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ISOMETRIC ISOMORPHISMS BETWEEN NORMED SPACES

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In memory of our very good friend, Robert Thompson

ABSTRACT. Two normed spaces $(\mathbf{V}_1, \|\cdot\|_1)$ and $(\mathbf{V}_2, \|\cdot\|_2)$ (over $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$) are *isometrically isomorphic* if there is a linear isomorphism $L : \mathbf{V}_1 \to \mathbf{V}_2$ such that

$$\|L(v)\|_2 = \|v\|_1 \quad \text{for all} \quad v \in \mathbf{V}_1.$$

For various finite dimensional normed spaces, we determine whether there are isometric isomorphisms between them and characterize these mappings if they exist. The results are then applied to solve some related problems involving the dual norms and norms induced by invertible linear operators. These answer some open problems and give conceptual proofs for some results on norms.

1. Introduction. Let $(\mathbf{V}_1, \|\cdot\|_1)$ and $(\mathbf{V}_2, \|\cdot\|_2)$ be normed spaces over $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. They are *isometrically isomorphic* if there is a linear isomorphism $L : \mathbf{V}_1 \to \mathbf{V}_2$ such that

$$\|L(v)\|_2 = \|v\|_1 \quad \text{for all} \quad v \in \mathbf{V}_1.$$

Such an L is called an *isometric isomorphism* from $\mathbf{V}_1$ to $\mathbf{V}_2$. Clearly one can identify the two normed spaces if they are isometrically isomorphic. Therefore, it is interesting to determine when this will happen and characterize the isometric isomorphisms between the two normed spaces, if they exist. This problem has been considered in infinite dimensional spaces including various sequence spaces and function spaces, e.g., see [**14, 15**]. However, it seems that the same problem has not

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been studied systematically in the finite dimensional case. The purpose of this paper is to fill this gap. It is worth mentioning that the finite dimensional case may not behave as the infinite dimensional case, especially when $\mathbf{F} = \mathbf{R}$. More evidence will be seen in our study.

From now on we shall confine our attention to finite dimensional normed spaces. Since two isometrically isomorphic normed spaces must be linearly isomorphic, we may assume that $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}$ in our discussion. We shall say that two norms on $\mathbf{V}$ are isometrically isomorphic if the two normed spaces are. With this terminology, our problem can be stated as:

Problem A. Let $\mathbf{V}$ be a finite dimensional linear space equipped with norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Determine the conditions for the existence of isometric isomorphisms between these two norms, and characterize such mappings if they exist.

Recall that the *isometry group* of a norm is the group of linear isometries for the norm. The following result shows that knowing the isometry groups of $\|\cdot\|_1$ and $\|\cdot\|_2$ is very useful in the study of Problem A.

Proposition 1.1. *Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathbf{V}$ with isometry groups G_1 and G_2, respectively. If L is an isometric isomorphism from $\|\cdot\|_1$ to $\|\cdot\|_2$, then $L^{-1}G_2L = G_1$. In particular, if $G_1 = G_2 = G$, then L is in the normalizer $N(G)$ of G in the general linear group $GL(\mathbf{V})$ of $\mathbf{V}$.*

Proof. Suppose that L is an isometric isomorphism from $\|\cdot\|_1$ to $\|\cdot\|_2$. Then L^{-1} is an isometric isomorphism from $\|\cdot\|_2$ to $\|\cdot\|_1$. If $T \in G_2$, then

$$\|L^{-1}TL(v)\|_1 = \|TL(v)\|_2 = \|L(v)\|_2 = \|v\|_1$$

for all $v \in V$. Thus $L^{-1}G_2L$ is a subset of G_1. Similarly, one can prove that LG_1L^{-1} is a subset of G_2.

The second assertion is clear. □

We shall use Proposition 1.1 and some known results on isometry groups to answer Problem A for various kinds of norms in the next few sections.

Besides Problem A we also obtain answers for problems involving the dual norms, and norms induced by invertible linear operators. To describe these questions, we suppose that $\mathbf{V}$ is equipped with a fixed inner product (x, y). In this paper we shall use the inner product $(x, y) = \operatorname{tr}(xy^*) = \operatorname{tr}(y^*x)$ for column vectors or matrices.

Let $U(\mathbf{V})$ denote the group of orthogonal or unitary operators on $\mathbf{V}$ depending on $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Suppose G is a closed subgroup of $U(\mathbf{V})$. A norm $\|\cdot\|$ on $\mathbf{V}$ is a *G-invariant norm* if $\|g(v)\| = \|v\|$ for all $v \in \mathbf{V}$ and for all $g \in G$, i.e., G is a subgroup of the isometry group of $\|\cdot\|$. If $S \in GL(\mathbf{V})$, then $\|\cdot\|_S$ defined by $\|S(v)\|$ is also a norm. We shall study the following problem, which has drawn the attention of several authors in the last few years, e.g., see [**8**, Section 3.2 and its references].

Problem B. If $\|\cdot\|$ is a G-invariant norm, what is the condition on S so that $\|\cdot\|_S$ is also a G-invariant norm?

It is clear that S is an isometric isomorphism from $\|\cdot\|$ to $\|\cdot\|_S$. Thus the answer of Problem A is useful to study Problem B. Furthermore, we have the following result that can be easily verified.

Proposition 1.2. *Suppose that $\|\cdot\|$ is a G-invariant norm on $\mathbf{V}$ with isometry group H, and $S \in GL(\mathbf{V})$. Then*

(a) *$S^{-1}HS$ is the isometry group of $\|\cdot\|_S$;*

(b) *$\|\cdot\|_S$ is a G-invariant norm if and only if $G < S^{-1}HS$;*

(c) *$\|\cdot\|_S$ is a G-invariant norm if $S \in N(H)$.*

Next we turn to another problem. The *dual norm* of a norm $\|\cdot\|$ on $\mathbf{V}$ is defined (and denoted by)

$$\|x\|^* = \sup\{|(x, y)| : \|y\| \leq 1\}.$$

It is known, e.g., see [**7**, (5.4.16)] that there exists $\gamma > 0$ such that $\|x\| = \gamma\|x\|^*$ for all $x \in \mathbf{V}$ if and only if $\|x\|^2 = \gamma(x, x)$ for all $x \in \mathbf{V}$. A more general question is the following.

Problem C. When is a given norm isometrically isomorphic to its dual norm?

For $p \geq 1$, the l_p-norm on $\mathbf{F}^n$ is defined by $l_p(x) = (\sum_{i=1}^n |x_i|^p)^{1/p}$. It is well known that the l_p-norm and the l_q-norm are dual to each other if and only if $1/p + 1/q = 1$. The l_2-norm, which is also known as the Euclidean norm, is self-dual, and hence is trivially isometrically isomorphic to its dual norm. For other l_p-norms, it turns out that only the l_1-norm on $\mathbf{R}^2$ is isometrically isomorphic to its dual norm, the l_∞-norm. As will be seen, this is actually a special instance of a more general result, cf. Corollary 2.4, in Section 2.

In connection to Problem C, we have the following result that generalizes [**7**, (5.4.16)].

Proposition 1.3. *Suppose that* $\mathbf{V}$ *is equipped with an inner product* (x, y)*. Then there is a positive definite operator* S *on* $\mathbf{V}$ *such that* $\|x\| = \|Sx\|^*$ *for all* $x \in \mathbf{V}$ *if and only if* $\|x\| = (Sx, x)^{1/2}$ *for all* $x \in \mathbf{V}$.

Proof. Suppose $\|x\| = \|Sx\|^*$ for all $x \in \mathbf{V}$. Let $T = S^{-1/2}$. Then, for any $x \in \mathbf{V}$,

$$\begin{aligned}\|x\|_T^* &= \max\{|(x,y)| : \|y\|_T \leq 1\} \\ &= \max\{|(x,y)| : \|Ty\| \leq 1\} \\ &= \max\{|(x,T^{-1}z)| : \|z\| \leq 1\} \\ &= \max\{|(T^{-1}x,z)| : \|z\| \leq 1\} \\ &= \|T^{-1}x\|^*.\end{aligned}$$

It follows that

$$\|x\|_T = \|Tx\| = \|STx\|^* = \|T^{-1}x\|^* = \|x\|_T^*$$

for all $x \in \mathbf{V}$, i.e., $\|\cdot\|_T$ is self-dual. So $\|x\|_T = (x,x)^{1/2}$ and $\|x\| = (Sx,x)^{1/2}$ for all $x \in \mathbf{V}$.

Conversely, if $\|x\| = (Sx,x)^{1/2}$, then by the Cauchy-Schwartz inequality, we have

$$\|Sx\|^* = \max\{|(Sx,y)| : (Sy,y)^{1/2} \leq 1\} = (Sx,x)^{1/2} = \|x\|.$$

The result follows. □

Here is another observation related to Problem C.

Proposition 1.4. *Suppose that* $\|\cdot\|$ *is a norm on* $\mathbf{V}$ *with isometry group* G. *If* L *is an isometric isomorphism from* $\|\cdot\|$ *to* $\|\cdot\|^*$, *then* $LGL^{-1} = G^*$, *the group of all the dual transformations of* $g \in G$. *In particular, if* $G = G^*$, *then* $L \in N(G)$.

As mentioned before, we shall study Problems A–C for different normed spaces. In particular, Section 2 deals with symmetric gauge functions on $\mathbf{F}^n$, Section 3 is concerned with unitarily invariant norms on $\mathbf{F}^{m\times n}$, Section 4 deals with unitary similarity invariant norms on $\mathbf{H}_n$, the real linear space of $n \times n$ Hermitian matrices and Section 5 contains results on other types of norms on (skew-)symmetric matrices. Some remarks and related problems are mentioned in Section 6.

If H is a subgroup of a group K, we write $H < K$. If K is generated by subgroups $H_1, \dots, H_s$, and elements $h_1, \dots, h_t$, we write $K = \langle H_1, \dots, H_s, h_1, \dots, h_t\rangle$. If $G < U(\mathbf{V})$, denote by $N_U(G)$ the normalizer of G in $U(\mathbf{V})$. As can be seen, we often need to deal with the normalizers of groups in our study. For a connected Lie group G, it is easy to obtain $N(G)$ from $N_U(G)$, e.g., see [**3**, Theorem 2.5]. The group of positive numbers under multiplication is denoted by $\mathbf{R}_+$, which is also identified as a subgroup of $GL(\mathbf{V})$.

2. Symmetric gauge functions.

Let $GP(n)$ be the group of generalized permutation matrices, i.e., those matrices of the form DP, where D is a diagonal matrix in $U(\mathbf{F}^n)$ and P is a permutation matrix. A norm $\|\cdot\|$ on $\mathbf{F}^n$ is a *symmetric gauge function* if $\|Px\| = \|x\|$ for all $x \in \mathbf{F}^n$, and for all $P \in GP(n)$. We have the following result.

Theorem 2.1. *If* G *is the isometry group of a symmetric gauge function on* $\mathbf{F}^n$, *then one of the following happens*:

(a) G *is* $U(\mathbf{F}^n)$ *or* $GP(n)$.

(b) $\mathbf{F}^n = \mathbf{R}^4$ *and* G *is* $\mathcal{B} = \langle GP(4), B\rangle$ *or* $\mathcal{A} = \langle GP(4), A\rangle$, *where*

$$A = I - (1,1,1,1)^t(1,1,1,1)/2$$

and

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\} / \sqrt{2}.$$

(c) $\mathbf{F}^n = \mathbf{R}^2$ *and* G *is the dihedral group* $\mathcal{D}_{8k}$ *with* $8k$ *elements for some positive integer* k.

Furthermore,

$$N(G) = \begin{cases} \mathbf{R}_+ \cdot \mathcal{B} & \text{if } G = \mathcal{A} \text{ or } \mathcal{B}, \\ \mathbf{R}_+ \cdot \mathcal{D}_{16k} & \text{if } G = \mathcal{D}_{8k}, \\ \mathbf{R}_+ \cdot G & \text{otherwise.} \end{cases}$$

Proof. By Theorem 4.1 in [**4**], we get conditions (a)–(c). It is well known that $N(U(\mathbf{F}^n)) = \mathbf{R}_+ \cdot U(\mathbf{F}^n)$.

Suppose $G = GP(n)$ and $L \in N(G)$. Since $L^{-1}GL = G < U(\mathbf{F}^n)$, it follows that (see the proof of Lemma 2.3 in [**10**]) $L = \gamma U$ for some $U \in U(\mathbf{F}^n)$ and $\gamma > 0$. Thus $\gamma^{-1}L \in N_U(G)$. Since, for $n \neq 2, 4$, G is a maximal (closed) subgroup of $U(\mathbf{F}^n)$, see [**4**, Theorems 2.3, 2.4], and since $N_U(G) \neq U(\mathbf{F}^n)$, we have $N_U(G) = G$.

For $n = 4$, the only closed overgroups of $G = GP(4)$ are $\mathcal{A}, \mathcal{B}$ and $U(\mathbf{R}^4)$, and none of them is the normalizer of G. Hence $N_U(G) = G$ and $L \in \gamma G$.

As in the preceding case, one can show that $N(G) = \mathbf{R}_+ \cdot N_U(G)$ if $G = A$ or $\mathcal{B}$. By the results in [**4**] (see the proof and discussion after Theorem 3.2), we have $N_U(\mathcal{A}) = \mathcal{B}$ and $N_U(\mathcal{B}) = \mathcal{B}$

As before, one can show that $N(\mathcal{D}_{8k} = \mathbf{R}_+ \cdot N_U(\mathcal{D}_{8k})$. By the fact that $N_U(\mathcal{D}_{8k}) = \mathcal{D}_{16k}$, the result follows. □

By Proposition 1.1 and Theorem 2.1, we have the following result that answers Problem A for symmetric gauge functions.

Theorem 2.2. *Suppose that* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are symmetric gauge functions on* $\mathbf{F}^n$ *with isometry groups* G_1 *and* G_2, *respectively. There* *is an isometric isomorphism L from $\|\cdot\|_1$ to $\|\cdot\|_2$ if and only if there exists $\gamma > 0$ such that one of the following holds*:

(a) $G_1 = G_2$, $\|x\|_1 = \gamma\|x\|_2$ *for all* $x \in \mathbf{F}^n$, *and* $\gamma^{-1}L \in G_1$.

(b) $\mathbf{F}^n = \mathbf{R}^4$, $G_1 = G_2 = \mathcal{A}$, $\|x\|_1 = \gamma\|Bx\|_2$ *for all* $x \in \mathbf{R}^4$, *and* $\gamma^{-1}L \in \mathcal{B}\backslash\mathcal{A}$.

(c) $\mathbf{F}^n = \mathbf{R}^2$, $G_1 = G_2 = \mathcal{D}_{8k}$, $\|x\|_1 = \gamma\|R_{\pi/4k}x\|_2$ *for all* $x \in \mathbf{R}^2$, *and* $\gamma^{-1}L \in \mathcal{D}_{16k}\backslash\mathcal{D}_{8k}$, *where*

$$R_{\pi/4k} = \begin{pmatrix} \cos(\pi/4k) & \sin(\pi/4k) \\ -\sin(\pi/4k) & \cos(\pi/4k) \end{pmatrix}.$$

We remark that only condition (a) can happen in infinite dimensional spaces, e.g., see [**14**]. In the finite dimensional case, there are examples, cf. Examples 6.1 and 6.2, of $\|\cdot\|_1$ and $\|\cdot\|_2$ satisfying condition (b) or (c).

Next we turn to Problem B for symmetric gauge functions. This problem has been studied in [**6**], and a characterization of S is given, cf. 3.1(c), 3.2(c), 3.3(c) and 3.4(c). As mentioned by the authors of that paper, their proofs are computational, and it would be nice to have a conceptual proof. By Theorem 2.2 we easily obtain the following corollary that answers Problem B for symmetric gauge functions. Note that our description of S is more explicit than that in [**6**].

Corollary 2.3. *Suppose that $\|\cdot\|$ is a symmetric gauge function on $\mathbf{F}^n$ with isometry group G. For $S \in GL(\mathbf{F}^n)$, $\|\cdot\|_S$ is a symmetric gauge function if and only if there exists $\gamma > 0$ such that one of the following holds*:

(a) $\gamma^{-1}S \in G$.

(b) $\mathbf{F}^n = \mathbf{R}^4$, $G = \mathcal{A}$ *and* $\gamma^{-1}S \in \mathcal{B}\backslash\mathcal{A}$.

(c) $\mathbf{F}^n = \mathbf{R}^2$, $G = \mathcal{D}_{8k}$ *and* $\gamma^{-1}S \in \mathcal{D}_{16k}\backslash\mathcal{D}_{8k}$.

Proof. If $\|\cdot\|$ is a symmetric gauge function, then S is an isometric isomorphism from $\|\cdot\|$ to $\|\cdot\|_S$, and so we can conclude that one of the conditions (a)–(c) of Theorem 2.2 holds. For the converse, notice that the isometry group of the norm $\|\cdot\|_S$ is $G_S = S^{-1}GS$

by Proposition 1.2. Since $S \in N(G)$ in all cases (a)–(c), the result follows from Proposition 1.2. □

Next we consider Problem C for symmetric gauge functions.

Corollary 2.4. *Suppose that* $\|\cdot\|$ *is a symmetric gauge function on* $\mathbf{F}^n$ *with isometry group* G. *There is an isometric isomorphism* L *from* $\|\cdot\|$ *to its dual norm* $\|\cdot\|^*$ *if and only if there exists* $\gamma > 0$ *such that one of the following holds*:

(a) $G = U(\mathbf{F}^n)$, $\|x\| = \gamma l_2(x)$ *for all* $x \in \mathbf{F}^n$, *and* $\gamma^{-1}L \in U(\mathbf{F}^n)$.

(b) $\mathbf{F}^n = \mathbf{R}^4$, $G = \mathcal{A}$, $\|x\| = \gamma\|Bx\|^*$ *for all* $x \in \mathbf{R}^4$, *and* $\gamma^{-1}L \in \mathcal{B}\backslash\mathcal{A}$.

(c) $\mathbf{F}^n = \mathbf{R}^2$, $G = \mathcal{D}_{8k}$, $\|x\| = \gamma\|R_{\pi/4k}x\|^*$ *for all* $x \in \mathbf{R}^2$, *and* $\gamma^{-1}L \in \mathcal{D}_{16k}\backslash\mathcal{D}_{8k}$.

Proof. Note that $\|\cdot\|^*$ is also a symmetric gauge function with isometry group G. Hence an isometric isomorphism L from $\|\cdot\|$ to $\|\cdot\|^*$ must satisfy one of the conditions (a)–(c) in Theorem 2.2. If condition (a) of Theorem 2.2 holds, then $\|\cdot\|^*$ is a multiple of $\|\cdot\|$. This can happen if and only if it is a positive multiple of the Euclidean norm l_2, e.g., see [**7**, 5.4.16]. Thus we get the conclusion. □

One may wonder whether conditions (b) and (c) in Corollary 2.4 can indeed occur. Actually, the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in Examples 6.1 and 6.2 are dual to each other and satisfy conditions (b) and (c) in Corollary 2.4, respectively.

It is interesting to note that, by Proposition 1.3 and Corollary 2.4, except for $\mathbf{F}^n = \mathbf{R}^2$ or $\mathbf{R}^4$, a symmetric gauge function on $\mathbf{F}^n$ is isometrically isomorphic to its dual norm if and only if it is induced by an inner product. It would be interesting to know whether this conclusion holds for other norms (not necessarily symmetric gauge functions) on $\mathbf{F}^n$.

3. Unitarily invariant norms. A norm $\|\cdot\|$ on $\mathbf{F}^{m\times n}$ is *unitarily invariant* if $\|UXV\| = \|X\|$ for all $X \in \mathbf{F}^{m\times n}$, and for all $U \in U(\mathbf{F}^m)$ and $V \in U(\mathbf{F}^n)$. Let Γ be the group of linear operators of the form

$A \mapsto UAV$ for some fixed $U \in U(\mathbf{F}^m)$ and $V \in U(\mathbf{F}^n)$, and let τ be the transposition operator on $\mathbf{F}^{n\times n}$, i.e., $\tau(A) = A^t$. By the results in [**10**], see also [**3**], we have the following result.

Theorem 3.1. *If G is the isometry group of a unitarily invariant norm, then one of the following happens*:

(*a*) $G = U(\mathbf{F}^{m\times n})$.

(*b*) $G = \begin{cases} \Gamma & \text{if } m \neq n, \\ \langle \Gamma, \tau \rangle & \text{if } m = n. \end{cases}$

(c) $\mathbf{F}^{m\times n} = \mathbf{R}^{4\times 4}$, *and* $G = \Phi = \langle \Gamma, \tau, \phi \rangle$, *where ϕ is defined by*

$$\phi(A) = \{A + B_1AC_1 + B_2AC_2 + B_3AC_3\}/2,$$

with

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Furthermore,

$$N(G) = \begin{cases} \mathbf{R}_+ \cdot \Phi & \text{if } G = \langle \Gamma, \tau \rangle, \\ \mathbf{R}_+ \cdot G & \text{otherwise.} \end{cases}$$

Using Proposition 1.1 and Theorem 3.1, we have the following result that answers Problem A for unitarily invariant norms, which was raised in [**10**].

Theorem 3.2. *Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are unitarily invariant norms on $\mathbf{F}^{m\times n}$ with isometry groups G_1 and G_2, respectively. There* *is an isometric isomorphism L from $\|\cdot\|_1$ to $\|\cdot\|_2$ if and only if there exists $\gamma > 0$ such that one of the following holds*:

(a) $G_1 = G_2$, $\|X\|_1 = \gamma\|X\|_2$ *for all* $X \in \mathbf{F}^{m\times n}$, *and* $\gamma^{-1}L \in G_1$.

(b) $\mathbf{F}^{m\times n} = \mathbf{R}^{4\times 4}$, $G_1 = G_2 = \langle \Gamma, \tau\rangle$, $\|X\|_1 = \gamma\|\phi(X)\|_2$ *for all* $X \in \mathbf{R}^{4\times 4}$ *and* $\gamma^{-1}L \in \Phi\backslash G_1$.

There exist examples, cf. Example 6.3, satisfying condition (b) of Theorem 3.2. Next we turn to Problem B for unitarily invariant norms. This problem has been considered in [**8**], cf. Theorem 3.5. Our description of S is more explicit.

Corollary 3.3. *Suppose that $\|\cdot\|$ is a unitarily invariant norm on $\mathbf{F}^{m\times n}$ with isometry group G. For $S \in GL(\mathbf{F}^{m\times n})$, $\|\cdot\|_S$ is a unitarily invariant norm if and only if there exists $\gamma > 0$ such that one of the following holds*:

(a) $\gamma^{-1}S \in G$.

(b) $\mathbf{F}^{m\times n} = \mathbf{R}^{4\times 4}$, $G = \langle \Gamma, \tau\rangle$, *and* $\gamma^{-1}S \in \Phi\backslash\langle \Gamma, \tau\rangle$.

Proof. Similar to that of Corollary 2.3. □

Now we consider Problem C for unitarily invariant norms.

Corollary 3.4. *Suppose that $\|\cdot\|$ is a unitarily invariant norm on $\mathbf{F}^{m\times n}$ with isometry group G. There is an isometric isomorphism L from $\|\cdot\|$ to its dual norm $\|\cdot\|^*$ if and only if there exists $\gamma > 0$ such that one of the following holds*:

(a) $G = U(\mathbf{F}^{m\times n})$, $\|X\| = \gamma(X,X)^{1/2}$ *for all* $X \in \mathbf{F}^{m\times n}$, *and* $\gamma^{-1}L \in U(\mathbf{F}^{m\times n})$.

(b) $\mathbf{F}^{m\times n} = \mathbf{R}^{4\times 4}$, $G = \langle \Gamma, \tau\rangle$, $\|X\| = \gamma\|\phi(X)\|^*$ *for all* $X \in \mathbf{R}^{4\times 4}$, *and* $\gamma^{-1}L \in \Phi\backslash\langle \Gamma, \tau\rangle$.

Proof. Similar to that of Corollary 2.4. □

By Example 6.3, one sees that condition (b) of Corollary 3.4 can actually happen.

4. Unitary similarity invariant norms. A norm $\|\cdot\|$ on $\mathbf{H}_n$, the real linear space of $n \times n$ complex Hermitian matrices, is *unitary similarity invariant* if $\|U^*XU\| = \|X\|$ for all $X \in \mathbf{H}_n$, and for all $U \in U(\mathbf{C}^n)$. Consider the following subgroups of $GL(\mathbf{H}_n)$:

Γ_1 is the group of linear operators on $\mathbf{H}_n$ of the form $X \mapsto U^*XU$ for some fixed unitary U, and

K is the group of invertible operators of the form $X \mapsto \alpha X + (\beta - \alpha)(\operatorname{tr} X)I/n$ for some nonzero $\alpha, \beta \in \mathbf{R}$.

It is not difficult to check that K is the centralizer of Γ_1 in $GL(\mathbf{H}_n)$. Furthermore, we have the following result.

Theorem 4.1. *If G is the isometry group of a unitary similarity invariant norm on $\mathbf{H}_n$, then one of the following holds*:

(a) $G = TU(\mathbf{H}_n)T^{-1}$ *for some* $T \in K$.

(b) $G = U'(\mathbf{H}_n) = \{T \in U(\mathbf{H}_n) : T(I) = \pm I\}$.

(c) $G = \langle \Gamma_1, \tau, T_0 \rangle$, *where T_0 is defined by* $T_0(A) = A - (2\operatorname{tr} A)I/n$.

(d) $G = \langle \Gamma_1, \tau \rangle$.

Furthermore, $N(G) = \begin{cases} \mathbf{R}_+ \cdot G & \text{if } G = TU(\mathbf{H}_n)T^{-1}, \\ \langle G, K \rangle & \text{otherwise.} \end{cases}$

Proof. The isometry group result is proved in [**11**]. The result on $N(G)$ can be found in [**5**]. We give a different proof in the following.

Now consider $N(G)$. If $G = TU(\mathbf{H}_n)T^{-1}$ and $L^{-1}GL = G$, then $T^{-1}LT \in N(U(\mathbf{H}_n))$ is a scalar operator. Thus, L is a scalar operator. It follows that $N(G) < \mathbf{R}_+ \cdot G$. The reverse inclusion is clear.

Suppose $G = U'(\mathbf{H}_n)$ and $L^{-1}GL = G$. By using an orthonormal basis of $\mathbf{H}_n$ with $I/\sqrt{n}$ as the first member, the matrix representation of every element in G is of the form $[\pm 1] \oplus U$ for some orthogonal matrix U. Let the matrix representation of L be $\tilde{L}$. Then, for every matrix $\tilde{U}$ of the form $[\pm 1] \oplus U$, the product $\tilde{L}\tilde{U}$ will be of the form $\tilde{V}\tilde{L}$, where $\tilde{V} = [\pm 1] \oplus V$ for some orthogonal matrix V. One easily concludes that this can happen if and only if $\tilde{L} = [a] \oplus bI_{n^2-1}$, i.e., $L \in K$. It follows that $N(G) < \langle G, K\rangle$. The reverse inclusion is clear.

Next suppose $G = \langle \Gamma_1, \tau, T_0\rangle$ and $L \in N(G)$. Then $LTL^{-1} \in G$ for any $T \in G$. As a result, for any unitary U there is a unitary V such that one of the following happens:

(i) $L(UXU^*) = VL(X)V^*$ for all $X \in \mathbf{H}_n$.

(ii) $L(UXU^*) = VL(X)^tV^*$ for all $X \in \mathbf{H}_n$.

(iii) $L(UXU^*) = V(T_0L(X))V^*$ for all $X \in \mathbf{H}_n$.

(iv) $L(UXU^*) = V(T_0L(X))^tV^*$ for all $X \in \mathbf{H}_n$.

Let $\mathcal{U}(Y) = \{WYW^* : WW^* = I\}$ be the unitary similarity orbit of $Y \in \mathbf{H}_n$. By (i)–(iv), we have $L(\mathcal{U}(E_{11})) \subseteq \mathcal{T}_1 \cup \mathcal{T}_2$, where $\mathcal{T}_1 = \mathcal{U}(Y_1) = \mathcal{U}(Y_1^t)$ with $Y_1 = L(E_{11})$, and $\mathcal{T}_2 = \mathcal{U}(Y_2) = \mathcal{U}(Y_2^t)$ with $Y_2 = T_0L(E_{11})$. Since $\mathcal{U}(E_{11})$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are connected, $L(\mathcal{U}(E_{11})) \subseteq \mathcal{T}_i$ for $i = 1$ or 2. Since $\mathcal{U}(E_{11})$ is an algebraic set, e.g., see [**9**], and since L is invertible, a result of Dixon [**2**] ensures that $L(\mathcal{U}(E_{11})) = \mathcal{T}_i$. Now L maps a unitary similarity orbit onto another one. By the result of [**13**], we have $L \in \langle G, K\rangle$. The inclusion $\langle G, K\rangle < N(G)$ is clear.

Similar to the above case, if $G = \langle \Gamma_1, \tau\rangle$, one can prove that $N(G) = \langle G, K\rangle$. □

We are ready to prove the following result that answers Problem A for unitary similarity invariant norms on $\mathbf{H}_n$.

Theorem 4.2. *Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are unitary similarity invariant norms on $\mathbf{H}_n$ with isometry groups G_1 and G_2, respectively. An isometric isomorphism L exists between them if and only if there exists $T \in K$ such that $\|X\|_1 = \|T(X)\|_2$ for all $X \in \mathbf{H}_n$ and $T^{-1}L \in G_1$.*

Proof. Note that G_1 and G_2 must satisfy conditions (a)–(d) of Theorem 4.1. Suppose L is an isomorphic isomorphism between the two norms. Then $L^{-1}G_2L = G_1$ by Proposition 1.1.

If $G_1 = L_1U(\mathbf{H}_n)L_1^{-1}$ for some $L_1 \in K$, then $G_2 = L_2U(\mathbf{H}_n)L_2^{-1}$ for some $L_2 \in K$. Thus $L_2^{-1}LL_1 \in N(U(\mathbf{H}_n))$ and hence is of the form γR for some $R \in U(\mathbf{H}_n)$. Thus, $\|A\|_1 = \|L(A)\|_2 = \|\gamma L_2RL_1^{-1}(A)\|_2 = \|(L_2RL_2^{-1})T(A)\|_2 = \|T(A)\|_2$, where $T = \gamma L_2L_1^{-1} \in K$, and $T^{-1}L =$

$L_1RL_1^{-1} \in G_1$ as asserted.

If $G_1 = U'(\mathbf{H}_n)$, $\langle \Gamma_1, \tau, T_0\rangle$ or $\langle \Gamma_1, \tau\rangle$, then G_2 must be of the same structure. Thus $L \in N(G_1) = \langle G_1, K\rangle$, and the result follows from Theorem 4.1.

The converse is clear. □

By Theorem 4.2, one easily obtains the answer of Problem B for unitary similarity invariant norms on $\mathbf{H}_n$.

Corollary 4.3. *Suppose that $\|\cdot\|$ is a unitary similarity invariant norm on $\mathbf{H}_n$ with isometry group G. For $S \in GL(\mathbf{H}_n)$, $\|\cdot\|_S$ is a unitary similarity invariant norm if and only if $ST \in G$ for some $T \in K$.*

Now we consider Problem C for unitary similarity invariant norms on $\mathbf{H}_n$.

Corollary 4.4. *Suppose that $\|\cdot\|$ is a unitary similarity invariant norm on $\mathbf{H}_n$. There is an isometric isomorphism L from $\|\cdot\|$ to its dual norm $\|\cdot\|^*$ if and only if*

(a) *$G = TU(\mathbf{H}_n)T^{-1}$ for some $T \in K$ such that $\|T(X)\| = (X, X)^{1/2}$ for all $X \in \mathbf{H}_n$ and $T^*LT \in U(\mathbf{H}_n)$, or*

(b) *$G = \langle \Gamma_1, \tau\rangle$, $\|A\| = \|T_0(A)\|^*$ for all A, and $T_0L \in G$, where $T_0(A) = A - (2\mathrm{tr}\, A)I/n$.*

Proof. Suppose L is an isometric isomorphism between $\|\cdot\|$ and its dual norm $\|\cdot\|^*$. Then L is in the normalizer of G, that satisfies one of the conditions (a)–(d) in Theorem 4.1. If G satisfies (a), the result follows from Proposition 1.3. If G satisfies (b) or (c),then there exist $R, S \in G$ such that RLS is a positive definite operator on $\mathbf{H}_n$ and is still an isometric isomorphism between $\|\cdot\|$ and its dual norm. But then, by Proposition 1.3, $\|\cdot\|$ should be induced by an inner product, and the isometry group of $\|\cdot\|$ should satisfy (a), which is a contradiction. Thus (b) and (c) cannot hold. If Theorem 4.1(d) holds, we get condition (b) of the corollary.

The converse is clear. □

We suspect that case (b) in Corollary 4.4 cannot happen. However, we are not able to prove it at the present. In general it would be interesting to know whether $\|\cdot\|$ and $\|\cdot\|^*$ can be related by a *reflection* R on a normed space $\mathbf{V}$, i.e., R is defined by $R(x) = x - 2(x, v)v$ for a fixed vector $v \in \mathbf{V}$ with $(v, v) = 1$.

One may also consider unitary similarity invariant norms on $\mathbf{C}^{n\times n}$. The corresponding isometric isomorphism problem has been solved in [**5**] very recently.

5. Results on symmetric and skew-symmetric matrices. In this section we consider the following matrix spaces: $S_n(\mathbf{F})$ is the linear space of all $n \times n$ symmetric matrices over $\mathbf{F}$, and $K_n(\mathbf{F})$ is the linear space of all $n \times n$ skew-symmetric matrices over $\mathbf{F}$.

We consider *unitary congruence invariant norms* on $\mathbf{V} = S_n(\mathbf{C})$ or $K_n(\mathbf{C})$, i.e., those norms $\|\cdot\|$ on $\mathbf{V}$ satisfying $\|U^tXU\| = \|X\|$ for all $X \in \mathbf{V}$ and for all $U \in U(\mathbf{C}^n)$. We have the following result.

Theorem 5.1. *If G is the isometry group of a unitary congruence invariant norm on $\mathbf{V} = S_n(\mathbf{C})$ or $K_n(\mathbf{C})$, then one of the following holds*:

(a) $G = U(\mathbf{V})$.

(b) $G = \Gamma_2$, *the group of invertible operators on* $\mathbf{V}$ *of the form* $X \mapsto U^tXU$ *for some* $U \in U(\mathbf{C}^n)$.

(c) $\mathbf{V}$ *is* $K_4(\mathbf{C})$ *and* $G = \langle \Gamma_2, \psi\rangle$, *where* $\psi(X)$ *is obtained from* X *by interchanging its* $(1,4)$ *and* $(2,3)$ *entries and interchanging its* $(4,1)$ *and* $(3,2)$ *entries.*

Furthermore, $N(G) = \mathbf{R}_+ \cdot G$.

Proof. The assertions on isometry groups follow from the results in [**8**] and [**12**].

We consider the normalizer of G. If $G = U(\mathbf{V})$, the result is clear. If G satisfies the conditions (b) or (c), then clearly $\mathbf{R}_+ \cdot G < N(G)$.

For the reverse inclusion, suppose $L \in N(G)$. Then, for every $U \in U(\mathbf{C}^n)$, there exists $V \in U(\mathbf{C}^n)$, such that $L(UAU^t) = VL(A)V^t$ for all $A \in \mathbf{V}$, or $L(UAU^t) = V\psi L(A)V^t$ for all $A \in \mathbf{V}$. Let $\tilde{\mathcal{U}}(X) = \{UXU^t : U \in U(\mathbf{C}^n)\}$. For a special choice of $X \in \mathbf{V}$, we conclude that $L(\tilde{\mathcal{U}}(X)) = \tilde{\mathcal{U}}(Y)$ for some $Y \in \mathbf{V}$. By the results in [**12**], we see that $\gamma^{-1}L \in G$ for some $\gamma > 0$. The result follows. ◻

By Proposition 1.1 and Theorem 5.1, we have the following result.

Theorem 5.2. *Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are unitary congruence invariant norms on $\mathbf{V} = S_n(\mathbf{C})$ or $K_n(\mathbf{C})$ with isometry groups G_1 and G_2. An isometric isomorphism L from $\|\cdot\|_1$ to $\|\cdot\|_2$ exists if and only if $G_1 = G_2$, $\|A\|_1 = \gamma\|A\|_2$ for some $\gamma > 0$ and $\gamma^{-1}L \in G_1$.*

For Problems B and C, we have the following corollaries.

Corollary 5.3. *Suppose that $\|\cdot\|$ is a unitary congruence invariant norm on $\mathbf{V} = S_n(\mathbf{C})$ or $K_n(\mathbf{C})$ with isometry groups G. For $S \in GL(\mathbf{V})$, $\|\cdot\|_S$ is a unitary congruence invariant norm if and only if there exists $\gamma > 0$ such that $\gamma^{-1}S \in G$.*

Corollary 5.4. *Suppose that $\|\cdot\|$ is a unitary congruence invariant norm on $\mathbf{V} = S_n(\mathbf{C})$ or $K_n(\mathbf{C})$ with isometry groups G. There is an isometric isomorphism L from $\|\cdot\|$ to its dual norm $\|\cdot\|^*$ if and only if $G = U(\mathbf{V})$, and there exists $\gamma > 0$ such that $\|X\| = \gamma(X,X)^{1/2}$ for all $X \in \mathbf{V}$ and $\gamma^{-1}L \in U(\mathbf{V})$.*

One can consider unitary congruence invariant norms on $\mathbf{C}^{n\times n}$. The corresponding isometric isomorphism problem has been solved in [**5**] very recently.

A norm $\|\cdot\|$ on $\mathbf{V} = S_n(\mathbf{R})$ or $K_n(\mathbf{R})$ is *orthogonal similarity invariant* if $\|U^tXU\| = \|X\|$ for all $X \in \mathbf{V}$ and for all $U \in U(\mathbf{R}^n)$. Orthogonal similarity invariant norms on $S_n(\mathbf{R})$ behave in the same way as unitary similarity invariant norms on $\mathbf{H}_n$. One can modify the proofs in Section 4 to get the corresponding results on orthogonal similarity invariant norms on $S_n(\mathbf{R})$. Orthogonal similarity invariant norms on $K_n(\mathbf{R})$ behave in the same way as unitary similarity invariant norms on $K_n(\mathbf{C})$. One can modify the proofs in this section to get the corresponding results on orthogonal similarity invariant norms on $K_n(\mathbf{R})$.

One may also consider orthogonal similarity invariant norms on $\mathbf{R}^{n\times n}$. Problems A–C for such norms are still open.

6. Remarks and related problems. In our discussion, we adopt an algebraic approach to determine the existence of isometric isomorphism between two norms. As we see, it is in general very rare to have an isometric isomorphism. In many cases, the two norms have to be multiples of each other.

There is a geometric approach to the isometric isomorphism problem. Suppose that $\mathcal{B}_i$ is the unit norm ball of $\|\cdot\|_i$ for $i = 1, 2$. Then L is an isometric isomorphism from $\|\cdot\|_1$ to $\|\cdot\|_2$ if and only if $L(\mathcal{B}_1) = \mathcal{B}_2$. Furthermore, if $\mathcal{E}_i$ denotes the set of extreme points of $\mathcal{B}_i$ for $i = 1, 2$, then the above conditions are equivalent to $L(\mathcal{E}_1) = \mathcal{E}_2$. Thus, our results are related to linear operators mapping certain sets onto certain sets. Such questions are special instances of *linear preserver problems*. One may see the monograph [**17**] for a nice survey of this topic. The geometric approach is especially useful in constructing examples as shown in the following.

Example 6.1. Suppose that the unit norm ball of $\|\cdot\|_1$ on $\mathbf{R}^4$ equals the convex hull of

$$\mathcal{E}_1 = \{P(1,1,0,0)^t/\sqrt{2} : P \in GP(4)\}.$$

Then, e.g., see [**4**], the unit norm ball of the dual norm $\|\cdot\|_2$ of $\|\cdot\|_1$ equals the convex hull of

$$\mathcal{E}_2 = \{P(1,1,1,1)^t/2 : P \in GP(4)\} \cup \{P(1,0,0,0)^t : P \in GP(4)\}.$$

One can verify that $B(\mathcal{E}_1) = \mathcal{E}_2$ and condition (b) of Theorem 2.2 holds.

Example 6.2. Suppose that the boundary of the unit norm ball of $\|\cdot\|_1$ on $\mathbf{R}^2$ equals the regular convex polygon with $4k$ sides centered at the origin with $(1,0)^t$ as one of the vertices. Then the unit norm ball of the dual norm $\|\cdot\|_2$ of $\|\cdot\|_1$ is obtained by rotating the unit norm ball of $\|\cdot\|_1$ by an angle $2\pi/8k$. One can check that condition (c) of Theorem 2.2 holds.

Example 6.3. Suppose that the unit norm ball of $\|\cdot\|_1$ on $\mathbf{R}^{4\times 4}$ equals the convex hull of the set $\mathcal{E}_1$ of all matrices in $\mathbf{R}^{4\times 4}$ with singular values 1,1,0,0. Then, e.g., see [**12**], the dual norm $\|\cdot\|_2$ of $\|\cdot\|_1$ is the Ky-Fan 2-norm, and the unit norm ball of $\|\cdot\|_1$ on $\mathbf{R}^{4\times 4}$ equals the convex hull of $\phi(\mathcal{E}_1)$, the set of all matrices in $\mathbf{R}^{4\times 4}$ with singular values 1,0,0,0, or with singular values 1/2,1/2,1/2,1/2. One can verify that condition (b) of Theorem 3.2 holds.

A more general problem is studying the possibility of embedding a normed space $\mathbf{V}_1$ into another normed space $\mathbf{V}_2$ of higher dimension by an injective linear map that preserves norms. This seems to be much more difficult. One may see [**16**] for a special instance of this problem.

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