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RIEMANNIAN MANIFOLDS WITH CONICAL SINGULARITIES

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ABSTRACT. We study Riemannian manifolds with isolated conical singularities, in particular, the relationship between the curvature near singularities and the geometry of the tangent cones. We obtain some local and global rigidity theorems for singular metrics.

1. Introduction. Singular spaces appear naturally in many areas in both mathematics and physics. In general, it is difficult to study the global geometry of singular spaces. There is a special class of singular spaces, namely, Riemannian manifolds with conical singularities, which have been investigated by several people. See, e.g., [**3, 4, 7, 8, 15, 16**], etc. In this paper we shall study the geometric structure of singular Riemannian manifolds from a different point of view.

Throughout this paper, $\mathbf{S}^{n-1}$ and $\mathbf{B}^n(r)$ denote the standard unit sphere and the standard r-ball in the Euclidean space $\mathbf{R}^n$, respectively. Let Σ be an $(n-1)$-dimensional connected closed C^∞ manifold. The *topological cone* $C(\Sigma)$ over Σ is defined by

$$C(\Sigma) := [0,\infty)\times\Sigma/(\{0\}\times\Sigma).$$

Denote points in $C(\Sigma)$ by $[t,x]$ and the vertex by o. For $r>0$, put $C_r(\Sigma)=\{[t,x]\in C(\Sigma) : t<r\}$. Thus $\mathbf{B}^n(r)=C_r(\mathbf{S}^{n-1})$. From now on, "=" means the canonical pointed-isometry (preserving the vertices). A lens space is the quotient space $\mathbf{S}^{n-1}/\Gamma$, where Γ is a finite group acting freely on $\mathbf{S}^{n-1}$ by isometries. We shall always denote by $d\theta^2$ the canonical quotient metric on $\mathbf{S}^{n-1}/\Gamma$. The action of Γ on $\mathbf{S}^{n-1}$ can be lifted to an action of Γ on $\mathbf{B}^n(r)$ such that $\mathbf{B}^n(r)/\Gamma = C_r(\mathbf{S}^{n-1}/\Gamma)$. Thus $C_r(\mathbf{S}^{n-1}/\Gamma)$ is a topological orbifold.

An n-dimensional C^∞ *manifold with isolated conical singularities* is a Hausdorff space with countably many points, called singular

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points, S, such that $M \setminus S$ is an n-dimensional C^∞ manifold without boundary, and for each $p \in S$, there is a pointed-diffeomorphism $\varphi_p : (C_{r_p}(\Sigma_p), o) \to (U_p, p)$ with $\varphi_p(o) = p$. $(C_{r_p}(\Sigma_p), \varphi_p)$ is called a *conical chart* at p. If p is a regular point, let $\Sigma_p = \mathbf{S}^{n-1}$, and $\varphi_p : \mathbf{B}^n(r_p) \to U_p \subset M \setminus S$ is a coordinate map.

Let g be an arbitrary Riemannian metric on the regular part $M \setminus S$. Without any condition on g, the geometry near singularities can be complicated. For example, take a two-dimensional cone. Embed it into $\mathbf{R}^3$ so that its tip spirals around z-axis infinitely many times, and its vertex is the origin. With the induced metric, the resulting surface does not have natural tangent cone at the singular point. Therefore we will impose some reasonable conditions on the metrics near singularities. For the sake of simplicity, our conditions on the metrics are slightly stronger than that given by D. Stone [**15**].

Definition 1.1. Let M be a C^∞ manifold with isolated singularities S. A Riemannian metric g on $M \setminus S$ is called C^0-*conical*, respectively C^2-*conical*, at a singular point $p \in S$, if there is a conical chart $(C_{r_p}(\Sigma_p), \varphi_p)$ and a Riemannian metric h_p on Σ_p such that (1.1) and (1.2), respectively (1.1) and (1.2′), hold.

(1.1) $\varphi_p^* g = dt^2 \oplus h_t$, where h_t is a family of metrics on Σ_p;

(1.2) $(1/t^2)h_t \to h_p$, $(1/(2t))(\partial/\partial t)h_t \to h_p$ in the C^0 topology on Σ_p;

(1.2′) $(1/t^2)h_t \to h_p$, $(1/(2t))(\partial/\partial t)h_t \to h_p$, $(1/2)(\partial^2/\partial t^2)h_t \to h_p$ in the C^2 topology on Σ_p.

$(C_{r_p}(\Sigma_p), \varphi_p)$ is called a *metric-conical chart* at p, (Σ_p, h_p) is called the *space of directions* at p and $(C(\Sigma_p), g_p)$ is called the *tangent cone* at p, where $g_p := dt^2 \oplus t^2 h_p$.

In this paper we begin with studying the relationship between the geometry of (Σ_p, h_p) and the curvature near singularities. Thus we shall restrict our attention to $C_r(\Sigma)$ equipped with a singular metric $g = dt^2 \oplus h_t$ such that g is C^k-conical at the vertex o. Let (Σ, h) be the space of directions at o. Let $K_-(t)$, respectively $\mathrm{Ric}_-(t)$, denote the minimum of the sectional curvature K, respectively the Ricci curvature Ric, on the t-sphere $S(o,t) \subset C_r(\Sigma)$ around the vertex o. Let $K(t)$, respectively $\mathrm{Ric}\,(t)$, denote the maximum of the absolute value of K, respectively Ric, on $S(o,t)$. We have the following rigidity theorem for (Σ, h).

Theorem A. *Let $n = \dim M > 2$. Let $g = dt^2 \oplus h_t$ be a C^k-conical Riemannian metric on $C(\Sigma)$, $k = 0$ in the sectional curvature case and $k = 2$ in the Ricci curvature case.*

(a) *If*

$$\liminf_{t \to 0^+} t^2 K_-(t) \geq 0,$$

respectively

$$\lim_{t \to 0^+} t^2 \mathrm{Ric}_-(t) \geq 0, \tag{1.3}$$

then (Σ, h) has sectional curvature $K_\Sigma \geq 1$, respectively Ricci curvature $\mathrm{Ric}_\Sigma \geq n-2$;

(b) *If*

$$\lim_{t \to 0^+} t^2 K(t) = 0,$$

respectively

$$\lim_{t \to 0^+} t^2 \mathrm{Ric}\,(t) = 0, \tag{1.4}$$

then (Σ, h) has sectional curvature $K_\Sigma = 1$, respectively Ricci curvature $\mathrm{Ric}_\Sigma = n-2$.

If the sectional curvature near a singular point p satisfies (1.4), then the topology at p is of a very restricted type. Let $(C_{r_p}(\Sigma_p), \varphi_p)$ be a metric-conical chart at p. Suppose that $\varphi_p^* g$ satisfies (1.4). By Theorem A (b), there is an isometry $\bar{\psi}_p : \mathbf{S}^{n-1}/\Gamma_p \to (\Sigma_p, h_p)$, which induces a pointed-diffeomorphism $\psi_p : \mathbf{B}^n(r_p)/\Gamma_p = C_{r_p}(\mathbf{S}^{n-1}/\Gamma_p) \to C_{r_p}(\Sigma_p)$. Let $\tilde{\varphi}_p = \varphi_p \circ \psi_p$. The above argument shows that U_p is a topological orbifold with respect to $\tilde{\varphi}_p$. In general, the lift of the metric $\tilde{\varphi}_p^* g$ to $\mathbf{B}^n(r_p)$ cannot be smoothly extended across the origin and the lifted action of Γ_p on $\mathbf{B}^n(r_p)$ is not by isometries. Thus (U_p, g) is not a Riemannian orbifold with respect to $\tilde{\varphi}_p : \mathbf{B}^n(r_p)/\Gamma_p \to U_p$.

When does (U_p, g) become a Riemannian orbifold with respect to the above chart $\tilde{\varphi}_p : \mathbf{B}^n(r_p)/\Gamma_p \to U_p$? Let $\sin_\lambda(t)$ denote the unique solution of $y'' + \lambda y = 0$, $y(0) = 0, y'(0) = 1$. Assume that $\varphi_p^* g$ has the form

$$\varphi_p^* g = dt^2 \oplus \sin_\lambda^2(t) h_p. \tag{1.5}$$

Then $\tilde{\varphi}_p^* g$ has the form $\tilde{\varphi}_p^* g = dt^2 \oplus \sin_\lambda^2(t)\, d\theta^2$. In this case, the lift of $\tilde{\varphi}_p^* g$ to $\mathbf{B}^n(r_p)$ is a smooth metric (across the origin) of constant curvature λ and Γ_p acts on $\mathbf{B}^n(r_p)$ by isometries. Hence (U_p, g) is a Riemannian orbifold. We assert that (1.5) is satisfied if $K_{M\setminus S} = \lambda$. More precisely, we have the following

Theorem B. *Let (M^n, g), $n \geq 3$, be a Riemannian manifold with isolated C^0-conical singularities S. Assume that $K_{M\setminus S} = \lambda$. Then, for every singular point p, there is a metric-conical chart $\tilde{\varphi}_p : C_{r_p}(\mathbf{S}^{n-1}/\Gamma_p) \to U_p$ such that $\tilde{\varphi}_p^* g = dt^2 \oplus \sin_\lambda^2(t)\, d\theta^2$. Hence (U_p, g) is a Riemannian orbifold. With the above metric-conical charts, (M, g) is isometric to $\mathbf{M}^n(\lambda)/\Gamma$ for some discrete group Γ acting on the space form $\mathbf{M}^n(\lambda)$ by isometries with finite isotropic subgroup at isolated points.*

Let (M, g) be a Riemannian manifold with isolated conical singularities S. Let d^* denote the induced metric on $M \setminus S$ by g. Hence the topology on $M \setminus S$ defined by d^* coincides with the manifold topology of $M \setminus S$. (M, g) is called *complete* if every *bounded* subset $A \subset M \setminus S$ contains a sequence convergent to a point in M. It is easy to see that there is a unique metric d on M such that (M, d) is the completion of $(M \setminus S, d^*)$. Further, the topology determined by d coincides with the topology of M. In [**1**], (M, d) is called *finitely compact*.

A natural question is if the Bishop-Gromov volume comparison theorem still holds for open metric r-balls $B(p, r)$. The answer is negative in dimension 2. Consider the cone $C(\mathbf{S}^1)$ with the metric $g = dt^2 \oplus 2t^2 d\theta^2$, where $d\theta^2$ is the canonical metric on the unit circle S^1. Let $p = [1, x] \in C(\mathbf{S}^1)$. It is easy to see that g is flat and the ratio $\mathrm{vol}\,[B(p, r)]/r^2$ is strictly increasing across $r = 1$.

Nevertheless, we have the following

Theorem C. *Let (M^n, g) be a complete Riemannian n-manifold with isolated C^2-conical singularities S. Suppose that* $\mathrm{Ric}_{M\setminus S} \geq (n-1)\lambda$

(and $\operatorname{diam}(\Sigma_p) \le \pi$ *if* $n = 2$*). Then for any* $p \in M$*, the ratio* $v_p(r) := \operatorname{vol}[B(p,r)]/V_\lambda(r)$ *is nonincreasing. In particular, we have*

$$\operatorname{vol}(B(p,r)) \le \frac{\operatorname{vol}(\Sigma_p)}{\operatorname{vol}(\mathbf{S}^{n-1})} V_\lambda(r). \tag{1.6}$$

Here $V_\lambda(r)$ *denotes the volume of the* r*-ball in the space form* $\mathbf{M}^n(\lambda)$ *of constant curvature* λ*. If the equality in* (1.6) *holds, for some* $r > 0$*, then* $B(p,r)$ *is isometric to the cone* $C_r(\Sigma_p)$ *with the metric* $g = dt^2 \oplus \sin^2_\lambda(t) h_p$*,* $0 < t < r$*.*

The following is an important application of Theorem C. It is the conical singularity version of Bonnet-Myers's theorem and Cheng's maximal diameter theorem.

Theorem D. *Let* (M,g) *be a complete Riemannian manifold of dimension* $n \ge 3$ *with isolated* C^2*-conical singularities* S*. Suppose that* $\operatorname{Ric}_{M\setminus S} \ge (n-1)$*. Then the diameter* $\operatorname{diam}_M = d(p,q) \le \pi$*. The equality holds if and only if* (M,g) *is isometric to the standard metric sine-suspension* $S(\Sigma)$ *over a connected closed Riemannian* $(n-1)$*-manifold* (Σ,h) *with* $\operatorname{Ric}_\Sigma \ge (n-2)$*.*

The sine-suspension $S(\Sigma)$ over (Σ,h) is the quotient space $[0,\pi] \times \Sigma/(\{0\}\times\Sigma \cup \{\pi\}\times\Sigma)$ equipped with the metric $dt^2 \oplus \sin^2_\lambda(t)h$.

2. Curvature and space of directions. In this section we shall study the role played by the curvature near a singular point in the topology as well as in the geometry of the tangent cone. In the process, we prove Theorems A and B.

Proof of Theorem A. Let Π_t denote the second fundamental form of the submanifold $i_t : \Sigma \to C(\Sigma)$ where $i_t(x) := [t,x]$. Notice that $(i_t)^*g = h_t$ and $T := \partial/\partial t$ is the normal vector to $i_t(\Sigma)$. Let (x^i) be a local coordinate system for Σ, and (t,x^i) be the standard local coordinate system for $C(\Sigma)$. By an easy calculation, using the definition of the Levi-Civita connection, we have

$$(\Pi_t)_{ij} = \frac{1}{2}\frac{\partial}{\partial t}(h_t)_{ij}. \tag{2.1}$$

We first prove Theorem A in the sectional curvature case. Let R^t denote the curvature tensor of (Σ, h_t). By the Gauss equation

$$(R^t)_{ijkl} = R_{ijkl} + \frac{1}{4}\left\{ \frac{\partial}{\partial t}(h_t)_{il}\frac{\partial}{\partial t}(h_t)_{jk} - \frac{\partial}{\partial t}(h_t)_{ik}\frac{\partial}{\partial t}(h_t)_{jl} \right\}. \tag{2.2}$$

If we assume (1.3), then we see that

$$\liminf_{t\to 0^+} K_{(\Sigma, t^{-2}h_t)} \geq 1.$$

Since the lower sectional curvature bound is preserved in the Hausdorff-Gromov convergence, we have $K_{(\Sigma,h)} \geq 1$.

On the other hand, if we assume (1.4), then

$$\lim_{t\to 0^+} K_{(\Sigma, t^{-2}h_t)}(\sigma) = 1.$$

Since $t^{-2}h_t \to h$ in the C^0 topology, $t^{-2}h_t$ does not collapse, hence $t^{-2}h_t \to h$ in the $C^{1,\alpha}$ topology and the limit metric h has constant curvature 1 in the sense of Alexandrov. Hence h must be of constant curvature 1 in the usual sense.

To prove Theorem A in the Ricci curvature case, we need the following identities.

$$R_{ittj} = -\frac{1}{2}\frac{\partial^2}{\partial t^2}(h_t)_{ij} + \frac{1}{4}\frac{\partial}{\partial t}(h_t)_{ik}(h_t)^{kl}\frac{\partial}{\partial t}(h_t)_{lj}. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned}(R^t)_{ij} = R_{ij} &+ \frac{1}{2}\frac{\partial^2}{\partial t^2}(h_t)_{ij} \\ &+ \frac{1}{4}\frac{\partial}{\partial t}(h_t)_{ij}(h_t)^{kl}\frac{\partial}{\partial t}(h_t)_{kl} \\ &- \frac{1}{2}\frac{\partial}{\partial t}(h_t)_{ik}(h_t)^{kl}\frac{\partial}{\partial t}(h_t)_{jl}.\end{aligned}$$

The rest of the proof is similar to that in the sectional curvature case. Note that, since we do not have Hausdorff-Gromov convergence in the Ricci case, C^2-conical condition has to be assumed instead of C^0-conical. □

The following example shows that assumptions (1.3) and (1.4) are necessary. Let (Σ, h) be a connected Riemannian manifold. Suppose that $\min K_\Sigma < 1$ or $\max K_\Sigma > 1$. Let $g = dt^2 \oplus \sin_\lambda^2(t)h$ be the standard metric on $C(\Sigma)$. Let $K_+(t) = \max_{S(o,t)} K_\Sigma$ and $K_-(t) = \min_{S(o,t)} K_{C(\Sigma)}$, where $S(o,t)$ denote the t-sphere around the vertex o. It is easy to see that

$$\limsup_{t\to 0^+} t^2 K_+(t) = \max K_\Sigma - 1, \quad \liminf_{t\to 0^+} t^2 K_-(t) = \min K_\Sigma - 1.$$

We now prove Theorem B.

Proof of Theorem B. By Theorem A, (Σ_p, h_p) has constant curvature 1. Thus (Σ_p, h_p) is isometric to a lens space $\mathbf{S}^{n-1}/\Gamma_p$. The main task is to show that $(\varphi_p)^* g$ has the following form

$$(\varphi_p)^* g = dt^2 \oplus \sin_\lambda^2(t) h_p.$$

Let (x^i) be a local coordinate system for Σ_p and (t, x^i) be the standard coordinate system for U_p given by the metric-conical chart. Let H_t denote the $(n-1)\times(n-1)$ matrix $((h_t)_{ij})$ and H denote the $(n-1)\times(n-1)$ matrix $((h_p)_{ij})$. Since $K_{M\setminus S} = \lambda$,

$$R_{ittj} = \lambda (h_t)_{ij}. \tag{2.4}$$

It follows from (2.3) and (2.4) that H_t satisfies the following ODE which is *singular at* $t = 0$, because of the term H_t^{-1},

$$\lambda H_t = -\frac{1}{2} H_t'' + \frac{1}{4} H_t' H_t^{-1} H_t', \tag{2.5}$$

with the initial conditions

$$\frac{1}{t^2} H_t \longrightarrow H, \quad \frac{1}{2t}\frac{\partial}{\partial t} H_t \longrightarrow H.$$

Let

$$F_t = \sin_\lambda(t)^{-2} H_t.$$

Then Equation (2.5) becomes

$$2F'' + 4\frac{\sin_\lambda'(t)}{\sin_\lambda(t)} F' - F' F^{-1} F' = 0, \tag{2.6}$$

with initial conditions:

$$\lim_{t\to 0^+} F_t = H, \qquad \lim_{t\to 0^+} tF'_t = 0. \tag{2.7}$$

We want to show that $F_t = H$ for small $t > 0$.

On the set of $(n-1)\times(n-1)$ matrices, define an inner product $\langle A, B\rangle = \Sigma a_{i,j} b_{i,j}$, with associated L_2-norm $\|A\| = \sqrt{\langle A, A\rangle}$. It is easy to check that $\|AB\| \le \|A\|\,\|B\|$.

By (2.7), there are numbers $\delta > 0$, $C > 0$ and $\varepsilon > 0$ with $\varepsilon C < 2$, such that

$$\|F_t^{-1}\| \le C, \qquad \frac{\sin_\lambda(t)}{\sin'_\lambda(t)}\|F'_t\| \le \varepsilon, \quad 0 < t \le \delta. \tag{2.8}$$

Let $f(t) = \|F'_t\|^2$. Note that by (2.7) and (2.8),

$$f^{1/2}(t) \le \varepsilon \frac{\sin'_\lambda(t)}{\sin_\lambda(t)}, \qquad \lim_{t\to 0^+} t f^{1/2}(t) = 0. \tag{2.9}$$

By (2.6), we have

$$\begin{aligned} f'(t) &= -4\frac{\sin'_\lambda(t)}{\sin_\lambda(t)}\langle F'_t, F'_t\rangle + \langle F'_t, F'_t F_t^{-1} F'_t\rangle \\ &\le -4\frac{\sin'_\lambda(t)}{\sin_\lambda(t)} f + C f^{3/2}. \end{aligned}$$

Thus

$$f'(t) \le -4\frac{\sin'_\lambda(t)}{\sin_\lambda(t)} f + C\varepsilon \sin'_\lambda(t)\sin_\lambda(t) f = -2\sin'_\lambda(t)\sin_\lambda(t) f.$$

It follows that, for $0 < \tau \le t \le \delta$,

$$f(t) \le \left(\frac{\sin_\lambda(\tau)}{\sin_\lambda(t)}\right)^2 f(\tau).$$

For any fixed $t > 0$, letting $\tau \to 0^+$ and using (2.9), we obtain $f(t) = 0$. That is, $F'_t = 0$. Thus $F_t = F_0 = H$.

Now we see that M is a *Riemannian* orbifold with constant curvature. It is now well-known that such spaces are isometric to a quotient $\mathbf{M}^n(\lambda)/\Gamma$ for some discrete group Γ acting on the space form $\mathbf{M}^n(\lambda)$ by isometries with finite isotropic subgroup at isolated points. See, for example, [**14**, Chapter 13]. (We gave a more direct, geometric proof of this fact in a preliminary version of this paper. It is omitted here for brevity.) □

3. The index lemma. In order to study the geometry of singular Riemannian manifolds by comparison method, we need to establish the basic index lemma.

Let (M, g) be a Riemannian manifold with isolated conical singularities S. Let d^* denote the induced metric by g on $M \setminus S$. Let $c : I \to M \setminus S$ be a continuous curve. c is called a *geodesic* with respect to d^* if, for any $t_0 \in I$, there is an open subinterval $t_0 \in I' \subset I$ such that, for some $\mu > 0$,

$$d^*(c(t_1), c(t_2)) = \mu|t_1 - t_2|, \quad \forall\, t_1, t_2 \in I'. \tag{3.1}$$

c is called a *geodesic* with respect to g, if c is smooth and satisfies the following *geodesic* equation

$$\frac{d^2x^a}{dt^2} + \Gamma^a_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} = 0. \tag{3.2}$$

A well-known fact is that both definitions are equivalent.

Let $(C_{r_p}(\Sigma_p), \varphi_p)$ be a metric-conical chart at p and (Σ_p, h_p) the space of directions. It is easy to verify that, for any $x \in \Sigma_p$, $d^*(\varphi_p[t_1, x], \varphi_p[t_2, x]) = |t_1 - t_2|$, for all $t_1, t_2 \in (0, r_p)$. Thus $B(p, r) = \varphi_p[C_r(\Sigma_p)]$ for $r < r_p$. We shall always denote by $\gamma_x : (0, a_x) \to (M \setminus S, g)$ the geodesic defined on the *maximal* interval such that $\gamma_x(t) = \varphi_p[t, x]$, $0 < t < r_p$.

Let $i_t : \Sigma_p \to C(\Sigma_p)$ denote the natural embedding given by $i_t(x) = [t, x]$. For a vector $u \in T_x\Sigma_p$, let $U(t) = (\varphi_p \circ i_t)_* u$. Clearly, U is a Jacobi field along $\gamma_x|_{(0,r_p)}$. U can be uniquely extended to a Jacobi field J_u along $\gamma_x|_{(0,a_x)}$. The map $\mathcal{I}_t : u \in T_x\Sigma_p \to J_u(t) \in \gamma_x'(t)^\perp$ is a linear map for any $0 < t < a_x$.

Let (x^i) be a local coordinate system in Σ_p at x and (t, x^i) the standard coordinate system in U_p along $\gamma_x|_{(0,r_p)}$. Then $J_i(t) = (\partial/\partial x^i)|_{[t,x]}$

is the Jacobi field corresponding to $(\partial/\partial x^i)|_x \in T_x\Sigma_p$. Further,

$$g(J_i(t), J_j(t)) = (h_t)_{ij}. \tag{3.3}$$

By Schwarz inequality and (3.3),

$$|g(E(t), J_i(t))| \le g(E(t), E(t))^{1/2} g(J_i(t), J_i(t))^{1/2} = O(t). \tag{3.4}$$

Let $\mathcal{W}$ denote the vector space of piecewise smooth vector fields along $\gamma_x|_{(0,r]}$. The *index form* on $\gamma_x|_{(0,r]}$ is defined by the following *improper* integral

$$I_r(W, W) = \int_0^r \{g(W', W') - g(R(W, \gamma_x')\gamma_x', W)\}\, dt, \quad \forall\, W \in \mathcal{W}.$$

By (1.2), we have

$$g(J_u'(t), J_v(t)) = \frac{1}{2}\frac{\partial}{\partial t}(h_t)(u, v) = O(t). \tag{3.5}$$

Integration by parts yields

$$I_r(J_u, J_u) = g(J_u'(r), J_u(r)).$$

Let $\{e_i\}_{i=1}^{n-1}$ denote a basis for $T_x\Sigma_p$ and $J_i = J_{e_i}$ denote the corresponding Jacobi field along γ_x. Fix $0 < r < a_x$.

Lemma 3.1 (Index Lemma). *Let $J_u = u^i J_i$ be a Jacobi field along γ_x. Then for any $W = f^i J_i$ along $\gamma_x|_{(0,r]}$, where $f_i : (0, r] \to \mathbf{R}$ are piecewise smooth functions with $f_i(t) = o(t^{-1/2})$ and $f_i(r) = u^i$,*

$$I_r(J_u, J_u) \le I_r(W, W),$$

where the equality holds if and only if $W = J$ along $\gamma_x|_{(0,r]}$.

Proof. We follow [**5**] closely. We may let $e_i = (\partial/\partial x^i)|_x$, hence $J_i(t) = (\partial/\partial x^i)|_{[t,x]}$. An easy computation yields

$$J_i'(t) = \frac{1}{2}(h_t)^{jk}\frac{\partial}{\partial t}(h_t)_{ij} J_k(t).$$

From (3.5) we see that

$$g(J_i', J_j) = g(J_j', J_i) = O(t). \tag{3.6}$$

Hence integration by parts yields

$$\begin{aligned} I_r(W,W) = I_r(J,J) + \int_0^r g(f_i' J_i, f_j' J_j)\, dt \\ - \lim_{t\to 0^+} f_i(t) f_j(t) g(J_i'(t), J_j(t)). \end{aligned} \tag{3.7}$$

By assumption and (3.6), we get

$$I_r(W,W) = I_r(J,J) + \int_0^r g(f_i' J_i, f_j' J_j)\, dt.$$

The rest is trivial. □

Let $W \in \mathcal{W}$ be a piecewise smooth vector field along γ_x. We say $W(t) = o(t^\alpha)$ if, for every parallel vector field E, $g(W(t), E(t)) = o(t^\alpha)$. In applications we need the following special version of Lemma 3.1.

Lemma 3.2. *Let J_u, W be a Jacobi field and a piecewise smooth vector field along γ_x, respectively, such that $W(r) = J_u(r)$ and $W(t) = o(t^{1/2})$. Then*

$$I_r(J_u, J_u) \le I_r(W,W), \tag{3.8}$$

where the equality holds if and only if $W = J$ along $\gamma_x|_{(0,r]}$.

Proof. Let $W(t) = f^i(t) J_i(t) = h^i(t) E_i(t)$, where $J_i(t) = (\partial/\partial x^i)|_{[t,x]}$ and E_i are parallel vector fields along γ_x. Observe that

$$g(W(t), J_j(t)) = f^i(t)(h_t)_{ij} = h^k(t) g(J_j(t), E_k(t)).$$

By assumption and (3.4), we have $f^i(t) = o(t^{-1/2})$. Then (3.8) follows from Lemma 3.1. □

4. Exponential map and volume comparison. Let (M, g) be a complete Riemannian manifold with isolated conical singularities S.

Let (M, d) denote the completion of $(M \backslash S, d^*)$. By the definition of d, a continuous curve $c : I \to M\backslash S$ is a geodesic with respect to d if and only if c is a geodesic with respect to d^*. By [**2**] the following Hopf-Rinow theorem holds. For any $p, q \in M$ with $a = d(p, q)$, there is a continuous curve $\gamma : [0, a] \to M$ with $\gamma(0) = p$, $\gamma(a) = q$ such that γ is minimizing geodesic with respect to d, that is, $d(\gamma(t_1), \gamma(t_2)) = |t_1, -t_2|$, for all $t_1, t_2 \in [0, a]$. It is possible that a geodesic passes through some singular points.

Let $(C_{r_p}(\Sigma_p), \varphi_p)$ be a metric-conical chart at p. Recall that $\gamma_x : (0, a_x) \to M\backslash S$ denotes the smooth geodesic defined on the maximal interval such that $\gamma_x(t) = \varphi_p[t, x], 0 < t < r_p$. Notice that $(C_{r_p}(\Sigma_p), \varphi_p^* g)$ is approximated by the tangent cone $(C_{r_p}(\Sigma_p), g_p)$. Thus, if $x, y \in \Sigma_p$ satisfies $d_{h_p}(x, y) < \pi$, then there is a number $r < r_p$ such that

$$d(\gamma_x(t), \gamma_y(t)) < 2t, \quad t \leq r.$$

Therefore, the curve $\gamma_x \cup \gamma_y$ is not a geodesic with respect to d.

Let $C^*(\Sigma_p) = \{[t, x] \in C(\Sigma_p), t < a_x\}$. Define $\exp_p : C^*(\Sigma_p) \to M$ by $\exp_p[t, x] = \gamma_x(t)$. By definition, $\exp_p[t, x] = \varphi_p[t, x]$, $t < r_p$. Since $x \to a_x$ is lower semi-continuous, $C^*(\Sigma_p)$ is open in $C(\Sigma_p)$. From ODE, one can also see that $\exp_p$ is smooth (away from the vertex).

For $x \in \Sigma_p$, let c_x to be the least upper bound of all those $r < a_x$ such that $\gamma_x|_{[0,r]}$ is minimizing. c_x is called the *cut-value* of x. Let $\Sigma_p^* := \{x \in \Sigma_p;\ c_x < a_x\}$. By the standard argument, one can show that Σ_p^* is open in Σ_p and the map $x \in \Sigma_p^* \to c_x$ is continuous. Put $\mathcal{C}_p := \{[c_x, x],\ x \in \Sigma_p^*\}$ and $\Omega_p = \{[t, x],\ 0 \leq t < c_x, x \in \Sigma_p\}$. Clearly, $\dim_H(\exp_p \mathcal{C}_p) \leq n - 1$.

A natural question is whether or not $\exp_p : \Omega_p \to M$ is almost onto. Let us look at the following example again. Let $C(\mathbf{S}^1)$ be the cone with $g = dt^2 \oplus 2t^2 d\theta^2$. Let $p = [1, x] \in C(\mathbf{S}^1)$. Clearly, $\exp_p : \Omega_p \to C(\mathbf{S}^1)$ is not almost onto.

In order to exclude this case, we introduce a notion of convex points. A point $p \in M$ is called *convex* if, for every $x \in \Sigma_p$, there is at most one point $y \in \Sigma_p$ such that $d(x, y) \geq \pi$. In dimension 2, if length$\,(\Sigma_p, h_p) \leq 2\pi$, then p is convex. In higher dimensions, if $\mathrm{Ric}_{M\backslash S} \geq (n - 1)\lambda$, then by Theorem A, $\mathrm{Ric}_{\Sigma_p} \geq n - 2$. Thus p is convex by Cheng's maximal diameter theorem [**9**].

Lemma 4.1. *Let (M, g) be a complete Riemannian n-manifold with isolated C^0-conical convex singularities S. Then, at each point p, the exponential map $\exp_p : \Omega_p \to M$ is almost onto.*

Proof. By the Hopf-Rinow theorem, every point can be joined to p by a minimizing geodesic. Let $q \in S$ be contained in the *interior* of a minimizing geodesic from p. Since q is convex, there is a minimizing geodesic γ from p, passing through q, such that γ is *maximal*, i.e., if σ is another minimizing geodesic from p, passing through q, then $\sigma \subset \gamma$. Let l_q denote the part of γ from q on. If $q \in S$ is not contained in the interior of any minimizing geodesic from p, put $l_q = q$. Let $L_p = \cup_{q \in S} l_q$. Clearly, $\dim L_p \leq 1$.

It suffices to prove that

$$M = \exp_p \Omega_p \cup \exp_p \mathcal{C}_p \cup L_p.$$

Let $z \in M$ and $\gamma : [0, r] \to M$ be a minimizing geodesic from p to z. If γ contains a point $q \in S$, then $z \in L_p$. Now assume that $r < a_x$. If $r = c_x$, then $z \in \exp_p \mathcal{C}_p$, otherwise $z \in \exp_p \Omega_p$. □

An outline of the proof of Theorem C. First by Lemma 4.1, we have

$$\begin{aligned} \mathrm{vol}[B(p, r)] &= \int_{\Omega_p \cap C_r(\Sigma_p)} \det(\exp_p)_* dv_p \\ &= \int_0^r \left\{ \int_{\Sigma_p} \Theta[t, x] dA_p \right\} dt, \end{aligned}$$

where dv_p denotes the volume form on the tangent cone $(C(\Sigma_p), g_p)$, dA_p denotes the volume form on (Σ_p, h_p), and $\Theta[t, x]$ is defined as follows. Let $\{e_i\}_{i=1}^{n-1}$ be an orthonormal basis for $T_x \Sigma_p$, and let $J_i = J_{e_i}$ be the corresponding Jacobi field along γ_x. Then

$$\Theta[t, x] = \sqrt{\det g(J_i(t), J_j(t))}, \quad 0 < t < \min(c_x, r).$$

Otherwise put $\Theta[t, x] = 0$.

For $0 < r' < \min(c_x, r)$, take $\{e_i\}$ such that $g(J_i(r'), J_j(r')) = 0$, $i \neq j$. Thus

$$d \log(\Theta[t, x])|_{t=r'} = \sum_{i=1}^{n-1} \frac{I_{r'}(J_i, J_i)}{g(J_i(r'), J_i(r'))}.$$

Let E_i be parallel vector fields along γ_x with $E_i(r') = J_i(r')$. Let $W_i(t) = (\sin_\lambda(t)/\sin_\lambda(r'))E_i(t)$. By Lemma 3.2,

$$\begin{aligned} d\log(\Theta[t,x])|_{t=r'} &\le \sum_{i=1}^{n-1} \frac{I_{r'}(W_i,W_i)}{g(J_i(r'),J_i(r'))} \\ &= \int_0^{r'} \left\{(n-1)\lambda - \mathrm{Ric}(\gamma_x'(t))\left(\frac{\sin_\lambda(t)}{\sin_\lambda(r')}\right)^2\right\} dt. \end{aligned}$$

By the standard argument one can prove (1.6), see [**2, 12**].

Suppose that the equality in (1.6) holds for some $r > 0$. Then $c_x \ge \min(a_x, r)$, for all $x \in \Sigma_p$. Thus

$$\Omega_p \cap C_r(\Sigma_p) = C^*(\Sigma_p) \cap C_r(\Sigma_p). \tag{4.1}$$

Fix any $r' < \min(a_x, r)$. By the proof of (1.6), we see that, for any $u \in T_x\Sigma_p$,

$$J_u(t) = \frac{\sin_\lambda(t)}{\sin_\lambda(r)} E_u(t), \quad 0 < t \le r',$$

where E_u is a parallel vector field along γ_x with $E_u(r') = J_u(r')$. Since J_u is a Jacobi field,

$$R(E_u(t), \gamma_x'(t))\gamma_x'(t) = \lambda E_u(t), \quad 0 < t < r'.$$

Since $\gamma_x'(r')^\perp = \mathrm{span}\,\{J_u(r'), u \in T_x\Sigma_p\}$ for any $r' < \min(a_x, r)$,

$$R(E(t), \gamma_x'(t))\gamma_x(t) = \lambda E(t), \quad 0 < t < \min(a_x, r),$$

where E is an arbitrary parallel vector field along $\gamma_x|_{(0,\min(a_x,r))}$. By the proof of Theorem B, we get

$$\begin{gathered} g((\exp_p)_*U, (\exp_p)_*U) = \sin_\lambda^2(t) h_p(u,u), \\ 0 < t < \min(a_x, r), \end{gathered} \tag{4.2}$$

where $U_{[t,x]} = (i_t)_*u$.

By (4.2) $\exp_p : \Omega_p \cap C_r(\Sigma_p) \to B(p,r)$ is an embedding preserving the Riemannian metrics, i.e., $(\exp_p)^*g = dt^2 \oplus \sin_\lambda^2(t)h_p$.

By (4.1), it suffices to prove that $\exp_p$ can be extended to a homeomorphism $\Psi : C_r(\Sigma_p) \to B(p,r)$. First we assert that, for any $x \in \Sigma_p$, γ_x can be uniquely extended to a minimizing geodesic $\tilde{\gamma}_x$ defined on $[0,r]$. To see this, let $x_i \in \Sigma_p$ with $a_{x_i} \geq r$ such that $x_i \to x$. Passing to a subsequence, if necessary, we may assume that $\gamma_{x_i}|_{[0,r]}$ are minimizing geodesic converging to a minimizing geodesic $\gamma : [0,r] \to M$ from p. Since $\gamma_{x_i}(t) = \varphi_p[t,x_i], 0 \leq t < r_p$, we must have $\gamma(t) = \varphi_p[t,x] = \gamma_x(t), 0 \leq t < r_p$. Let $\tilde{\gamma}_x = \gamma$. Define $\Psi : C_r(\Sigma_p) \to B(p,r)$ by $\Psi[t,x] = \tilde{\gamma}_x(t), 0 \leq t < r$. It is easy to see that Ψ is an onto homeomorphism. □

5. Proof of Theorem D. We first show that if $\mathrm{Ric}_{M\setminus S} \geq (n-1)\lambda$ for some constant $\lambda > 0$, then $\mathrm{diam}_M \leq \pi/\sqrt{\lambda}$.

Suppose there are p and q in M such that $\mathrm{diam}_M = d(p,q) > \pi/\sqrt{\lambda}$. We claim that there is a minimizing geodesic $\gamma : [0,a] \to M$, joining p and some q' near q such that (i) $a > \pi/\sqrt{\lambda}$, (ii) $q' \in \Omega_p$. This is possible because $M \setminus \Omega_p$ has null measure. On the other hand, by the index lemma, there is a point $\gamma(r)$, $r \leq \pi/\sqrt{\lambda}$, which is conjugate to p along γ. By a standard argument, we conclude that $\gamma|_{[0,r+\varepsilon]}$ is not minimizing for any small $\varepsilon > 0$. It is a contradiction, see [**5**, Chapter 1].

It remains to show that if $\mathrm{diam}_M = \pi$; then M is isometric to the standard metric sine-suspension $S(\Sigma)$ over a connected manifold with $\mathrm{Ric}_\Sigma \geq n-2$.

Let $p,q \in M$ be two points with $d(p,q) = \pi$. Let $e_{pq}(z) = d(p,z) + d(q,z) - \pi$. Since we have Theorem C, the elegant volume comparison argument due to Eschenburg, see [**11**, p. 746], shows that $e_{pq} \equiv 0$. Let $z \neq p,q$ be a (possibly singular) point. Let γ_1, γ_2 be two minimizing geodesics from p to z and z to q, respectively. Since $e_{pq} = 0$, $\tilde{\gamma} = \gamma_1 \cup \gamma_2$ is a minimizing geodesic from p to q. By the argument in Section 4, $\mathrm{diam}\,(\Sigma_z) \geq \pi$. On the other hand, by Theorem A, $\mathrm{Ric}_{\Sigma_z} \geq n-2$. Thus (Σ_z, h_z) is isometric to the standard unit sphere $\mathbf{S}^{n-1}$ by Cheng's maximal diameter theorem. For every $x \in \Sigma_p$, the geodesic γ_x can be uniquely extended to a minimizing geodesic $\tilde{\gamma}_x$ from p to q. Let $\Psi : S(\Sigma_p) \to M$ be defined by $\Psi[t,x] = \tilde{\gamma}_x(t)$. Clearly, Ψ is the homeomorphism.

Following the standard volume argument, see, e.g., [**12**, p. 204] and, using Theorem C again, we have

$$\frac{\mathrm{vol}\,(B(p,r))}{V_1(r)} = \frac{\mathrm{vol}\,(B(q,r))}{V_1(r)} = \text{const.}, \quad 0 < r < \pi. \tag{5.1}$$

By the same argument as in the proof of Theorem C, we conclude that $\Omega_p \cap C_\pi(\Sigma_p) = C^*(\Sigma_p) \cap C_\pi(\Sigma_p)$, and $\exp_p : \Omega_p \cap C_\pi(\Sigma_p) \to M$ is an embedding preserving the Riemannian metrics, i.e., $(\exp_p)^* g = dt^2 \oplus \sin^2(t) h_p$. Notice that $\Psi = \exp_p$ on $\Omega_p \cap C_\pi(\Sigma_p) \subset S(\Sigma_p)$. We are done. □

Let us remark that, by Theorem D and Lemma 4.1, we can also generalize some other well-known theorems in Riemannian geometry, for example, Gromov's theorem on the upper bound of the first Betti number, [**10**], and Milnor's theorem on the fundamental group of a Riemannian manifold with nonnegative Ricci curvature, [**13**].

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