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SPHERE-FOLIATED CONSTANT MEAN CURVATURE SUBMANIFOLDS

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1. Introduction. In this article we will consider constant mean curvature submanifolds M of space forms ($\mathbf{R}^n$, $\mathbf{H}^n$ and $\mathbf{S}^n$). M will be of codimension one and will be 'foliated by spheres,' in a sense made precise below.

We begin with a few examples of constant mean curvature surfaces in $\mathbf{R}^3$ that are built up from *circles* in the sense we will be considering. The first is the catenoid, which is a minimal surface of revolution, given by the equation $r = \cosh z$ in cylindrical coordinates. The intersection of the catenoid with a plane $z = z_0$ is a circle with center on the z axis. The circles in two different such planes, say $z = z_1$ and $z = z_2$, are *coaxial*, meaning that the line joining their centers (the z axis) is orthogonal to both planes.

If we allow *nonzero* constant mean curvature, there are round spheres, such as $x^2+y^2+z^2 = 1$ in rectangular coordinates. Any family of planes (parallel or not) intersects the sphere in circles.

The well-known example that we shall call the "Riemann staircase" is also a complete minimal surface in $\mathbf{R}^3$. The intersections of a family of parallel planes (we take the planes $z = z_0$ again) with this surface are round circles, with the exception of a discrete set of straight lines. It is not a surface of revolution; if z_1 and z_2 are close together, the circles in planes $z = z_1$ and $z = z_2$ are not coaxial. As mentioned, for z near evenly spaced values z_j, the radius $r(z)$ goes to infinity as $z \to z_j$, and the plane $z = z_j$ intersects the surface in a straight line. There is a detailed discussion of this surface (and some lovely pictures) in an article by Hoffman and Meeks [**1**].

We are naturally led to ask whether the Riemann staircase is an isolated example—could something similar be found elsewhere, perhaps by allowing nonzero constant mean curvature, or by considering submanifolds of hyperbolic space?

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In Theorems 1 and 2, and Corollary 1, we consider the possible extensions of these examples to higher dimensional Euclidean space. That is, M^n will be a constant mean curvature submanifold of $\mathbf{R}^{n+1}$, with $n \geq 3$. M^n will be required to be *foliated by spheres*, meaning that there is a one-parameter family of hyperplanes which meet M^n in round $(n-1)$-spheres. The hyperplanes are not assumed parallel, and if two spheres should lie in hyperplanes that happen to be parallel, the spheres are not assumed coaxial. M^n is not assumed complete. We do not assume the existence of any group action on M^n. Finally, we do not require that the spheres themselves be complete; we consider only a small neighborhood in M^n, so that only 'pieces of spheres' are involved. A priori, an attempt to 'fill in' the remainder of each sphere could introduce self intersections. Our Theorem 1 states simply that the hyperplanes of the family are parallel unless M^n is itself a subset of a round n-sphere. Theorem 2 continues with the case of parallel planes; the conclusion is that the spheres of the foliation must be coaxial, meaning that M is a subset of a hypersurface of revolution. Taken together, these two results give Corollary 1, which gives the complete answer to our main question in Euclidean space. Theorem 2 shows a striking contrast to the situation in $\mathbf{R}^3$, where there is the Riemann example. The case of surfaces in $\mathbf{R}^3$ was finished by Nitsche [**5**]. The case of *minimal* submanifolds in $\mathbf{R}^{n+1}$, for $n \geq 3$, was considered by the present author [**2**].

There is ongoing work on submanifolds with boundary that relate to the present investigation. Max Shiffman [**7**] showed that an *annular* minimal surface $\mathbf{R}^3$ bounded by two circles in parallel planes must be a part of either a catenoid or a staircase. This is quite remarkable: the results of [**2**] show that Shiffman's result fails for $\mathbf{R}^{n+1}$, $n \geq 3$. Shiffman also showed the analogous result for convexity: an annular minimal surface with boundary made up of two convex curves in parallel planes meets intermediate parallel planes in convex curves. Meeks [**4**] has conjectured that the topological hypothesis of annularity is not necessary. Schoen [**6**] developed a version of Alexandrov reflection for minimal submanifolds with boundary. His method shows quickly that a connected minimal submanifold M^n in $\mathbf{R}^{n+1}$, for which the boundary consists of two *coaxial* $(n-1)$-spheres in parallel hyperplanes, must be a catenoid. No additional topological assumption on M^n is required in Schoen's results.

The remainder of this article gives partial results for the other space forms. Theorem 3 considers submanifolds of hyperbolic space $\mathbf{H}^{n+1}$, in this case allowing surfaces, i.e., $n \geq 2$. There is a restriction on the family of hyperplanes analogous to that in Theorem 2; the $(n-1)$-spheres are required to lie in 'asymptotic horospheres,' by which we mean the planes $\{x_{n+1} = \text{constant}\}$ in the upper half-space model. We end with Theorems 4 and 5, about $\mathbf{S}^{n+1}$ and $\mathbf{H}^{n+1}$ respectively. These last are proved simultaneously, using the ball model for $\mathbf{H}^{n+1}$ and stereographic projection for $\mathbf{S}^{n+1}$. The rather complicated restrictions in Theorems 4 and 5 amount to requiring parallel hyperplanes in the conformal models used.

2. Algebraic tools. The method of investigation, suggested by R. Schoen, consists of providing an open neighborhood in the ambient space, with a chosen coordinate system for which M^n is the level set of a particularly useful function. It becomes necessary to deal with some very detailed algebraic questions. The main tool used is a lemma from commutative algebra over the real numbers, found in an article of Lam [**3**].

We describe Lemma 6.14 of [**3**] as it would read over the field of real numbers. Suppose that U is a polynomial in n variables that assumes both positive and negative values on $\mathbf{R}^n$. Suppose that U is also irreducible over $\mathbf{R}$. Finally, suppose that W is a polynomial that vanishes whenever U does. Then U must divide W. In the present work, U will be a fixed polynomial $v_1^2+\cdots+v_n^2-1$. (Remark: for this U there is a short elementary proof of the lemma.) U achieves a negative value at the origin and a positive value at the point $(2, 0, 0, \ldots, 0)$. U is irreducible over $\mathbf{R}$ for $n \geq 2$. The specific results we find will come from applying the lemma to a polynomial called Q, and gaining information from the condition that $v_1^2 + \cdots + v_n^2 - 1$ must divide Q. We will also find occasion to use the fact that the homogeneous degree two part of this U, that is, $v_1^2 + \cdots + v_n^2$, is irreducible over $\mathbf{R}$ for $n \geq 2$.

It is appropriate to describe the uses and limitations of computers in an investigation of this level of complexity. In each of three main settings, one establishes expressions for the n-variable polynomial Q. The calculation of mean curvature in each setting becomes that of perceiving useful patterns in Q, then selectively applying the algebraic lemma. In the more mechanical stages that follow writing out Q, the author was able to use symbolic manipulation programs called Macsyma and Mathematica to check his work. It never became possible to feed in the problem and ask for an answer; the computer was used instead to compare pairs of expressions by expanding both and printing out any difference. In any event, after arriving at the definition of Q, the author performed each calculation several times, giving an understandable expression to be checked, then used the computer programs to confirm that portion of the work. The reader interested in details about these symbolic computations is urged to contact the author.

3. Euclidean space. We are able to answer the main question completely when dealing with submanifolds of $\mathbf{R}^{n+1}$, $n \geq 3$. We show first (Theorem 1) that if M^n is an (open) submanifold of $\mathbf{R}^{n+1}$, foliated by (pieces of) spheres, then either M is a subset of a round $\mathbf{S}^n$, or the hyperplanes containing said spheres are parallel. Next (Theorem 2) we show that in the latter case the spheres of the foliation are coaxial, i.e., M^n is a hypersurface of revolution.

We settle on some conventions and notation. Unless otherwise specified, the word "plane" will refer to an n-dimensional plane in $\mathbf{R}^{n+1}$, while the word "sphere" will mean a round $\mathbf{S}^{n-1}$ lying in a plane of $\mathbf{R}^{n+1}$. Notice that in the case $n = 2$, the word "sphere" would refer to a circle. Given some smooth function r of the variable t, we will use the notations $\dot{r}$ and $\ddot{r}$ for the first and second derivatives of r by t. Finally we will be using a subscript 0, so we will use the notation below to avoid ambiguity.

$$p * q = \sum_{k=1}^{n} p_k q_k.$$

Theorem 1. *If M^n is a submanifold of constant mean curvature in $\mathbf{R}^{n+1}$ with $n \geq 3$, M is nonplanar, and M is foliated by pieces of spheres lying in a one-parameter family of planes, then either M is a subset of a round $\mathbf{S}^n$ or the planes in the family are parallel.*

Proof. We need to recall most of the definitions from [**2**]. M^n is an (open) submanifold of $\mathbf{R}^{n+1}$, such that a family of planes intersects

M in subsets of round spheres. We perform a lengthy construction to provide a coordinate system on an open neighborhood around M^n. First we construct a smooth unit normal vector field N_0 to the planes of the family. Next we pick some integral curve $\gamma(t)$ of the field N_0, that is, $\dot{\gamma} = N_0(\gamma(t))$. We write the rest of the Frenet frame for γ, beginning with $\dot{\gamma} = N_0$, $\dot{N}_0 = \kappa_0 N_1$, and $\dot{N}_1 = -\kappa_0 N_0 + \kappa_1 N_2$; we continue with $\dot{N}_i = -\kappa_{i-1} N_{i-1} + \kappa_i N_{i+1}$ for $1 \le i \le n-1$, and finally $\dot{N}_n = -\kappa_{n-1} N_{n-1}$.

Let us use the notation Π_t for the plane containing the point $\gamma(t)$. Referring to the sphere in Π_t of which $\Pi_t \cap M^n$ is a subset, let $r(t)$ be the radius and $c(t)$ be the center. Since $\gamma(t)$ and $c(t)$ are both in the plane Π_t and $\dot{\gamma} = N_0$ is constructed perpendicular to Π_t, it follows that N_0 is perpendicular to the vector $c(t) - \gamma(t)$. From the construction of the Frenet frame above, we also find that N_i lies within Π_t for $1 \le i \le n$ (see Figure 1).

In what follows, we compare positions in the plane Π_t to the point $\gamma(t)$. We define functions $a_1, \dots, a_n$ to satisfy the equation

$$c(t) - \gamma(t) = \sum_{i=1}^{n} a_i(t) N_i(t).$$

We also introduce quantities requiring the subscript 0, namely, $\alpha_0, \dots, \alpha_n$, with

$$\dot{c}(t) = \sum_{k=0}^{n} \alpha_k(t) N_k(t).$$

We introduce variables that will become coordinates in a local neighborhood. These are $v_1, \dots, v_n$. Let U be an open subset of $\mathbf{R} \times \mathbf{R}^n$. We define the mapping $\mathbf{X} : U \to \mathbf{R}^{n+1}$ by

$$\mathbf{X}(t, v_1, \dots, v_n) = c(t) + r(t) \sum_{i=1}^{n} v_i N_i.$$

The inverse of this mapping defines a coordinate system on an open subset W of $\mathbf{R}^{n+1}$ that contains an open subset of M^n. We rename M^n to refer to this possibly smaller open submanifold. Notice that when $v_1^2 + \cdots + v_n^2 = 1$, the result of the mapping $\mathbf{X}$ is a point lying in M, and the converse applies. To summarize,

$$M^n = \{\vec{x} \in \mathbf{R}^{n+1} : v_1^2 + \cdots + v_n^2 = 1\}.$$

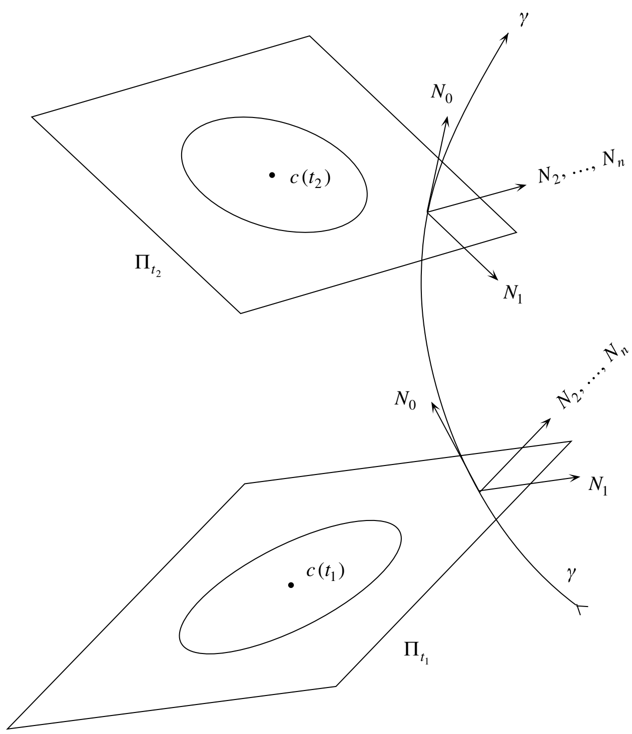

FIGURE 1.

Using notation defined earlier, we may refer to $v_1^2 + \cdots + v_n^2$ as $v * v$.

We need to be able to recognize when either of the conclusions desired in the theorem apply. The planes $\{\Pi_t\}$ would be parallel if $\dot{N}_0$ were 0, which occurs when $\kappa_0 = 0$. In that case, $\gamma(t)$ would be a straight line to which all the Π_t were perpendicular. We have:

Lemma 1. *The planes Π_t defining the foliation of M^n by spheres are parallel if and only if $\kappa_0 = 0$.*

The other possibility is more difficult to recognize. M is a subset of an $\mathbf{S}^n$ if and only if each point of M is the same distance from a fixed point

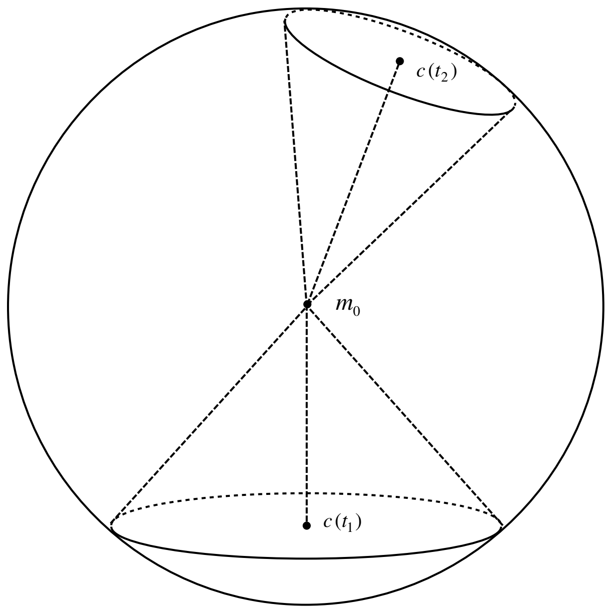

FIGURE 2.

$\mathbf{m}_0$ in $\mathbf{R}^{n+1}$. This could happen in a few different ways. First, M would be part of a sphere if $c(t) = \mathbf{m}_0$ for all t, with $r(t)$ also being constant. Notice that if c and r are constant it is not possible for κ_0 to be the constant 0. Second, we could have $\kappa_0 = 0$ and $r^2 + (t - t_0)^2 =$ constant.

A third way for M^n to be a part of a sphere is for each $\mathbf{S}^{n-1}$ in the foliation to be the base of a cone, and the axis of this cone be perpendicular to its base (see Figure 2). Since the axis of the cone is a segment from $\mathbf{m}_0$ to $c(t)$, and the base of the cone lies in the plane Π_t, we see that $c(t) - \mathbf{m}_0$ must be perpendicular to Π_t. This shows that $c(t) - \mathbf{m}_0$ must be parallel to N_0, so there is a scalar $q(t)$ with

$$c(t) - \mathbf{m}_0 = q(t)N_0.$$

Differentiating, we find the requirement $\dot{c} = \dot{q}N_0 + q\kappa_0 N_1$. Recalling

$$\dot{c}(t) = \sum_{k=0}^{n} \alpha_k(t)N_k(t),$$

we see that, for M^n to be part of an $\mathbf{S}^n$, it is necessary that $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$, along with $\alpha_0 = \dot{q}$ and $\alpha_1 = q\kappa_0$. Since $|q|$ is also the distance between $c(t)$ and $\mathbf{m}_0$, and $r(t)$ is the radius of the base of the cone, and any segment generating the conical surface is the same length as the radius ρ of the $\mathbf{S}^n$ that contains M^n, we find that $r^2 + q^2 = \rho^2$, or $r^2 + q^2$ is constant. If the planes π_t are not parallel, so that $\kappa_0 \neq 0$, then $q = \alpha_1/\kappa_0$. It follows that

$$r^2 + \alpha_1^2/\kappa_0^2$$

must be constant. Since $\alpha_0 = \dot{q}$, we find also that

$$\alpha_0 = \frac{d}{dt}\frac{\alpha_1}{\kappa_0}.$$

On the other hand, given the three requirements above, we may define

$$\mathbf{m} = c - \frac{\alpha_1}{\kappa_0} N_0,$$

and calculate easily that $\dot{\mathbf{m}} = 0$. We have

Lemma 2. *M^n is a subset of a round $\mathbf{S}^n$ if one of the following conditions holds*:

1) *$c(t)$ and $r(t)$ are both constant*,

2) *$\kappa_0 \equiv 0$ and r varies appropriately with t, or*

3) *$\kappa_0 \neq 0$, $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$, $r^2 + \alpha_1^2/\kappa_0^2$ is constant, and $\alpha_0 = (d/dt)(\alpha_1/\kappa_0)$.*

3.1. Calculation of the mean curvature. We use the fact that the submanifold M^n is the level set of a smooth scalar function. Let us save some writing by defining

$$V = v * v = v_1^2 + \cdots + v_n^2,$$

so that M is the set

$$M^n = \{\vec{x} \in \mathbf{R}^{n+1} : V(\vec{x}) = 1\}.$$

Then we know that the mean curvature H of M^n is given, up to sign, by

$$H = \frac{-1}{n}\operatorname{div}\left(\frac{\nabla V}{|\nabla V|}\right),$$

with ∇ denoting the induced connection on M. We wish to write H in the coordinate system $\{t, v_1, \dots, v_n\}$. The subscript 0 will be used for the coordinate t when applicable. When possible, subscripts with values from 1 to n will be denoted by the letters i, j, k, l. Subscripts with values from 0 to n will be denoted by the letters β, μ, ν.

We define the usual metric terms, $g_{00} = \langle \partial\mathbf{X}/\partial t, \partial\mathbf{X}/\partial t\rangle$, $g_{0i} = \langle \partial\mathbf{X}/\partial t, \partial\mathbf{X}/\partial v_i\rangle$, and $g_{ij} = \langle \partial\mathbf{X}/\partial v_i, \partial\mathbf{X}/\partial v_j\rangle$ for $1 \le i,\ j \le n$. Further values are presented with the understanding that, should these ever arise,

$$\kappa_{-1} = 0, \qquad \kappa_n = 0, \qquad v_0 = 0, \qquad v_{n+1} = 0.$$

We are able to explicitly calculate the metric coefficients in this coordinate system as follows:

$$\begin{aligned} g_{ij} &= \delta_{ij} r^2, \\ g_{01} &= r\alpha_1 + r\dot{r}v_1 - r^2\kappa_1 v_2, \\ g_{0i} &= r\alpha_i + r\dot{r}v_i + r^2(\kappa_{i-1}v_{i-1} - \kappa_i v_{i+1}) \\ &\qquad \text{for } 2 \le i \le n-1, \\ g_{0n} &= r\alpha_n + r\dot{r}v_n + r^2\kappa_{n-1}v_{n-1}, \end{aligned}$$

and

$$g_{00} = (\alpha_0 - r\kappa_0 v_1)^2 + \frac{1}{r^2}\sum_{j=1}^{n} g_{0j}^2.$$

Let G be the matrix $g_{\mu\nu}$, and let $g = \det(G)$. Then cofactor expansion along the row with index 0 shows that $g = r^{2n}(\alpha_0 - r\kappa_0 v_1)^2$, so that

$$\sqrt{g} = r^n|\alpha_0 - r\kappa_0 v_1|.$$

As $g \equiv 0$ would imply that $\mathbf{X}$ need not be an immersion, but that M actually lies within a single plane, we require that M be nonplanar and arrange $g \neq 0$ by restricting to a smaller open set. We introduce the quantity

$$\alpha * v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

and note that it does not involve the function α_0. We find an identity that will be ubiquitous in what follows,

$$\sum_{i=1}^{n} g_{0i} v_i = r[(a * v) + \dot{r}(v * v)]. \tag{1}$$

In our coordinate system, with $V = v * v = v_1^2 + \cdots + v_n^2$, we have

$$\frac{1}{2}\nabla V = \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij} v_i \frac{\partial}{\partial v_j} + \sum_{i=1}^{n} g^{i0} v_i \frac{\partial}{\partial t},$$

and

$$\frac{1}{4}|\nabla V|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij} v_i v_j.$$

We do not write explicit values for $g^{\mu\nu}$. Instead, we evaluate the evident sums, a task equivalent to carrying out an underlying matrix computation. We define the variables $w^0, \dots, w^n$ by

$$w^\mu = \sum_{k=1}^{n} g^{\mu k} v_k.$$

We rewrite quantities related to ∇V using the w^μ, such as

$$\frac{1}{2}\nabla V = \sum_{i=1}^{n} w^i \frac{\partial}{\partial v_i} + w^0 \frac{\partial}{\partial t}, \tag{2}$$

and

$$\frac{1}{4}|\nabla V|^2 = \sum_{i=1}^{n} w^i v_i, \tag{3}$$

or

$$\frac{1}{2}|\nabla V| = \sqrt{\sum_{i=1}^{n} w^i v_i}. \tag{4}$$

Simply placing (4) in the definition of divergence gives

$$\operatorname{div}\left(\frac{\nabla V}{|\nabla V|}\right) = \frac{1}{\sqrt{g}}\left\{\begin{array}{l}(\partial/\partial t)[\sqrt{g}(w^0/\sqrt{\sum_{k=1}^n w^k v_k})] \\ +(\partial/\partial v_i)[\sqrt{g}(w^i/\sqrt{\sum_{k=1}^n w^k v_k})]\end{array}\right\}.$$

It follows from the definitions that

$$\sum_{\mu=0}^{n} g_{0\mu} w^{\mu} = \sum_{\mu=0}^{n}\sum_{k=1}^{n} g_{0\mu} g^{\mu k} v_k = \delta_0^k v_k = 0, \tag{5}$$

while

$$\sum_{\mu=0}^{n} g_{i\mu} w^{\mu} = \sum_{\mu=0}^{n}\sum_{k=1}^{n} g_{i\mu} g^{\mu k} v_k = \delta_i^k v_k = v_i. \tag{6}$$

Repeated use of the identities (1), (5) and (6) leads to values for the w^μ, these being

$$w^0 = \frac{-[(\alpha * v) + \dot{r}(v * v)]}{r(\alpha_0 - r\kappa_0 v_1)^2} \tag{7}$$

and

$$w^i = \frac{1}{r^2}\left(\frac{r(\alpha_0 - r\kappa_0 v_1)^2 + [(\alpha * v) + \dot{r}(v * v)]g_{0i}}{r(\alpha_0 - r\kappa_0 v_1)^2}\right). \tag{8}$$

Substituting (8) into (4) and using the identity (1) gives

$$\frac{1}{4}|\nabla V|^2 = \frac{1}{r^2}\left(\frac{r(\alpha_0 - r\kappa_0 v_1)^2(v * v) + [(\alpha * v) + \dot{r}(v * v)]^2}{(\alpha_0 - r\kappa_0 v_1)^2}\right). \tag{9}$$

Recall that we may restrict t and perhaps v_1 so that $\alpha_0 - r\kappa_0 v_1 \neq 0$. We define

$$A = r\kappa_0 v_1 - \alpha_0 \tag{10}$$

$$B = (\alpha * v) + \dot{r}(v * v) \tag{11}$$

$$D = A^2(v * v) + B^2 \tag{12}$$

$$T_0 = -r^n B \tag{13}$$

$$T_i = r^{n-1}A^2 v_i + r^{n-2} B g_{0i}. \tag{14}$$

Using the defined symbols above, we may now rewrite

$$\frac{\sqrt{g}w^0}{\sqrt{\sum_{i=1}^n w^i v_i}} = \frac{T_0}{\sqrt{D}} \tag{15}$$

$$\frac{\sqrt{g}w^j}{\sqrt{\sum_{i=1}^n w^i v_i}} = \frac{T_j}{\sqrt{D}} \tag{16}$$

and

$$\sqrt{g} = r^n |A|. \tag{17}$$

Altogether, we find that

$$\operatorname{div}\left(\frac{\nabla V}{|\nabla V|}\right) = \frac{\pm P}{2r^n|A|D^{3/2}}, \tag{18}$$

if we define

$$P = \left(2\frac{\partial T_0}{\partial t}D - T_0\frac{\partial D}{\partial t}\right) + \sum_{i=1}^n \left(2\frac{\partial T_i}{\partial v_i}D - T_i\frac{\partial D}{\partial v_i}\right). \tag{19}$$

We now choose the mean curvature of M to be

$$H = \frac{-P}{2r^n|A|D^{3/2}}.$$

For the example of the constant radius right spherical cylinder with inward normal, this choice gives mean curvature $(n-1)/(nr)$.

If H is constant, equal to some number h, we may square both sides of the equation $H \equiv h$, then move the denominator to arrive at

$$P^2 = 4n^2h^2r^{2n}A^2D^3. \tag{20}$$

This condition must apply on M. We proceed to define the polynomial

$$Q = P^2 - 4n^2h^2r^{2n}A^2D^3. \tag{21}$$

The rest of the effort goes into restricting Q to M, which is to say that we require

$$V - 1 = 0 \Longrightarrow Q = 0.$$

Two processes alternate in the ensuing algebra. After all partial derivatives have been evaluated, the expression $v * v$ may be replaced by 1 whenever it appears. In order to use Lam's lemma, we identify the highest degree homogeneous terms in Q that remain at any given time.

We examine the definitions of A, B, D, T_0, T_i, P for terms of high degree in $v_1, \dots, v_n$: the highest degree term in A^2 is the quadratic $r^2\kappa_0^2 v_1^2$. The expression D is also quadratic once $v * v$ is replaced by 1. The quadratic terms in D are those in $r^2\kappa_0^2 v_1^2 + (\alpha * v)^2$. The expression B is merely linear, as is T_0. A rather longer calculation is required to find that P (also restricted to M) is of degree four, and that the degree four part of P is $2nr^{n+1}\kappa_0^2 v_1^2(r^2\kappa_0^2 v_1^2 + (\alpha * v)^2) \equiv 2nr^{n+1}\kappa_0^2 v_1^2 D$. A detailed calculation of the highest-degree homogeneous part of P is written out in the Appendix to [**2**]. The degree 8 part of Q is then

$$Q_8 = 4n^2 r^{2n+2}\kappa_0^4 v_1^4 D_2^2 - 4n^2 h^2 r^{2n} r^2 \kappa_0^2 v_1^2 D_2^3,$$

where the symbol D_2 refers to the quadratic terms in D, and we may factor this as

$$Q_8 = 4n^2 r^{2n+2}\kappa_0^2 v_1^2 D_2^2(\kappa_0^2 v_1^2 - h^2 D_2).$$

Since $V - 1$ must divide Q, in particular $V = v_1^2 + \cdots + v_n^2$ must divide Q_8 with a homogeneous quotient. As V is irreducible, we know that we have three choices: V divides $4n^2 r^{2n+2}\kappa_0^2 v_1^2$ or V divides D^2 or V divides $\kappa_0^2 v_1^2 - h^2 D_2$.

The first choice can only occur if $\kappa_0 = 0$, which means that the planes are parallel. The second choice implies that V divides the quadratic part of D, again because V is irreducible over $\mathbf{R}$. This is the same case that occurred at the end of [**2**]. For $n \geq 3$, we eventually find that $r^2\kappa_0^2 + v_1^2 = 0$, from which we again conclude that $\kappa_0 = 0$. The third case is new and gives the possibility that $\kappa_0 \neq 0$ but that M^n is part of a round sphere. There are many cases to consider, and these are discussed in the Appendix.

4. Euclidean space, parallel planes. With the previous theorem in hand, we need only complete the consideration of the case when the planes Π_t are parallel, that is, $\kappa_0 = 0$.

Theorem 2. *If M^n is a submanifold of constant mean curvature in $\mathbf{R}^{n+1}$ with $n \geq 3$, M is nonplanar, and M is foliated by pieces of spheres lying in parallel planes, then M is a subset of a hypersurface of revolution.*

Corollary 1. *If M^n is a submanifold of constant mean curvature in $\mathbf{R}^{n+1}$ with $n \geq 3$, M is connected, complete, and an open subset of M is foliated by pieces of spheres, then M is one of the known examples: plane, sphere, cylinder, catenoid or Delaunay type.*

Proof. It is only necessary to examine the polynomial Q in the situation when $\kappa_0 = 0$, so that all the $\kappa_\beta = 0$. We will summarize the calculation of the highest degree terms in Q, which are of degree six when $\kappa_0 = 0$. First, from the expression for the degree four terms in P, we see that P is of degree at most three when $\kappa_0 = 0$. More work is involved in showing that the cubic terms in P cancel completely when the restriction $V \equiv 1$ is invoked. What remains is of degree two, and the quadratic part of P is $2(n-2)r^{n-1}(\alpha * v)^2$.

On the other hand, there are still terms of degree six in the expression $4n^2h^2r^{2n}A^2D^3$. While A^2 is now 1, there are still quadratic terms in D, these being $D_2 = (\alpha * v)^2$. As a result, Q is now of degree six, P^2 does not contribute to the degree six terms, and

$$Q_6 = -4n^2h^2r^{2n}(\alpha * v)^6.$$

If $h = 0$, M is actually minimal, and we know that M is a hypersurface of revolution [**2**]. If $h \neq 0$, then $V-1$ divides Q and V divides Q_6. Since V is irreducible, V divides $(\alpha * v)$, which implies that $\alpha_1 = \cdots = \alpha_n = 0$. This last condition means that M is coaxial. This concludes the proof of the theorem. The corollary follows from the maximum principle. □

5. Other ambient spaces—preliminaries. We prove three theorems that are analogous to the case of parallel planes in Euclidean space. We will simply introduce a conformal factor F^{-2} into the metric on Euclidean space and thereby use conformal models for hyperbolic space and for the sphere. We eventually define a new value for Q and proceed from there.

We will need some abbreviations: define $s_1 = -r\kappa_1 v_2$, $s_i = r\kappa_{i-1}v_{i-1} - r\kappa_i v_{i+1}$ for $2 \le i \le n-1$, and $s_n = r\kappa_{n-1}v_{n-1}$. Then define $z_i = \alpha_i + \dot{r}v_i + s_i$ for $1 \le i \le n$. We record the usual kind of cancellation, in this case

$$z * v = \sum_{k=1}^{n} z_k v_k = B.$$

Suppose we use the superscript (euc) to describe the value of various quantities on Euclidean space. The abbreviations just defined allow us to write

$$\begin{aligned} g_{ij}^{\mathrm{euc}} &= \delta_{ij} r^2 \quad \text{for } 1 \le i, j \le n, \\ g_{0i}^{\mathrm{euc}} &= r\alpha_i + r\dot{r}v_i + rs_i \quad \text{for } 1 \le i \le n, \end{aligned}$$

and

$$g_{00}^{\mathrm{euc}} = A^2 + \sum_{k=1}^{n} z_k^2.$$

As before

$$\sqrt{g^{\mathrm{euc}}} = \mathbf{R}^n |A|.$$

In our models, with any subscripts, the inner products become

$$g_{\mu\nu} = F^{-2} g_{\mu\nu}^{\mathrm{euc}}.$$

The determinant is now $g = F^{-2n-2} g^{\mathrm{euc}} = r^{2n} A^2 F^{-2n-2}$, so that

$$\sqrt{g} = F^{-n-1} r^n |A|.$$

Once again, we define summations $u^0, \ldots, u^n$ by

$$u^\nu = \sum_{k=1}^{n} g^{\nu k} v_k.$$

As the terms $g^{\nu k}$ are equal to the original Euclidean terms multiplied by F^2, this shows that

$$u^\nu = F^2 w^\nu.$$

The effects on the terms related to derivatives of V are

$$\frac{1}{2}\nabla V = \sum_{i=1}^{n} u^i \frac{\partial}{\partial v_i} + u^0 \frac{\partial}{\partial t} \quad \text{for } 1 \leq i \leq n \tag{22}$$

and

$$\frac{1}{2}|\nabla V| = \sqrt{\sum_{i=1}^{n} u^i v_i}. \tag{23}$$

To save space, we use the summation convention and the abbreviations

$$\partial_0 = \frac{\partial}{\partial t}$$

and

$$\partial_i = \frac{\partial}{\partial v_i} \quad \text{for } 1 \leq i \leq n.$$

This allows us to abbreviate the expression for mean curvature as

$$H = \left(\frac{-1}{n\sqrt{g}}\right)\partial_\nu\left(\frac{u^\nu\sqrt{q}}{\sqrt{u^i v_i}}\right).$$

We may compare this to the expression found earlier. The main ingredient is the appearance of the factor F in various places. For example,

$$H = \left(\frac{-F^{n+1}}{nr^n|A|}\right)\partial_\nu\left(\frac{r^n|A|F^2 w^\nu}{F^{n+1}F\sqrt{w^i v_i}}\right)$$

or

$$H = \left(\frac{-F^{n+1}}{n\sqrt{g^{\text{euc}}}}\right)\partial_\nu\left(\frac{F^{-n}w^\nu\sqrt{g^{\text{euc}}}}{\sqrt{w^i v_i}}\right).$$

From the product rule for derivatives,

$$H = \left(\frac{-F}{n\sqrt{g^{\mathrm{euc}}}}\right)\partial_\nu\left(\frac{w^\nu\sqrt{g^{\mathrm{euc}}}}{\sqrt{w^i v_i}}\right) - \left(\frac{F^{n+1}w^\nu}{n\sqrt{w^i v_i}}\right)\partial_\nu(F^{-n}),$$

or

$$H = -F\left\{\left(\frac{1}{n\sqrt{g^{\mathrm{euc}}}}\right)\partial_\nu\left(\frac{w^\nu\sqrt{g^{\mathrm{euc}}}}{\sqrt{w^i v_i}}\right)\right\} + \left(\frac{w^\nu}{n\sqrt{w^i v_i}}\right)\partial_\nu F.$$

The long first term is precisely F times the Euclidean mean curvature of the subset $M \subset \mathbf{R^{n+1}}$. The second term may be rewritten by using the expressions T_ν, which satisfy

$$\frac{r^n|A|w^\nu}{\sqrt{w^i v_i}} = \frac{T_\nu}{\sqrt{D}}.$$

Using these facts, we write

$$H = -F\left\{\frac{P}{2nr^n|A|D^{3/2}}\right\} + \left(\frac{T_\nu}{r^n|A|\sqrt{D}}\right)\partial_\nu F.$$

Combining over a common denominator,

$$H = \frac{2nDT_\nu(\partial_\nu F) - FP}{2nr^n|A|D^{3/2}}.$$

There are some common factors in this last expression for H. It is useful to provide even more abbreviations, denoted T^ν, C, E and W. First, let

$$T^0 = -rB \quad \text{and} \quad T^i = A^2 v_i + Bz_i,$$

so that

$$T^\nu = r^{1-n}T_\nu.$$

Next, let

$$C = \frac{P}{2r^{n-1}},$$

and

$$E = \sum_{\nu=0}^{n}(\partial_\nu F)T^\nu = \frac{\partial F}{\partial t}T^0 + \sum_{i=1}^{n}\frac{\partial F}{\partial v_i}T^i.$$

With these abbreviations, we may write H as

$$H = \frac{nDE - FC}{nr|A|D^{3/2}}.$$

One more abbreviation will be

$$W = nDE - FC,$$

so that

$$H = \frac{W}{nr|A|D^{3/2}}.$$

Once again, if we require constant curvature h, we are setting

$$H = h \quad \text{whenever } v * v = 1.$$

We are led once more to define the polynomial Q

$$Q = W^2 - n^2 h^2 r^2 A^2 D^3,$$

and require that Q be 0 when $V - 1 = 0$.

6. Asymptotic horospheres in $\mathbf{H}^{n+1}$. We refer to horospheres that have a common point at infinity as "asymptotic" horospheres. In the ball model for $\mathbf{H}^{n+1}$, these would appear as spheres all tangent to a common point in the unit sphere, the unit sphere corresponding to the set of points at infinity of $\mathbf{H}^{n+1}$. We will instead use the upper half space model on $\mathbf{R}^{n+1}$, with the coordinate we will call x_0 satisfying $x_0 > 0$. Then we rotate the asymptotic family of horospheres within $\mathbf{H}^{n+1}$ so that they appear in the model as the parallel planes $x_0 =$ constant. Horospheres may also appear, in this model, as spheres tangent to the plane $x_0 = 0$. It will be possible to include surfaces in our result, allowing $n \geq 2$. Indeed, it is a shame that this result applies for $n = 2$, because it appears to make unlikely the possibility of a 'Riemann staircase' in hyperbolic space.

Theorem 3. *If M^n is a submanifold of constant mean curvature in $\mathbf{H}^{n+1}$ with $n \geq 2$, and M is foliated by pieces of spheres lying in asymptotic horospheres, then M is a subset of a plane, hypersphere,* *horosphere, equidistant hypersurface, or other hypersurface of revolution around a geodesic. If complete, M is itself one of the submanifolds mentioned.*

Proof. The conformal factor for the metric is $1/(x_0^2)$. To fit in with previous definitions, we may simply require that

$$t = x_0.$$

This gives the value of the function F as

$$F = t.$$

Meanwhile, we have

$$\kappa_\beta = 0, \qquad \alpha_0 = 1, \qquad \dot{\alpha}_0 = 0,$$
$$A = -1, \qquad s_i = 0,$$

and

$$z_i = \alpha_i + \dot{r} v_i.$$

All the partial derivatives we need, except for derivatives of F, have been performed already. We may therefore set $V = 1$ and $\kappa_0 = 0$ throughout. For example, we have $B = \dot{r} + (\alpha * v)$ and

$$D = (\alpha * v)^2 + 2\dot{r}(\alpha * v) + \dot{r}^2 + 1.$$

The only nonzero partial derivative of F is $\partial F/\partial t = 1$. This gives

$$E = T^0 = -rB = -r(\dot{r} + (\alpha * v)).$$

The factor C of P is considerably shorter than expected, with

$$\begin{aligned} C = {} & (n-2)(\alpha * v)^2 + 2\dot{r}(n-1)(\alpha * v) \\ & - r(\dot{a} * v) + (n-1)(1 + \dot{r}^2) - r\ddot{r} + (\alpha * \alpha). \end{aligned}$$

The polynomial Q turns out to be of degree 6. For this theorem, it is necessary to examine the homogeneous parts of Q from degree 6 all the way down to degree 2. We relegate the description of the remainder of the proof to the Appendix. □

7. In $\mathbf{S}^{n+1}$ or in $\mathbf{H}^{n+1}$. We are able to apply our apparatus to a pair of related situations. In $\mathbf{S}^{n+1}$, a geodesic plane is an $\mathbf{S}^n$ of maximum diameter, an equator or 'great sphere.' A family of spheres in $\mathbf{S}^{n+1}$ may be said to be tangent at a common point if the spheres of the family all pass through that point and share a common tangent plane there. Suppose we refer to the common point of tangency as the 'North Pole,' and apply stereographic projection from the North Pole. The plane that is the target of the projection will be the equatorial plane that passes through the center of the $\mathbf{S}^{n+1}$ and is orthogonal to a line through the center and the North Pole. Under this stereographic projection, the spheres with a common point of tangency at the North Pole become a family of parallel Euclidean planes in $\mathbf{R}^{n+1}$.

In the ball model for $\mathbf{H}^{n+1}$, parallel Euclidean planes correspond to a situation we will call a "distance-related and parallel family of equidistant hypersurfaces." The description of such a family within $\mathbf{H}^{n+1}$ is as follows: a single 'equidistant hypersurface' is a connected submanifold, and is the set of all points a fixed distance from a geodesic plane Π_0 and on one side of it. Having fixed Π_0, draw a geodesic γ that is perpendicular to Π_0. Parametrize γ by arclength s, and let the plane perpendicular to γ at the point $\gamma(s)$ be called Π_s. Let Σ_t denote the equidistant hypersurface that is a constant distance s from Π_s and lies on the opposite side of Π_s from Π_0.

It is not immediately obvious that the situation described in hyperbolic space actually corresponds to parallel Euclidean planes in the ball model. The concept is really two-dimensional, with the higher dimensional cases achieved by rotation (see Figure 3). In the Poincare disk model, let the y axis be Π_0, let the x axis be γ, and let Π_t be the circular arc (orthogonal to the unit circle) that intersects the x axis orthogonally at $x = t$. The distance between some point $(x, 0)$ and the origin $(0, 0)$ is given by $\log((1+x)/(1-x))$. The equidistant line Σ_t of interest is modeled by the Euclidean line segment $x = a$ that meets Π_t at two points of the unit circle. Inversion in the unit circle shows that $a = 2t/(t^2 + 1)$. The distance between $(0, 0)$ and $(t, 0)$ is then shown algebraically to be the same as the distance between $(t, 0)$ and $(a, 0)$.

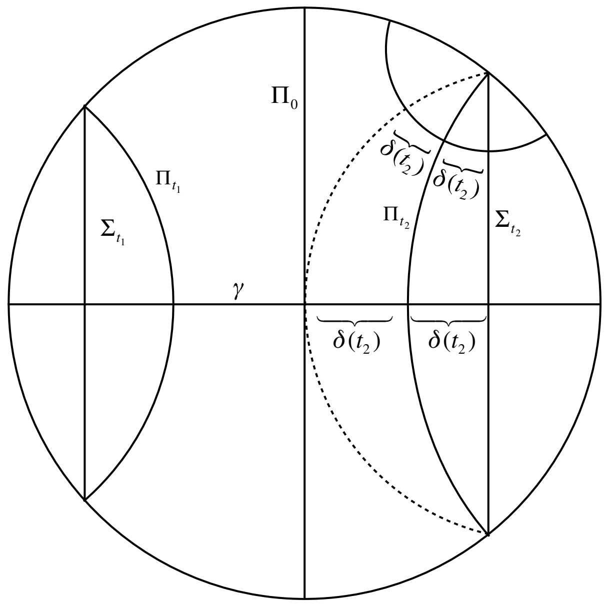

FIGURE 3.

As in the previous section, these results extend to $n \geq 2$, so that surfaces are included.

Theorem 4. *If M^n is a submanifold of constant mean curvature in $\mathbf{S}^{n+1}$ with $n \geq 2$, and M is foliated by pieces of $(n-1)$-spheres lying in n-spheres that share a common point of tangency, then M is a subset of an equator, n-sphere, or some other hypersurface of revolution around a geodesic. If complete, M is itself one of the submanifolds mentioned.*

Theorem 5. *If M^n is a submanifold of constant mean curvature in $\mathbf{H}^{n+1}$ with $n \geq 2$, and M is foliated by pieces of $(n-1)$-spheres lying in a distance-related and parallel family of equidistant hypersurfaces, then M is a subset of a geodesic plane, geodesic n-sphere, equidistant hypersurface, horosphere, or some other hypersurface of revolution around a geodesic. If complete, M is itself one of the submanifolds mentioned.*

Proof. We write the conformal factor simultaneously as

$$\frac{4}{(1+\varepsilon|X|^2)^2},$$

where X refers to position in $\mathbf{R}^{n+1}$, while ε is 1 for the sphere and -1 for hyperbolic space. It is necessary to distinguish the origin $\mathbf{0}$ in $\mathbf{R}^{n+1}$. Without loss of generality, we may demand that γ be a line through the origin, so that $\gamma(t) = tN_0$ for a fixed vector N_0. We also have the remaining $N_1, \dots, N_n$ fixed. The center of the (Euclidean) sphere in the plane Π_t is

$$c(t) = tN_0 + \sum_{i=1}^{n} a_i N_i,$$

while

$$\dot{c}(t) = N_0 + \sum_{i=1}^{n} \dot{a}_i N_i,$$

so that

$$\alpha_0 = 1$$

and

$$\alpha_i = \dot{a}_i \quad \text{for } 1 \le i \le n.$$

Other quantities simplify as follows:

$$\kappa_\beta = 0, \qquad \dot{\alpha}_0 = 0, \qquad A = -1, \qquad s_i = 0,$$

and

$$z_i = \alpha_i + \dot{r} v_i.$$

The mapping

$$\mathbf{X} : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1}$$

is therefore given by

$$\mathbf{X} = c + r\sum_{i=1}^{n} v_i N_i = tN_0 + \sum_{i=1}^{n} (a_i + r v_i N_i).$$

The expression F expands into

$$F = \frac{1 + \varepsilon(r^2 V + 2a * v + r\alpha * v + a * a + t)}{2}.$$

The partial derivatives of F are

$$\frac{\partial F}{\partial v_i} = \varepsilon(r^2 v_i + r a_i)$$

and

$$\begin{aligned}\frac{\partial F}{\partial t} &= \varepsilon(r\dot{r} + \dot{r}a * v + r\alpha * v + a * \alpha + t)\\ &= \varepsilon(rB + \dot{r}a * v + a * \alpha + t).\end{aligned}$$

The summation E becomes

$$E = \varepsilon\{r^2 + ra * v + rB(a * \alpha + \alpha * v - t)\},$$

which expands out to

$$\begin{aligned}E = \varepsilon\{&r(\alpha * v)^2 - tr\alpha * v + r(a * \alpha)(\alpha * v)\\ &+ ra * v + r\dot{r}\alpha * v + r^2 - tr\dot{r} + r\dot{r}a * \alpha\}.\end{aligned}$$

The remainder of the proof is short enough to reproduce here. When all is finished here, Q is seen to be of degree 8. The highest degree part of Q is just

$$Q_8 = n^2 r^2 (\alpha * v)^8.$$

Since V divides Q_8, this shows that all the $\alpha_i = 0$ for $i \geq 1$, which means that the a_i are constant. With this fact in hand, the degrees of all the expressions we need drop dramatically, and Q becomes quadratic, with

$$Q_2 = r^2(a * v)^2(1 + \dot{r}^2 + r\ddot{r})^2.$$

The first possibility is that V divides $a * v$. This would imply that all the a_i are 0 for $i \geq 1$, which is to say that

$$c = tN_0.$$

The curve $c(t)$ is a straight line that passes through the origin. In either model, this means that $c(t)$ is actually a geodesic in the original space.

This means, in turn, that M is a hypersurface of revolution around a geodesic.

The second possibility is that the a_i are constant, and we have the ordinary differential equation

$$1 + \dot{r}^2 + r\ddot{r} = 0.$$

Here the curve $c(t)$ is a straight line that need not pass through the origin in $\mathbf{R}^{n+1}$. The general solution to the differential equation above is

$$r^2 + (t - t_0)^2 = \text{constant}.$$

Therefore, as a subset of $\mathbf{R}^{n+1}$, M is part of a round sphere. A sphere is significant in either model: pulling back the stereographic projection to $\mathbf{S}^{n+1}$, M can be part of a sphere (constant geodesic distance from a point) or, indeed, an equator (totally geodesic). Pulling back to $\mathbf{H}^{n+1}$, M can be part of a plane, sphere, equidistant hypersurface, or horosphere.

In either case, the usual maximum principle argument applies, so M is one of the objects described if it is complete. $\square$

Appendix

A. $P|_M$ should be viewed as an equivalence class of polynomials modulo $v * v = 1$. Recall these definitions. Generally,

$$p * q = \sum_{k=1}^{n} p_k q_k.$$

In particular, let

$$q * g_0 = \sum_{k=1}^{n} q_k g_{0k},$$

and recall the variables s_i:

$$s_1 = -r\kappa_1 v_2, \qquad s_i = r\kappa_{i-1} v_{i-1} - r\kappa_i v_{i+1}, \qquad s_n = r\kappa_{n-1} v_{n-1}.$$

We may write P as a sum of products of previously defined polynomials:

$$\begin{aligned} P \equiv 2r^{n-2} A\{ & nrAD + r\dot{r}AB - rB\kappa_0 g_{01} + rA^2\alpha_0 \\ & + 2rB^2\alpha_0 + A\alpha * g_0 + r^2\dot{A}B - r^2 A\dot{B}\}. \end{aligned}$$

Gathering terms of like degree, we have

$$P \equiv 2r^{n-1}\left\{\begin{array}{l}
nr^2\kappa_0^2 v_1^2[r^2\kappa_0^2 v_1^2 + (\alpha * v)^2] \\
+r\kappa_0 v_1 \left[\begin{array}{l}(1-4n)r^2\alpha_0\kappa_0^2 v_1^2 + (2n+2)r\dot{r}\kappa_0 v_1(\alpha * v) \\ + (2-2n)\alpha_0(\alpha * v)^2 + r^2\dot{\kappa}_0 v_1(\alpha * v) \\ - r^2\kappa_0 v_1(\dot{\alpha} * v) + r^2\kappa_0\kappa_1 v_2(\alpha * v) \\ + r\kappa_0 v_1(\alpha * s)\end{array}\right] \\
+\left[\begin{array}{l}(6n-3)r^2\alpha_0^2\kappa_0^2 v_1^2 - 4nr\dot{r}\alpha_0\kappa_0 v_1(\alpha * v) \\ + (n+1)r^2\dot{r}^2\kappa_0^2 v_1^2 - 2r\alpha_0\kappa_0 v_1(\alpha * s) \\ + (n-2)\alpha_0^2(\alpha * v)^2 - r^2\alpha_0\dot{\kappa}_0 v_1(\alpha * v) \\ + r^3\dot{r}\kappa_0^2\kappa_1 v_1 v_2 - r^2\alpha_0\kappa_0\kappa_1 v_2(\alpha * v) \\ + r^3\dot{r}\kappa_0\dot{\kappa}_0 v_1^2 - r^2\dot{\alpha}_0\kappa_0 v_1(\alpha * v) \\ - r^3\ddot{r}\kappa_0^2 v_1^2 - r^2\alpha_1\kappa_0^2 v_1(\alpha * v) \\ + r^2\kappa_0^2(\alpha * \alpha)v_1^2 + 2r^2\alpha_0\kappa_0 v_1(\dot{\alpha} * v)\end{array}\right] \\
+\left[\begin{array}{l}(3-4n)r\alpha_0^3\kappa_0 v_1 - 2nr\dot{r}^2\alpha_0\kappa_0 v_1 \\ + 2r^2\ddot{r}\alpha_0\kappa_0 v_1 - r^2\dot{r}\alpha_1\kappa_0^2 v_1 \\ - 2r\alpha_0(\alpha * \alpha)\kappa_0 v_1 - r^2\dot{r}\dot{\alpha}_0\kappa_0 v_1 \\ + (2n-2)\dot{r}\alpha_0^2(\alpha * v) - r^2\dot{r}\alpha_0\kappa_0\kappa_1 v_2 \\ + \alpha_0^2(\alpha * s) - r^2\dot{r}\alpha_0\dot{\kappa}_0 v_1 \\ + r\alpha_0\dot{\alpha}_0(\alpha * v) - r\alpha_0^2(\dot{\alpha} * v) \\ + r\alpha_0\alpha_1\kappa_0(\alpha * v)\end{array}\right] \\
+\alpha_0\left[\begin{array}{l}(n-1)\alpha_0^3 + (n-1)\dot{r}^2\alpha_0 + r\dot{r}\alpha_1\kappa_0 \\ + \alpha_0(\alpha * \alpha) + r\dot{r}\dot{\alpha}_0 - r\ddot{r}\alpha_0\end{array}\right].
\end{array}\right.$$

B. We complete the discussion of the main theorem in Euclidean space, planes permitted nonparallel. The case to be discussed is that when

$$V|((h^2r^2-1)\kappa_0^2 v_1^2 + h^2(\alpha * v)^2).$$

We will show that this implies either that the planes are really parallel ($\kappa_0 = 0$) or that M^n is part of a round sphere. First, if $h =$

0, M is minimal, and this case was treated in [**2**]. Note that in $((h^2r^2-1)\kappa_0^2v_1^2+h^2(\alpha * v)^2)$, the coefficient of v_1^2 is $(h^2r^2-1)\kappa_0^2+h^2\alpha_1^2$, while the coefficient of any other v_i is α_i^2. This gives the equation

$$h^2r^2\kappa_0^2 - \kappa_0^2 + h^2\alpha_1^2 = \alpha_2^2 = \alpha_2^2 = \cdots = \alpha_n^2. \tag{24}$$

The coefficient of the product v_iv_j is $\alpha_i\alpha_j$, which shows that all products

$$\alpha_i\alpha_j = 0.$$

At most one of the α_i can be nonzero. In $\mathbf{R}^{n+1}$, we were restricting to $n \geq 3$; it follows that, say, either $\alpha_2 = 0$ or $\alpha_3 = 0$. Equation (24) tells us that

$$\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0, \tag{25}$$

and

$$h^2r^2\kappa_0^2\kappa_0^2 + h^2\alpha_1^2 = \kappa_0^2.$$

Should it be the case that $\kappa_0 = 0$, we are done, being back in the case of parallel planes again. If $\kappa_0 \neq 0$, we divide through, arriving at

$$r^2 + \frac{\alpha_1^2}{\kappa_0^2} = \frac{1}{h^2} = \text{constant}. \tag{26}$$

The effect of (25) on P is dramatic. Terms such as $\alpha * v$ or $\dot{\alpha} * v$ reduce to a coefficient times v_1. There are some terms containing a v_2, either explicit or in the sum $\alpha * s$. The latter term becomes $-r\alpha_1\kappa_1v_2$. It is also true that A and D retain only terms with v_1. All in all, Q now involves only v_1 and v_2. This has an important consequence; it is not necessary here to consider the homogeneous terms in any particular order. Any nonzero multiple of $V-1$ must have a term with a factor of v_3^2. Since Q has no v_3 terms anywhere, we know that $Q = 0(V-1) = 0$. This will allow us to jump from considering degree seven terms to degrees zero and one; all must be zero. We display the revised value of

P below:

$$
\begin{aligned}
P \equiv\ & 2nr^{n+1}\kappa_0^2 v_1^4[r^2\kappa_0^2+\alpha_1^2] \\
& + 2r^n\kappa_0 v_1^3\begin{bmatrix} (1-4n)r^2\alpha_0\kappa_0^2+(2n+2)r\dot{r}\kappa_0\alpha_1 \\ +(2-2n)\alpha_0\alpha_1^2+r^2\dot{\kappa}_0\alpha_1-r^2\kappa_0\dot{\alpha}_1 \end{bmatrix} \\
& + 2r^{n-1}v_1\begin{bmatrix} (6n-3)r^2\alpha_0^2\kappa_0^2 v_1-4nr\dot{r}\alpha_0\alpha_1\kappa_0 v_1 \\ +(n+1)r^2\dot{r}^2\kappa_0^2 v_1+(n-2)\alpha_0^2\alpha_1^2 v_1 \\ -r^2\alpha_0\alpha_1\dot{\kappa}_0 v_1+r^3\dot{r}\kappa_0^2\kappa_1 v_2 \\ +r^2\alpha_0\alpha_1\kappa_0\kappa_1 v_2+r^3\dot{r}\kappa_0\dot{\kappa}_0 v_1 \\ -r^2\dot{\alpha}_0\alpha_1\kappa_0 v_1-r^3\ddot{r}\kappa_0^2 v_1 \\ +2r^2\alpha_0\dot{\alpha}_1\kappa_0 v_1 \end{bmatrix} \\
& + 2r^{n-1}\begin{bmatrix} (3-4n)r\alpha_0^3\kappa_0 v_1-2nr\dot{r}^2\alpha_0\kappa_0 v_1 \\ +2r^2\ddot{r}\alpha_0\kappa_0 v_1-r^2\dot{r}\alpha_1\kappa_0^2 v_1 \\ -r\alpha_0\alpha_1^2\kappa_0 v_1-r^2\dot{r}\dot{\alpha}_0\kappa_0 v_1 \\ +(2n-2)\dot{r}\alpha_0^2\alpha_1 v_1-r^2\dot{r}\alpha_0\kappa_0\kappa_1 v_2 \\ -r\alpha_0^2\alpha_1\kappa_1 v_2-r^2\dot{r}\alpha_0\dot{\kappa}_0 v_1 \\ +r\alpha_0\dot{\alpha}_0\alpha_1 v_1-r\alpha_0^2\dot{\alpha}_1 v_1 \end{bmatrix} \\
& + 2r^{n-1}\alpha_0\begin{bmatrix} (n-1)\alpha_0^3+(n-1)\dot{r}^2\alpha_0+r\dot{r}\alpha_1\kappa_0 \\ +\alpha_0\alpha_1^2+r\dot{r}\dot{\alpha}_0-r\ddot{r}\alpha_0 \end{bmatrix}.
\end{aligned}
$$

With (25) and (26), (and $\kappa_0 \neq 0$), we have two of the three conditions needed to prove that M is itself a subset of an n-dimensional sphere. It remains to show that $\alpha_0 = (d/dt)(\alpha_1/\kappa_0)$.

If we differentiate (26), we arrive at

$$\alpha_1^2\dot{\kappa}_0 = r\dot{r}\kappa_0^3 + \alpha_1\dot{\alpha}_1\kappa_0. \tag{27}$$

From here, we conclude by examining several cases. In all that follows, we assume $\kappa_0 \neq 0$.

Case I. If $\alpha_1 = 0$, the highest degree term in Q is

$$Q_7 = 8nr^{2n+5}\alpha_0\kappa_0^7 v_1^7.$$

This must be equal to zero, so $\kappa_0 \neq 0$ implies that $\alpha_0 = 0$. Thus all the α's are zero, meaning that $c(t)$ is a fixed point in $\mathbf{R}^{n+1}$. It follows from (26) that r is constant as well. Therefore, M is part of a round sphere.

We define three more abbreviations:

$$\begin{aligned} Y &= r^2\kappa_0^2 + \alpha_1^2, \\ Z &= r^2\kappa_0^2 - (n-2)\alpha_1^2, \\ U &= \alpha_0\alpha_1 + r\dot{r}\kappa_0. \end{aligned}$$

Case II. $\alpha_1 \neq 0$. This time we multiply Q by α_1. A long calculation reveals that the highest degree terms factor as

$$(\alpha_1 Q)_7 = 8nr^{2n+1}\kappa_0^3 v_1^7 YUZ.$$

Since Y cannot be 0 unless $\kappa_0 = 0$, we know that either U or Z is equal to 0.

a) $\alpha_1 \neq 0$, but $U = 0$. Then $0 = \kappa_0^2 U = \alpha_0\alpha_1\kappa_0^2 + r\dot{r}\kappa_0^3$. Combining this with (27) gives $\alpha_0\alpha_1\kappa_0^2 + \alpha_1^2\dot{\kappa}_0 - \alpha_1\dot{\alpha}_1\kappa_0 = 0$, or

$$\alpha_0 = \frac{\dot{\alpha}_1\kappa_0 - \alpha_1\dot{\kappa}_0}{\kappa_0^2}.$$

This is the quotient rule form of

$$\alpha_0 = \frac{d}{dt}\left(\frac{\alpha_1}{\kappa_0}\right),$$

so this case is finished: M is part of a sphere.

b) $\alpha_1 \neq 0$, but $Z = 0$, and we use the fact that $\dot{Z} = 0$. Applying (27) to the linear combination $[(\kappa_0/2)\dot{Z} - \dot{\kappa}_0 Z]$ gives

$$(n-1)r\dot{r}\kappa_0^3 = 0.$$

Therefore,

$$\dot{r} = 0.$$

A number of other useful equations occur in the case $Z = 0$. As

$$r^2\kappa_0^2 = (n-2)\alpha_1^2,$$

the fact that r is constant forces a constant ratio between α_1 and κ_0. In turn, this means that

$$\frac{d}{dt}\left(\frac{\alpha_1}{\kappa_0}\right) = 0. \tag{28}$$

Equation (26) may always be abbreviated $Yh^2 = \kappa_0^2$. Another consequence of $Z = 0$ is $Y = (n-1)\alpha_1^2$ or

$$Y = \left(\frac{n-1}{n-2}\right) r^2 \kappa_0^2. \tag{29}$$

Combining (29) and (26) leads to

$$h^2 r^2 = \left(\frac{n-2}{n-1}\right),$$

so that

$$4n^2 h^2 r^{2n} = 4n^2 \left(\frac{n-2}{n-1}\right) r^{2n-2}. \tag{30}$$

As mentioned before, we are free to consider homogeneous parts of Q in any convenient order, now that Q has no terms with a factor of v_3. The fact that $\dot{r} = 0$ makes the degree zero part of Q comparatively simple to calculate. In order to use (30), it is convenient to multiply by $(n-1)$ first. The result is

$$(n-1)Q_0 = 4r^{2n-2}\alpha_0^4 \begin{bmatrix} (-n^2+3n-1)\alpha_0^4 \\ +(2n^2-4n+2)\alpha_0^2\alpha_1^2 \\ +(n-1)\alpha_1^4 \end{bmatrix},$$

and this must be equal to zero. If we consider the case $\alpha_0 = 0$, combining this with (28) gives

$$\alpha_0 = 0 = \frac{d}{dt}\left(\frac{\alpha_1}{\kappa_0}\right),$$

so that case is concluded. Otherwise, we consider the case

$$(-n^2+3n-1)\alpha_0^4 + (2n^2-4n+2)\alpha_0^2\alpha_1^2 + (n-1)\alpha_1^4 = 0. \tag{31}$$

We have restricted to a situation where $\kappa_0 \neq 0$, $\alpha_0 \neq 0$, $\alpha_1 \neq 0$. Therefore, all three keep a definite sign. It follows that α_0/α_1 is constant. Differentiating this leads to

$$\dot{\alpha}_1 \alpha_0 = \dot{\alpha}_0 \alpha_1. \tag{32}$$

With this in hand, we search Q for linear terms that are multiples of v_1 but not of v_2. From previous comments, even these are forced to total zero. Once again, in order to use (30), it is convenient to multiply by $(n-1)$ first. The result is

$$(n-1)Q_{v_1} = 8r^{2n-1}\alpha_0^3\kappa_0 v_1 \begin{bmatrix} (3n^2-10n+3)\alpha_0^4 \\ +(-5n^2+9n-4)\alpha_0^2\alpha_1^2 \\ +(-n+1)\alpha_1^4 \end{bmatrix}.$$

This gives

$$(3n^2-10n+3)\alpha_0^4 + (-5n^2+9n-4)\alpha_0^2\alpha_1^2 + (-n+1)\alpha_1^4 = 0. \tag{33}$$

We will show by contradiction that the case under consideration, $Z = 0$, but $\alpha_0 \neq 0$, is not really possible. If we add together equations (31) and (33), then divide by α_0^2, we find that

$$(2n^2-7n+2)\alpha_0^2 + (-3n^2+5n-2)\alpha_1^2 = 0.$$

Therefore, the ratio α_0^2/α_1^2 must be a rational number. In contrast, divide (31) by α_1^4; the result is a quadratic equation in the same ratio α_0^2/α_1^2. For this quadratic equation, the discriminant is

$$4n^2(n-2)(n-1).$$

The square root of this cannot be rational, because

$$n-2 < \sqrt{(n-2)(n-1)} < n - \frac{3}{2},$$

so that $\sqrt{(n-2)(n-1)}$ is not an integer and must be irrational. Therefore, the ratio α_0^2/α_1^2 must also be irrational, a contradiction. $\square$

C. We complete the consideration of asymptotic horospheres in $\mathbf{H}^{n+1}$. So far, we wrote

$$\begin{aligned}
&A = -1, \qquad F = t,\\
&E = -r(\dot{r} + (\alpha * v)),\\
&D = (\alpha * v)^2 + 2\dot{r}(\alpha * v) + \dot{r}^2 + 1,
\end{aligned}$$

and

$$\begin{aligned}
C = {} & (n-2)(\alpha * v)^2 + 2\dot{r}(n-1)(\alpha * v)\\
& - r(\dot{\alpha} * v) + (n-1)(1 + \dot{r}^2) - r\ddot{r} + (\alpha * \alpha).
\end{aligned}$$

The condition of constant mean curvature $H = h$ is the vanishing, when $V - 1 = 0$, of the polynomial

$$Q = W^2 - n^2h^2r^2A^2D^3.$$

As $A = -1$ here, we have

$$Q = W^2 - n^2h^2r^2D^3.$$

The numerator in the mean curvature is

$$W = nDE - FC.$$

As $F = t$ is degree zero and C is quadratic, W is cubic, with

$$W_3 = -nr(\alpha * v)^3.$$

Then Q is seen to be of degree six, with

$$Q_6 = n^2r^2(1 - h^2)(\alpha * v)^6.$$

There are two possibilities. First, α_i could be zero for all $i \geq 1$. That means that the two spheres are coaxial. The relationship of the model to $\mathbf{H}^{n+1}$ implies that M must be a hypersurface of revolution, although it is true that the planes $t =$ constant represent horospheres rather than planes in the model. Note that we can allow $n \geq 2$ in this theorem.

The second possibility is $h^2 = 1$. The degree of Q drops to 5. A modest calculation shows that now

$$Q_5 = 2nrt(n-2)(\alpha * v)^5.$$

This shows, for $n \geq 3$, that $\alpha = 0$ anyway. We restrict to $\alpha \neq 0$ and $n = 2$, meaning a surface foliated by circles in asymptotic horospheres of $\mathbf{H}^3$.

We continue with $h^2 = 1$ and $n = 2$. Quartic terms remain, giving

$$Q_4 = 4r(\alpha * v)^3\{(2t\dot{r} - r)(\alpha * v) - rt(\dot{\alpha} * v)\}.$$

Unless $\alpha = 0$, we find that the complicated factor is 0, or

$$\{(2t\dot{r} - r)\alpha - rt\dot{\alpha}\} * v = 0.$$

By varying the vector v on an open set, we show that the vector quantity

$$(2t\dot{r} - r)\alpha - rt\dot{\alpha} = 0,$$

or

$$\dot{\alpha} = \{(2t\dot{r} - r)/(rt)\}\alpha.$$

After much cancellation, Q is seen to have become cubic, with

$$Q_3 = 4r(\alpha * v)^3\{t(\alpha * \alpha) + t\dot{r}^2 + t - tr\ddot{r} - r\dot{r}\}.$$

Once again, we take the second factor as being 0. The final piece is the quadratic Q, with

$$Q_2 = -3r^2(\alpha * v)^2.$$

Since V must divide this, we cannot escape the fact that $\alpha = 0$. $\square$

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