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A NOTE ON DEDEKIND DOMAINS

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There are many equivalent conditions for one-dimensional Noetherian domains to be Dedekind domains, see [**2**, Theorem 6.20]. In this short note we shall prove another one. The terminology from [**3**] will be used freely.

Theorem. *Let R be a one-dimensional Noetherian domain. Then the following statements are equivalent*:

(1) *R is a Dedekind domain.*

(2) *For any finitely generated R-module M, the torsion submodule of M is a direct summand of M.*

(3) *For any finitely generated R-module M, we have*

$$T(M) \cap IM = I \cdot T(M)$$

where $T(M)$ is the torsion submodule of M and I is the intersection of all the nonzero associated prime ideals of M. If 0 is the only associated prime ideal of M, then we put $I = R$.

Proof. (1) $\Rightarrow$ (2) is well known. It is easily seen that (2) $\Rightarrow$ (3). We now show (3) $\Rightarrow$ (1) by a contrapositive argument. First of all, note that $I \cdot T(M) \subseteq T(M) \cap IM$ holds for any idea I of any commutative domain R and any R-module M. Suppose R is not a Dedekind domain. Then there exists a maximal ideal $\mathcal{M}$ such that $R_{\mathcal{M}}$ is not a DVR. Choose $x \in R$ with $x \in \mathcal{M}R_{\mathcal{M}} \backslash \mathcal{M}^2 R_{\mathcal{M}}$. As $R_{\mathcal{M}}$ is not a DVR, there exists $a \in R$ with $a \in \mathcal{M}R_{\mathcal{M}} \backslash (xR_{\mathcal{M}} + \mathcal{M}^2 R_{\mathcal{M}})$. Since $\dim R_{\mathcal{M}} = 1$, there exists $b \in R_{\mathcal{M}}$ and natural number n with $a^n = xb$ in $R_{\mathcal{M}}$. We may assume n is the least natural number with $a^n \in xR_{\mathcal{M}}$. By our choice of a, $n \geq 2$. As $x \in \mathcal{M}R_{\mathcal{M}} \backslash \mathcal{M}^2 R_{\mathcal{M}}$, $b \in \mathcal{M}R_{\mathcal{M}}$. By multiplying b with a suitable element of $R \backslash \mathcal{M}$, we may assume $b \in \mathcal{M}$.

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Put $y = a^{n-1}$. Let $\Lambda_R(x,y) = \{(r_1,r_2) \in R \oplus R : xr_2 = yr_1\}$ and $M = R \oplus R/(x,y)R$. It is easily checked that $(a,b) \in \Lambda_R(x,y)$ and $T(M) = \Lambda_R(x,y)/(x,y)R$. Now $T(M_{\mathcal{M}}) = (T(M))_{\mathcal{M}} = \Lambda_{R_{\mathcal{M}}}(x,y)/(x,y)R_{\mathcal{M}}$. Clearly, $(a,b) \in \Lambda_{R_{\mathcal{M}}}(x,y)\backslash(x,y)R_{\mathcal{M}}$ and hence $T(M_{\mathcal{M}}) \neq 0$. As $\dim R_{\mathcal{M}} = 1$ and $T(M_{\mathcal{M}}) \neq 0$, we have $\mathcal{M}R_{\mathcal{M}} \in \operatorname{Ass}_{R_{\mathcal{M}}}(M_{\mathcal{M}})$. It follows that $\mathcal{M} \in \operatorname{Ass}_R(M)$. It remains to show $T(M) \cap IM$ is not contained in $I \cdot T(M)$ where I is as defined in statement (3). Choose an element r of R such that r lies in all the associated prime ideals of M except for $\mathcal{M}$. If $\mathcal{M}$ is the only associated prime ideal of M, then we put $r = 1$. Then $(ra,rb) + (x,y)R \in T(M) \cap IM$ and $(ra,rb) + (x,y)R$ do not lie in $I \cdot T(M)$. For, otherwise, we would have $(a,b) + (x,y)R_{\mathcal{M}} \in \mathcal{M} \cdot T(M_{\mathcal{M}}) = \mathcal{M}(\Lambda_{R_{\mathcal{M}}}(x,y)/(x,y)R_{\mathcal{M}})$. By our choices of x and y, $\Lambda_{R_{\mathcal{M}}}(x,y) \subseteq \mathcal{M}R_{\mathcal{M}} \oplus \mathcal{M}R_{\mathcal{M}}$. Hence we would get $a \in xR_{\mathcal{M}} + \mathcal{M}^2R_{\mathcal{M}}$ which contradicts our choice of a. □

The equivalence of (1) and (2) is known. More precisely, it was shown in [**1**] that statement (2) is a necessary and sufficient condition for an integral domain, not necessarily Noetherian, to be a Prüfer ring.

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