak{t}$. It is shown by Atiyah and Hirzebruch [**1**, Section 4.3] that $\chi \mapsto e^{d\chi}$ extends linearly to an isomorphism of k algebras from $\widehat{\mathcal{O}}_T$ to $\widehat{\mathcal{O}}_\mathfrak{t}$, which we will denote by ϕ. Let $\phi^\#$ denote the map from $\mathrm{Der}_k(\widehat{\mathcal{O}}_T)$ to $\mathrm{Der}_k(\widehat{\mathcal{O}}_\mathfrak{t})$ defined by $\phi^\#(\theta) = \phi\theta\phi^{-1}$. Then $\phi^\#$ is bijective and $\phi^\#(D(\widehat{\mathcal{O}}_T, f)) = D(\widehat{\mathcal{O}}_\mathfrak{t}, \phi(f))$ for $f \in \widehat{\mathcal{O}}_T$.

Lemma 4.3. *Suppose $\phi^{\#}$ is as above and χ is a character of T, then*
(a) $\phi^{\#}(\widetilde{\mathrm{Der}}_k(\mathcal{O}_T)) = \widetilde{\mathrm{Der}}_k(\mathcal{O}_{\mathfrak{t}})$,
(b) $\phi^{\#}(D(\widehat{\mathcal{O}}_T, \chi - 1)) = D(\widehat{\mathcal{O}}_{\mathfrak{t}}, d\chi)$, *and*
(c) $\phi^{\#}(\tilde{D}(\mathcal{O}_T, \chi - 1)) = \tilde{D}(\mathcal{O}_{\mathfrak{t}}, d\chi)$.

Proof. To prove (a), suppose $\{z_i \mid 1 \leq i \leq l\}$ is a basis of $X(T)$. Then $\{z_i(\partial/\partial z_i) \mid 1 \leq i \leq l\}$ is a basis of $\mathrm{Der}_k(\mathcal{O}_T)$ and $\{z_i\widetilde{(\partial/\partial z_i)} \mid 1 \leq i \leq l\}$ is a basis of $\widetilde{\mathrm{Der}}_k(\mathcal{O}_T)$. Define $x_i = dz_i$ for $1 \leq i \leq l$. Then $\{\partial/\partial x_i \mid 1 \leq i \leq l\}$ is a basis of $\mathrm{Der}_k(\mathcal{O}_{\mathfrak{t}})$ and $\{\widetilde{\partial/\partial x_i} \mid 1 \leq i \leq l\}$ is a basis of $\widetilde{\mathrm{Der}}_k(\mathcal{O}_{\mathfrak{t}})$. It suffices to show that $\phi^{\#}(z_i\widetilde{(\partial/\partial z_i)}) = \widetilde{\partial/\partial x_i}$ for $1 \leq i \leq l$.

Suppose $z_1^{n_1} \cdots z_l^{n_l}$ is in $\mathcal{O}_T$. Then

$$\begin{aligned} \phi \circ \widetilde{z_i \frac{\partial}{\partial z_i}}(z_1^{n_1} \cdots z_l^{n_l}) &= \phi(n_i z_1^{n_1} \cdots z_l^{n_l}) \\ &= n_i e^{n_1 x_1 + \cdots + n_l x_l} \\ &= \widetilde{\frac{\partial}{\partial x_i}}(e^{n_1 x_1 + \cdots + n_l x_l}) \\ &= \widetilde{\frac{\partial}{\partial x_i}} \circ \phi(z_1^{n_1} \cdots z_l^{n_l}). \end{aligned}$$

Therefore, $\phi \circ z_i\widetilde{(\partial/\partial z_i)}$ and $\widetilde{\partial/\partial x_i} \circ \phi$ agree on $\mathcal{O}_T$ and it follows that $\phi^{\#}(z_i\widetilde{(\partial/\partial z_i)}) = \widetilde{\partial/\partial x_i}$.

To prove (b), put $f = \sum_{n=1}^{\infty} (d\chi)^{n-1}/n!$. Then f is a unit in $\hat{\mathcal{O}}_{\mathfrak{t}}$ and $\phi^{\#}(\chi - 1) = e^{d\chi} - 1 = d\chi f$. Hence $\phi^{\#}(D(\hat{\mathcal{O}}_T, \chi - 1)) = D(\hat{\mathcal{O}}_{\mathfrak{t}}, d\chi_f) = D(\hat{\mathcal{O}}_{\mathfrak{t}}, d\chi)$ by Proposition 2.4(a).

Finally, (c) follows from (a), (b), and Corollary 2.7. □

We can now complete the proof of Theorem 4.1. We need to show that $D(\mathcal{O}_{T,e}, y)$ is a free $\mathcal{O}_{T,e}$-module if and only if $D(\mathcal{O}_{\mathfrak{t}}, y')$ is a free $\mathcal{O}_{\mathfrak{t}}$-module.

It is well known that $\widehat{\mathcal{O}}_T$ is naturally isomorphic to the completion of $\mathcal{O}_{T,e}$ in the $\mathfrak{m}_e\mathcal{O}_{T,e}$-adic topology, and hence that $\mathcal{O}_T$ is a faithfully flat

$\mathcal{O}_{T,e}$-algebra. Therefore, $D(\mathcal{O}_{T,e}, y)$ is a free $\mathcal{O}_{T,e}$-module if and only if its completion, $\mathcal{O}_T \otimes_{\mathcal{O}_{T,e}} D(\mathcal{O}_{T,e}, y) \cong \tilde{D}(\mathcal{O}_{T,e}, y)$, is a free $\widehat{\mathcal{O}}_T$-module.

As mentioned in the introduction, Terao has shown that $D(\mathcal{O}_{\mathfrak{t}}, y')$ is a free $\mathcal{O}_{\mathfrak{t}}$-module if and only if $D(\mathcal{O}_{\mathfrak{t},0}, y')$ is a free $\mathcal{O}_{\mathfrak{t},0}$-module. Applying the argument in the preceding paragraph to $\mathcal{O}_{\mathfrak{t}}$, $\mathcal{O}_{\mathfrak{t},0}$ and $\widehat{\mathcal{O}}_{\mathfrak{t}}$, we conclude that $D(\mathcal{O}_{\mathfrak{t}}, y')$ is a free $\mathcal{O}_{\mathfrak{t}}$-module if and only if $\tilde{D}(\mathcal{O}_{\mathfrak{t}}, y')$ is a free $\widehat{\mathcal{O}}_{\mathfrak{t}}$-module. Thus, it is enough to show that $\tilde{D}(\mathcal{O}_{T,e}, y)$ is a free $\widehat{\mathcal{O}}_T$-module if and only if $\tilde{D}(\mathcal{O}_{\mathfrak{t}}, y')$ is a free $\widehat{\mathcal{O}}_{\mathfrak{t}}$-module.

Notice that, if f is in $\mathcal{O}_T$, then using Proposition 2.4(b), gives

$$\tilde{D}(\mathcal{O}_{T,e}, f) \cong \widehat{\mathcal{O}}_T \otimes_{\mathcal{O}_{T,e}} D(\mathcal{O}_T, f)_e \cong \widehat{\mathcal{O}}_T \otimes_{\mathcal{O}_T} D(\mathcal{O}_T, f) \cong \tilde{D}(\mathcal{O}_T, f).$$

Assume the characters in $\mathcal{A}$ are labeled so that $\mathcal{A} = \{\chi_{i,j} \mid 1 \leq i \leq r, 1 \leq j \leq m_i\}$ for some positive integers $m_1, \ldots, m_r$, where $\ker d\chi_{i,j} = \ker d\chi_{i',j'}$ if and only if $i = i'$. For each i, choose a character, $\chi_{i,0}$, so that $\ker^0(\chi_{i,j}) = \ker(\chi_{i,0})$ for all j. Thus $\chi_{i,j}$ is a power of $\chi_{i,0}$ for every j. If χ is any character and n is a nonzero integer, then $\chi^n - 1 = (\chi - 1)(1 + \chi + \cdots + \chi^{n-1})$ and $1 + \chi + \cdots + \chi^{n-1}$ is a unit in $\mathcal{O}_{T,e}$. Hence, if $y_1 = \prod_{i=1}^r (\chi_{i,0} - 1)^{m_i}$, then $y = y_1 u$ where u is a unit in $\mathcal{O}_{T,e}$. By Proposition 2.4(a), $D(\mathcal{O}_{T,e}, y) = D(\mathcal{O}_{T,e}, y_1)$. It follows from Proposition 3.2 and the fact that $\mathcal{O}_T$ is a unique factorization domain that $y_1 \mathcal{O}_T = \cap_{i=1}^r ((\chi_{i,0} - 1)^{m_i} \mathcal{O}_T)$. It is easily seen that this last statement implies $y_1 \mathcal{O}_{T,e} = \cap_{i=1}^r ((\chi_{i,0} - 1)^{m_i} \mathcal{O}_{T,e})$. Thus, by Corollary 2.5 and Proposition 2.4(d), $D(\mathcal{O}_{T,e}, y) = \cap_{i=1}^r D(\mathcal{O}_{T,e}, \chi_{i,0} - 1)$. Therefore, taking completions we get

$$\tilde{D}(\mathcal{O}_{T,e}, y) = \bigcap_{i=1}^r \tilde{D}(\mathcal{O}_{T,e}, \chi_{i,0} - 1) = \bigcap_{i=1}^r \tilde{D}(\mathcal{O}_T, \chi_{i,0} - 1).$$

Now $y' = \alpha \prod_{i=1}^r (d\chi_{i,0})^{m_i}$, for some non-zero α in k, and the $d\chi_{i,0}$'s are relatively prime, so using Corollary 2.5 and Proposition 2.4(d) we have $D(\mathcal{O}_{\mathfrak{t}}, y') = \cap_{i=1}^r D(\mathcal{O}_{\mathfrak{t}}, d\chi_{i,0})$. Once again, taking completions, we get $\tilde{D}(\mathcal{O}_{\mathfrak{t}}, y') = \cap_{i=1}^r \tilde{D}(\mathcal{O}_{\mathfrak{t}}, d\chi_{i,0})$.

Finally, by Lemma 4.3(c), $\phi^\#(\tilde{D}(\mathcal{O}_T, \chi_{i,0} - 1)) = \tilde{D}(\mathcal{O}_{\mathfrak{t}}, d_{\chi_{i,0}})$ for $1 \leq i \leq r$. Therefore, $\phi^\#(\tilde{D}(\mathcal{O}_{T,e}, y)) = \tilde{D}(\mathcal{O}_{\mathfrak{t}}, y')$. It follows that $\tilde{D}(\mathcal{O}_{T,e}, y)$ is a free $\widehat{\mathcal{O}}_T$-module if and only if $\tilde{D}(\mathcal{O}_{\mathfrak{t}}, y')$ is a free $\widehat{\mathcal{O}}_{\mathfrak{t}}$-module. This completes the proof the theorem. □

Proposition 4.4. *Assume that $\mathcal{A}$ is free, $\{\theta_1, \dots, \theta_l\}$ is a basis of $D(\mathcal{O}_{T,e}, y)$, and ν is defined as in Section 2. Then $\{\nu(\theta_1), \dots, \nu(\theta_l)\}$ is the set of exponents of $d\mathcal{A}$.*

Proof. Choosing coordinates for $\mathfrak{t}$ we can identify $\mathcal{O}_\mathfrak{t}$ with a polynomial ring $k[x_1, \dots, x_l]$, $\mathfrak{m} = \mathfrak{m}_0$, with the ideal generated by $x_1, \dots, x_l$, and $\widehat{\mathcal{O}}_\mathfrak{t}$ with ring of formal power series $k[[x_1, \dots, x_l]]$. Then $\widehat{\mathcal{O}}_\mathfrak{t}$ is filtered by $(\widehat{\mathcal{O}}_\mathfrak{t})_p = \mathfrak{m}^p \widehat{\mathcal{O}}_\mathfrak{t}$. If M is any $\widehat{\mathcal{O}}_\mathfrak{t}$-module, we will consider M as filtered by $M_p = \mathfrak{m}^p M$ for $p \geq 0$. Let $\mathrm{Gr}\,(M)$ be the associated graded $\mathrm{Gr}\,(\widehat{\mathcal{O}}_\mathfrak{t})$-module. Then $\mathrm{Gr}\,(\widehat{\mathcal{O}}_\mathfrak{t}) \cong \mathcal{O}_\mathfrak{t}$ and Gr is an additive functor from the category of $\widehat{\mathcal{O}}_\mathfrak{t}$-modules to the category of $\mathcal{O}_\mathfrak{t}$-modules. In particular, if M is a free $\widehat{\mathcal{O}}_\mathfrak{t}$-module, then $\mathrm{Gr}\,(M)$ is a free $\mathcal{O}_\mathfrak{t}$-module.

Suppose M is a free $\widehat{\mathcal{O}}_\mathfrak{t}$-module and $\{m_1, \dots, m_n\}$ is a basis of M. Say $\nu(m_i) = p_i$ and define $\overline{m_i} = m_i + \mathfrak{m}^{p_i+1} M$ in $\mathrm{Gr}\,(M)$. Then $\overline{m_i}$ is homogeneous with degree $\nu(m_i)$ and it is easily checked that $\{\overline{m_1}, \dots, \overline{m}_n\}$ spans $\mathrm{Gr}\,(M)$. Now applying results of Orlik and Terao [**6**, Theorem A.19, Proposition A.24], we have that $\{\overline{m_1}, \dots, \overline{m_n}\}$ is a basis of $\mathrm{Gr}\,(M)$ and $\{\deg(\overline{m_1}), \dots, \deg(\overline{m_n})\}$ does not depend on $\{\overline{m_1}, \dots, \overline{m_n}\}$. Therefore, $\{\nu(m_1), \dots, \nu(m_n)\}$ does not depend on the choice of $\{m_1, \dots, m_n\}$.

For $1 \leq i \leq l$, let $\tilde{\theta}_i$ be the extension of θ_i to a derivation of $\widehat{\mathcal{O}}_T$ so $\{\tilde{\theta}_1, \dots, \tilde{\theta}_l\}$ is a basis of $\tilde{D}(\mathcal{O}_{T,e}, y)$. It follows from the proof of Theorem 4.1 that $\{\phi^\#(\tilde{\theta}_1), \dots, \phi^\#(\tilde{\theta}_l)\}$ is a basis of $\tilde{D}(\mathcal{O}_\mathfrak{t}, y')$. Now, by Proposition 2.10,

$$\begin{aligned}\{\nu(\theta_1), \dots, \nu(\theta_l)\} &= \{\nu(\tilde{\theta}_1), \dots, \nu(\tilde{\theta}_l)\} \\ &= \{\nu(\phi^\#(\tilde{\theta}_1)), \dots, \nu(\phi^\#(\tilde{\theta}_l))\}.\end{aligned}$$

Suppose $\{\theta'_1, \dots, \theta'_l\}$ is a homogeneous basis of $D(\mathcal{O}_\mathfrak{t}, y')$. Then the arguments in the preceding paragraphs apply as well to $D(\mathcal{O}_\mathfrak{t}, y')$ and so

$$\{\deg(\theta'_1), \dots, \deg(\theta'_l)\} = \{\nu(\phi^\#(\tilde{\theta}_1)), \dots, \nu(\phi^\#(\tilde{\theta}_l))\}.$$

Thus $\{\nu(\theta_1), \dots, \nu(\theta_l)\}$ is the set of exponents of $d\mathcal{A}$.

We conclude with an example of a toral arrangement $(T, \mathcal{A})$ where $D(\mathcal{A})_e$ is a free $\mathcal{O}_{T,e}$-module, but $D(\mathcal{A})$ is not a free $\mathcal{O}_T$-module. Our example is essentially the extension of [**6**, Example 4.36] to toral arrangements.

Suppose T is three-dimensional and z_1, z_2 and z_3 is a basis of the character group of T. Then every character of T is of the form $z_1^{n_1}z_2^{n_2}z_3^{n_3}$ for some integers n_1, n_2 and n_3. Let $\mathcal{A} = \{z_1^3, z_2^3, z_3^3, z_1, z_2, z_1^3z_2^3z_3^{-3}\}$, so with the notation we've been using, $y = (z_1^3 - 1)(z_2^3 - 1)(z_3^3 - 1)(z_1z_2 - 1)(z_1^3z_2^3z_3^{-3} - 1)$.

For $1 \leq i \leq 3$, put $x_i = dz_i$, so $\{x_1, x_2, x_3\}$ is a basis of the dual space of $\mathfrak{t}$ and $d\mathcal{A} = \{\ker(x_1), \ker(x_2), \ker(x_3), \ker(x_1+x_2), \ker(x_1+x_2-x_3)\}$. It is shown in [**6**, Example 4.54] that $d\mathcal{A}$ is free. It follows from Theorem 4.1 that $D(\mathcal{O}_{T,e}, y)$ is a free $\mathcal{O}_{T,e}$-module so $D(\mathcal{A})_e$ is a free $\mathcal{O}_{T,e}$-module.

Let ω be a primitive cube root of unity and put $t = (\omega, \omega, \omega) \in T$. We will show that $D(\mathcal{A})$ is not a free $\mathcal{O}_T$-module by showing that $D(\mathcal{O}_T, y)_t$ is not a free $\mathcal{O}_{T,t}$-module.

Define $\ell_t : \mathcal{O}_T \to \mathcal{O}_T$ by $\ell_t(f)(t') = f(t^{-1}t')$ for $f \in \mathcal{O}_T$ and $t' \in T$. Then $\ell_t(\mathfrak{m}_e) = \mathfrak{m}_t$ and ℓ_t extends to a k-algebra isomorphism from $\mathcal{O}_{T,e}$ to $\mathcal{O}_{T,t}$, which we will also denote by ℓ_t. Let $\ell_t^{\#} : \mathrm{Der}_k(\mathcal{O}_{T,e}) \to \mathrm{Der}_k(\mathcal{O}_{T,t})$ by $\ell_t^{\#}(\theta) = \ell_t\theta\ell_t^{-1}$ for $\theta \in \mathrm{Der}_k(\mathcal{O}_{T,e})$ so $\ell_t^{\#}$ is a k-vector space isomorphism.

Clearly $t \in \ker(z_1^3) \cap \ker(z_2^3) \cap \ker(z_3^3) \cap \ker(z_1^3z_2^3z_3^{-3})$ and $t \notin \ker(z_1z_2)$. Put $f = (z_1^3 - 1)(z_2^3 - 1)(z_3^3 - 1)(z_1^3z_2^3z_3^{-3} - 1)$. Then $y = (z_1z_2 - 1)f$, $z_1z_2 - 1$ is a unit in $\mathcal{O}_{T,t}$, and $\ell_t(f) = f$. Thus, $D(\mathcal{O}_{T,t}, y) = D(\mathcal{O}_{T,t}, f) = D(\mathcal{O}_{T,t}, \ell_t(f)) = \ell_t^{\#}(D(\mathcal{O}_{T,e}, f))$. Let $\mathcal{A}_1 = \{z_1^3, z_2^3, , z_3^3, z_1^3, z_2^3z_3^{-3}\}$, so $d\mathcal{A}_1 = \{\ker(x_1), \ker(x_2), \ker(x_3), \ker(x_1 + x_2 - x_3)\}$. It is shown in [**6**, Example 4.34] that $d\mathcal{A}_1$ is not free, so, by Theorem 4.1, $D(\mathcal{O}_{T,e}, f)$ is not a free $\mathcal{O}_{T,e}$-module. It follows that $D(\mathcal{O}_{T,t}, y) \cong D(\mathcal{A})_t$ is not a free $\mathcal{O}_{T,t}$-module and therefore $D(\mathcal{A})$ is not a free $\mathcal{O}_T$-module.

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