# A BIRKHOFF THEOREM FOR RIEMANN SURFACES

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ABSTRACT. A classical theorem of Birkhoff asserts that there exists an entire function $f$ such that the sequence of function $\{f(z + n)\}_{n \geq 0}$ is dense in the space of entire functions. In this paper we give sufficient conditions on a Riemann surface $R$ and on a given sequence $\{\varphi_n\}_{n \geq 0}$ of holomorphic self-mappings of $R$ such that there exists a holomorphic function $f$ on $R$ such that $\{f \circ \varphi_n\}_{n \geq 0}$ is dense in the space of holomorphic functions on $R$. The necessity of these conditions is examined. In particular, we characterize the Riemann surfaces $R$ and the sequences $\{\varphi_n\}_{n \geq 0}$ of automorphisms of $R$ for which there exists a holomorphic function $f$ on $R$ with the property that the sequence $\{f \circ \varphi_n\}_{n \geq 0}$ is dense in the space of the holomorphic functions on $R$.

**1. Introduction and terminology.** As mentioned in the abstract, in 1929 G.D. Birkhoff [4] proved the following:

**Theorem.** *There exists an entire function $f(z)$ for which an arbitrary entire function $g(z)$ corresponds to a sequence $\{a_n\}_{n \geq 0}$ depending on $g(z)$ and satisfying*

$$\lim_{n \to \infty} f(z + a_n) = g(z)$$

*uniformly on compact sets.*

In 1941, W.P. Seidel and J.L. Walsh [31] established the following analogous theorem for the unit disk in which the Euclidean translations are replaced by the non-Euclidean translations.

**Theorem.** *There exists a holomorphic function on the unit disk*
$f(z)$ *for which an arbitrary holomorphic function on the unit disk* $g(z)$
*corresponds to a sequence* $\{\alpha_n\}_{n\geq 0}$ *such that*

$$\lim_{n\to\infty} f\left(\frac{z+\alpha_n}{1+\bar\alpha_n z}\right) = g(z)$$

*uniformly on compact subsets of the unit disk.*

These functions, whose existence asserts the above theorems, are
generally called *universal* functions.

In 1976, Luh [**23**] proved that, given a sequence $\{a_n\}_{n\geq 0}$ with limit
$\infty$, there exists an entire function $f$ such that, for every compact set $K$
with connected complement in the complex plane and for every function
$g$ holomorphic in the interior of $K$ and continuous on $K$, there exists
a subsequence $\{a_{n_k}\}_{k\geq 0}$ such that

$$\lim_{k\to\infty} f\left(z+a_{n_k}\right) = g(z)$$

uniformly on $K$.

In 1984, Duios-Ruis [**13**] proved that, in Birkhoff's theorem on
translations, the universal vectors can have "arbitrarily slow growth."
This result is refined further and given an operator-theoretic twist by
Chan and Shapiro in [**12**]. Gethner and Shapiro furnished in [**18**] a
single sufficient condition that provides a unified proof of universality in
several situations, including theorems of Birkhoff, MacLane, Seidel and
Walsh and many others. This same point of view is further advanced
in the papers of Godefroy and Shapiro [**19**, Sections 4 and 5] and of
Bourdon and Shapiro [**8**]. The definition of universality always shows
that the set of universal functions is either empty or a dense $G_\delta$. See [**3,
6, 7, 20, 21, 24, 25**] for additional interesting results on universality.
See also [**29**], especially Chapters 7 and 8, where further references can
be found.

The purpose of this paper is to generalize the results of Birkhoff,
Seidel and Walsh to Riemann surfaces.

Throughout this paper $R$ will stand for a noncompact Riemann
surface. By $\mathcal{O}(R)$ we denote the space of holomorphic functions on $R$,

endowed with the topology of uniform convergence on compact subsets. It is well known that $\mathcal{O}(R)$ is a second-countable Fréchet space and thus a Baire space. By $\mathcal{O}(R;R)$ we denote the space of holomorphic self-mappings $\varphi : R \to R$.

If $K$ is a compact subset of $R$, $A(K)$ denotes the set of functions which are holomorphic in the interior of $K$ and continuous on $K$. $\mathcal{K}_1(R)$ denotes the set of all compact subsets $K \subset R$ whose complement is connected and $\mathcal{K}(R)$ denotes the set of all compact subsets whose complement with respect to $R$ has no connected, relatively compact components, that is, $\mathcal{K}(R)$ denotes the set of compact subsets which are Runge in $R$. Clearly $\mathcal{K}_1(R) \subset \mathcal{K}(R)$. Most of the results that we have mentioned use Mergelyan's approximation theorem in their proofs (older results use Runge's approximation theorem). We shall use a version of Mergelyan's theorem for Riemann surfaces proved by E. Bishop [**5**] which states that $\mathcal{O}(R)$ is dense in $A(K)$ for all $K \in \mathcal{K}(R)$.

If $f \in \mathcal{O}(R)$, then $f$ is said to be *universal* with respect to a sequence $\{\varphi_n\}_{n \geq 0} \subset \mathcal{O}(R;R)$ in $\mathcal{O}(R)$ if the orbit $\{f \circ \varphi_n\}_{n \geq 0}$ is dense in $\Omega(R)$. Analogously, a function $f \in \mathcal{O}(R)$ is said to be *universal* with respect to a sequence $\{\varphi_n\}_{n \geq 0} \subset \mathcal{O}(R;R)$ in $A(K)$, where $K \subset R$ is compact, if the orbit $\{f \circ \varphi_n\}_{n \geq 0}$ is dense in $A(K)$. It is clear that the above results can be expressed in these terms.

In 1989 Zappa [**32**] proved a version of Birkhoff's theorem with the additive group of complex numbers $\mathbf{C}$ replaced by the multiplicative group $\mathbf{C}^*$, and he pointed out a generalization for a general noncompact Riemann surface $R$. He proved the following theorem

**Theorem.**    Zappa's theorem.    *Let $R$ be a noncompact Riemann surface with an infinite discrete group of automorphisms. Then there exists a function $f \in \mathcal{O}(R)$ such that, for every compact subset $K$ with a fundamental system of simply connected neighborhoods and for every $g \in A(K)$ and for every $\varepsilon > 0$, there exists an automorphism $\varphi$ of $R$ such that*

$$\max_K |f \circ \varphi - g| \leq \varepsilon.$$

The group of automorphisms $\varphi \in \mathrm{Aut}(R)$ of a Riemann surface $R$ is said to be *discrete* if for all compact subsets $K \subset R$, one has

$K \cap \varphi(K) = \varnothing$ for all but finitely many $\varphi \in \text{Aut}\,(R)$. The set of all compact subsets of a Riemann surface $R$, with a fundamental system of simply connected neighborhoods will be denoted by $Z(R)$.

As a corollary of our results, in the following sections we will have the following theorem.

**Theorem 1.1.** *Let $R$ be a noncompact Riemann surface with an infinite discrete group of automorphisms. Then there exists a function $f \in \mathcal{O}(R)$ and a sequence of automorphisms $\{\varphi_n\}_{n \geq 0}$ such that $f$ is universal with respect to $\{\varphi_n\}_{n \geq 0}$ in $A(K)$ for all $K \in \mathcal{K}_1(R)$.*

This theorem improves Zappa's theorem because $Z(R) \subset \mathcal{K}_1(R)$ and this inclusion is an equality if and only if $R$ is a plane surface.

Birkhoff's theorem and Seidel-Walsh's theorem can both be stated in a single theorem

**Theorem** Birkhoff-Seidel-Walsh theorem. *Let $\Omega \subset \mathbf{C}$ be a simply connected region (the complex plane or the hyperbolic plane, respectively), then there exists a universal function $f \in \mathcal{O}(\Omega)$ such that the set of its compositions with the automorphisms of the region $\Omega$ is dense in the space $\mathcal{O}(\Omega)$.*

We emphasize here that the Birkhoff-Seidel-Walsh theorem does not generalize to $\mathbf{C}^*$. It can be observed that Zappa's theorem for $\mathbf{C}^*$ is weaker than the result of the Birkhoff-Seidel-Walsh theorem, see [**32**] and Remark 4 in Section 3.

Universal functions on a complex region $\Omega \subset \mathbf{C}$ for a sequence of automorphisms are studied in [**2**]. In that paper a sequence of automorphisms $\{\varphi_n\}_{n \geq 0}$ of a complex region $\Omega$ is called a *run-away sequence* if, for every compact subset $K \subset \Omega$ there is $n_0(K)$ such that $K \cap \varphi_{n_0}(K) = \varnothing$. It is proved in [**2**] that if $\Omega$ has a run-away sequence of automorphisms and is not isomorphic to $\mathbf{C}^*$, then we can find a holomorphic function in $\mathcal{O}(\Omega)$ which is universal in $\mathcal{O}(\Omega)$ with respect to a sequence of automorphisms. That is, the Birkhoff-Seidel-Walsh theorem generalizes to plane regions which are not isomorphic to $\mathbf{C}^*$ and has a run-away sequence of automorphisms.

In the light of the above results, some questions arise. First, to which Riemann surfaces can the Birkhoff-Seidel-Walsh theorem be extended? Second, for which Riemann surfaces can we extend Zappa's theorem to a larger class of compact subsets? Third, in these results, can the automorphisms be replaced by more general self-mappings of the Riemann surface? In this paper we give some answers to these questions, thus extending the results of [2] to Riemann surfaces. By means of Freudenthal's compactification, to be described below, the theorem of Zappa is improved. For instance, if $R$ supports a sequence of holomorphic self-mappings $\{\varphi_n\}_{n\geq 0}$, not necessarily of automorphisms, which satisfies certain conditions, see Definition 2.1, then there exists a universal function with respect to $\{\varphi_n\}_{n\geq 0}$ in $A(K)$ for all compact subsets of $\mathcal{K}_1(R)$, thus enlarging the class of compact subsets for which the conclusion of Zappa's theorem holds. If $R$ is not "similar" to $\mathbf{C}^*$ in a topological sense given by Freudenthal's compactification and supports a sequence of holomorphic self-mappings $\{\varphi_n\}_{n\geq 0}$ which satisfies a certain additional condition, which always holds when dealing with sequences of automorphisms, then we can find universal functions in $\mathcal{O}(R)$ with respect to $\{\varphi_n\}_{n\geq 0}$. This would be the theorem analogous to the Birkhoff-Seidel-Walsh theorem for a Riemann surface. The results of this paper suggest that the property of a Riemann surface $R$ of having a sequence of holomorphic self-mappings $\{\varphi_n\}_{n\geq 0}$ for which there is a holomorphic function on $R$ which is universal with respect to $\{\varphi_n\}_{n\geq 0}$ imposes strong restrictions on the topological structure of $R$.

As a particular consequence of our results, we will find the Riemann surfaces to which the Birkhoff-Seidel-Walsh theorem generalizes for sequences of automorphisms, Theorem 1.2.

In Section 2 we introduce the concept of *run-away* sequence of holomorphic self-mappings in a Riemann surface, and we give some examples. We study some of the properties of the run-away sequences and prove some lemmas needed to establish the existence of universal functions. These lemmas may be interesting in their own right. In Section 3 we study the existence of universal functions. The proof of the main result in this section is a mixture of arguments from function theoretic approximation and topology.

In the next two sections Freudenthal's compactification will play a fundamental role. This compactification is also known as Stoilow's compactification, see [1] and [28]. Let us begin with a clarifying exam-

ple. If $R = \Omega$ is a plane surface, then it is embedded in the Riemann sphere $\mathbf{C}^\infty = \mathbf{C} \cup \{\infty\}$. The Freudenthal compactification of $\Omega$ will be constructed by letting one point correspond to each connected component of $\mathbf{C}^\infty \backslash \Omega$ and adding these points to $\Omega$. Hence the Freudenthal compactification of $\Omega$ is homeomorphic to $\mathbf{C}^\infty$. This procedure can also be viewed, in a rather imprecise language, in the following way. Take any exhaustive sequence of connected compact subsets $\{K_n\}$ of $\Omega$, $K_n \subset \text{int } K_{n+1}$ and $\Omega = \cup K_n$, as $n$ increases the number of the connected components of $\mathbf{C}^\infty \backslash K_n$ is nondecreasing, but these connected components are decreasing to the connected components of $\mathbf{C}^\infty \backslash \Omega$. To each decreasing sequence of these connected components we associate one point.

In general if $R$ is not a plane surface we do not have any space as nice as $\mathbf{C}^\infty$ in which $R$ is embedded so we need a more sophisticated construction. Now we describe briefly the Freudenthal compactification of a general Riemann surface and record some of its most important properties. What follows can be found in [**16, 17**] and [**11**, pages 81–87]. For the definitions of inverse system and the inverse limit, see [**14**, pages 427–434]. If $X$ is a noncompact, locally compact, connected, locally connected and second countable topological space, then we may consider an exhausting increasing sequence of compact subsets $\{K_n\}_{n \geq 0}$, i.e., $K_n \subset \text{int } K_{n+1}$ and $X = \cup_{n \geq 0} K_n$. The sequence of compact subsets can be directed by the order $K_m \geq K_n$ if $K_n \subset K_m$. If we denote the set of connected components of $X \backslash K_n$ by $\pi_0(X \backslash K_n)$, then we may consider the inverse limit $\mathcal{F}(X) = \varprojlim \pi_0(X \backslash K_n)$ of the inverse system $\{\pi_0(X \backslash K_n); i_{nm}, \{K_n\}_{n \geq 0}\}$ where $i_{nm} : K_n \to K_m$ denotes the natural inclusion when $m \geq n$. The set $\mathcal{F}(X)$ is independent of the sequence of compact subsets $\{K_n\}_{n \geq 0}$. Each element of $\mathcal{F}(X)$ is called a *Freudenthal end* or simply an *end*. By definition, an end is determined by a strictly decreasing sequence $\{U_n\}_{n \geq 0}$ where each $U_n$ is a connected component of $X \backslash K_n$. Two sequences $\{U_n\}_{n \geq 0}$ and $\{U'_n\}_{n \geq 0}$ corresponding to two exhausting sequences $\{K_n\}_{n \geq 0}$ and $\{K'_n\}_{n \geq 0}$ determine the same end if and only if each $U_n$ contains some $U'_n$ and vice versa.

The Freudenthal space associated with $X$ is defined as $\hat{X} = X \cup \mathcal{F}(X)$ with the topology generated by the basis of the open sets of the topology of $X$ and by the sets $\hat{U} = U \cup U^*$ where $U \in \pi_0(X \backslash K_n)$ and $U^*$ is the set of ends determined by some sequence $\{U_m\}_{m \geq 0}$ with some $U_m \subset U$

(we shall say that *U determines $U^*$*).  Since $X$ is second countable, so is $\hat{X}$.  The space $\hat{X}$ is a compact Hausdorff space that contains $X$ as a dense subset; it is called the Freudenthal compactification of $X$. Its remainder $\mathcal{F}(X)$ is homeomorphic to a closed subset of the Cantor set and is therefore compact, metrizable and totally disconnected.  In fact, $\hat{X}$ is the maximal compactification of $X$ with zero-dimensional remainder.

On the other hand, it is well known that every noncompact Riemann surface $R$ is an orientable open two-manifold, see [1], for instance, which satisfies all the above topological conditions, so it has a Freudenthal compactification $\hat{R}$.  It should be noted that a compact subset $K$ is Runge if and only if each connected component of $\hat{R}\backslash K$ contains at least one end; this, in turn, holds if and only if the set of ends determined by each connected component of $R\backslash K$ is not the emptyset.

An important example for us is the Freudenthal compactification of $\mathbf{C}^*$, which is $\mathbf{C}^\infty$, where $\mathcal{F}(\mathbf{C}^*) = \{0, \infty\}$.  What we meant earlier by a Riemann surface $R$ "similar" to $\mathbf{C}^*$ is precisely that the set $\mathcal{F}(R)$ has exactly two elements.  For instance, if $R$ is a punctured disk or an annulus, then $\mathcal{F}(R)$ has two elements.  An example of a nonplanar Riemann surface with two Freudenthal ends is the "infinite ladder," see 2.9 in the next section, Example 5b.  Another important example is that in which $R$ is a noncompact simply connected Riemann surface, then it is isomorphic to the unit disk or to the complex plane, and thus $\mathcal{F}(R)$ is a point set.  For an example of a nonplanar surface with one Freudenthal end, see 2.9 Example 5a.  Finally, we stress that, for a plane region $\Omega \subset \mathbf{C}$, the cardinal of $\mathcal{F}(\Omega)$ is just the connectivity of $\Omega$. We recall that the connectivity of a region $\Omega \subset \mathbf{C}$ is defined to be the number of connected components of $\mathbf{C}^\infty \backslash \Omega$.

The Freudenthal compactification allows us to compare the collection of compact subsets $Z(R)$ in the conclusion of Zappa's theorem with $\mathcal{K}_1(R)$ and $\mathcal{K}(R)$; we have

$$Z(R) \subset \mathcal{K}_1(R) \subset \mathcal{K}(R).$$

We have pointed out earlier that $Z(R) = \mathcal{K}_1(R)$ if and only if $R$ is a plane surface.  We also have that $\mathcal{K}_1(R) = \mathcal{K}(R)$ if and only if $\mathcal{F}(R)$ is a one-point set.  So $Z(R) = \mathcal{K}(R)$ if and only if $R$ is a plane surface and $\mathcal{F}(R)$ is a point-set, that is, $R$ is a noncompact simply

connected Riemann surface and thus isomorphic to the complex plane or isomorphic to the unit disk.

Having defined the Freudenthal compactification, we can state in very precise terms a theorem which generalizes the Birkhoff-Seidel-Walsh theorem to Riemann surfaces $R$ with infinite discrete groups.

**Theorem 1.2.** *Let $R$ be a noncompact Riemann surface with an infinite discrete group of automorphisms and such that $\mathcal{F}(R)$ is not a two-point set. Then there exists a holomorphic function $f \in \mathcal{O}(R)$ and a sequence of automorphisms $\{\varphi_n\}_{n\geq 0}$ such that $f$ is universal with respect to $\{\varphi_n\}_{n\geq 0}$ in $\mathcal{O}(R)$.*

Observe that although in Theorem 1.2 we have added a "small" extra hypothesis to Theorem 1.1, we have obtained a better result on universality. This is an easy consequence of Mergelyan's theorem which states that $\mathcal{O}(R)$ is dense in $A(K)$ for all $K \in \mathcal{K}(R)$ and the fact that $\mathcal{K}_1(R) \subset \mathcal{K}(R)$. So we have obtained a further improvement of Theorem 1.1 for those Riemann surfaces whose Freudenthal space of ends $\mathcal{F}(R)$ is not a two-point set.

In order to prove the topological lemmas in Section 2, we are interested in working with "very regular" exhaustive sequences of compact subsets. To this end, it is quite helpful to bear in mind that there is a classification of the open two-manifolds, see [**27**] or [**9**], in which, incidentally, Freudenthal's compactification also plays a fundamental role. In these last two references it is proved that an exhaustive sequence of compact subsets $\{K_n\}_{n\geq 0}$ may be chosen satisfying:

i) Each $K_n$ is a compact bordered surface.

ii) Each connected component of $R\backslash K_n$ is not relatively compact and is either planar or of infinite genus.

iii) The closure of each connected component of $R\backslash K_n$ intersected with $K_n$ is a topological circle.

We recall that, if $K \in \mathcal{K}(R)$, then $R\backslash K$ has finitely many connected components. Property iii) implies that the number of connected components of $R\backslash K_n$ is the same as the number of boundary components $K_n$. The geometrical meaning is that each $K_n$ is chosen without "cutting handles." According to Property i) each $K_n$ is connected and of

finite genus. In fact, by the classification theorem of compact bordered surfaces each $K_n$ is homeomorphic to a sphere with finitely many handles attached from which finitely many disjoint open topological disks have been removed. We recall that a compact bordered surface is of genus $g \in \mathbf{N}$ if it has $g$ handles attached.

The set of all compact subsets satisfying properties i), ii) and iii) will be denoted by $\mathcal{K}'(R)$. Clearly $\mathcal{K}'(R)$ is a subset of $\mathcal{K}(R)$. Implicitly, we shall always assume that all exhaustive sequences we use are contained in $\mathcal{K}'(R)$.

## 2. Run-away sequences of holomorphic self-mappings.

**Definition 2.1.** A sequence $\{\varphi_n\}_{n\geq 0} \subset \mathcal{O}(R; R)$ is said to be *run-away* if for each compact subset $K \subset R$ there exists a positive integer $n_0 = n_0(K)$ such that $K \cap \varphi_{n_0}(K) = \varnothing$ and $\varphi_{n_0}$ restricted to $K$ is one-to-one.

This definition is motivated by the following two theorems in which we examine the necessity of the conditions in Definition 2.1 for the existence of universal functions.

**Theorem 2.2.** *Let $\{\varphi_n\}_{n\geq 0} \subset \mathcal{O}(R; R)$ be a sequence of holomorphic self-mappings. If there is a universal function for $\{\varphi_n\}_{n\geq 0}$, then for every compact subset $K$ there exists a natural number $n_0$ such that $K \cap \varphi_{n_0}(K) = \varnothing$.*

*Proof.* Suppose that there is a compact subset $K \subset R$ such that $K \cap \varphi_n(K) \neq \varnothing$ for all natural numbers $n$. Without loss of generality, we may suppose that $K \in \mathcal{K}(R)$. Hence, we can choose $z_n \in K$ with $\varphi_n(z_n) \in K$. Let $f \in \mathcal{O}(R)$ be a universal function in $\mathcal{O}(R)$ for $\{\varphi_n\}_{n\geq 0}$. According to Mergelyan's theorem, $f$ is also universal in $A(K)$. Now consider the constant function $g(z) = 1 + \max_K |f(z)| \in \mathcal{O}(R)$. We find that, for every $n$, $\max_{z \in K} |g(z) - f(\varphi_n(z))| \geq |g(z_n)| - |f(\varphi_n(z_n))| = 1 + \max_{z \in K} |f(z)| - |f(\varphi_n(z_n))| \geq 1$, which contradicts the fact that $f$ is universal in $A(K)$. $\square$

The necessity of the one-to-one condition, at least for regions $\Omega \subset \mathbf{C}$, is given by the following theorem.

**Theorem 2.3.** *Let $\Omega \subset \mathbf{C}$ be a region, and let $\{\varphi_n\}_{n \geq 0} \subset \mathcal{O}(\Omega; \Omega)$ be a sequence of holomorphic self-mappings. If there is a universal function for $\{\varphi_n\}_{n \geq 0}$, then for every compact subset $K \subset \Omega$ there exists a natural number $n_0$ such that $\varphi_{n_0}$ is one-to-one on $K$.*

*Proof.* Let $K$ be any compact subset of $\Omega$. By putting a fine enough rectangular grid on $\mathbf{C}$, we may choose a compact subset $K_1 \subset \Omega$ which is a connected, finite union of closed squares such that $K \subset K_1$. For such a compact subset $K_1$, there is a positive number $M$ such that, for all $z_1, z_2 \in K_1$, there is an arc $\gamma$ contained in $K_1$ joining $z_1$ to $z_2$ and with length $(\gamma) < M|z_1 - z_2|$. To see this, let $m$ be the number of closed squares whose union is $K_1$ and $l$ the length of the edge of these squares. Then we may choose $M = \max\{\sqrt{2}, m\}$ since, if $|z_1 - z_2| > l$ we may join $z_1$ to $z_2$ with a polygonal of length at most $ml$ and if $|z_1 - z_2| < l$, then $z_1$ and $z_2$ lie in the same square or in adjacent squares, and we can either join them by an arc of length $|z_1 - z_2|$ or $|\mathrm{Re}\,(z_1 - z_2)| + |\mathrm{Im}\,(z_1 - z_2)| \leq \sqrt{2}|z_1 - z_2|$.

Since $K_1 \subset \Omega$ is compact, there is a positive number $r > 0$ such that $K_1 \subset \cup_{j=1}^{k} \overline{B}(a_j, r) \subset \cup_{j=1}^{k} \overline{B}(a_j, 2r) \subset \Omega$. If $f \in \mathcal{O}(\Omega)$ is universal, then there exists a natural number $n_0$ such that $\max_{K_2} |f \circ \varphi_{n_0}(z) - z| < r/(2M)$ where $K_2 = \cup_{i=1}^{k} \overline{B}(a_i, 2r)$. We set $g = f \circ \varphi_{n_0}$. Now if $z \in K_1$, then $z \in B(a_j, r)$ for some $a_j$. So, by Cauchy's formula for the derivative, for all $z \in K_1$ we have:

$$|g'(z) - 1| = \left| \frac{1}{2\pi i} \oint_{|\xi - a_j| = 2r} \frac{g(\xi) - \xi}{(\xi - z)^2} \, d\xi \right|$$
$$\leq \frac{1}{2\pi} \oint_{|\xi - a_j| = 2r} \frac{|g(\xi) - \xi|}{|\xi - z|^2} \, |d\xi| < \frac{1}{M}.$$

Let us suppose that there are $z_1, z_2 \in K_1$ with $z_1 \neq z_2$ and $g(z_1) = g(z_2)$. If we choose an arc $\gamma$ such that length $(\gamma) < M|z_1 - z_2|$, then

$$|z_1 - z_2| = \left| \int_{\gamma} (g'(z) - 1) \, dz \right| \leq \int_{\gamma} |g'(z) - 1| \, |dz|$$
$$< \frac{\mathrm{length}\,(\gamma)}{M} < |z_1 - z_2|,$$

which is a contradiction. Hence, $g = f \circ \varphi_{n_0}$ is one-to-one on $K_1$ and, consequently, so is $\varphi_{n_0}$, thus $\varphi_{n_0}$ is also one-to-one on $K$. $\quad\square$

It is an easy exercise to check that if $\psi$ is an isomorphism from $R$ onto $R_1$, then $\{\varphi_n\}_{n\geq 0}$ is run-away on $R$ if and only if $\{\psi \circ \varphi_n \circ \psi^{-1}\}_{n\geq 0}$ is run-away on $R_1$. It is clear from the definition that if $\{K_n\}_{n\geq 0}$ is an exhaustive sequence of compact subsets in $R$, then we only have to verify the condition on every $K_n$. In fact, by extracting a subsequence of $\{\varphi_n\}_{n\geq 0}$, if necessary, we may always assume that if $\{\varphi_n\}_{n\geq 0}$ is run-away on $R$, then there is an exhaustive sequence of compact subsets $\{K_n\}_{n\geq 0}$ such that $K_n \cap \varphi_n(K_n) = \varnothing$ and $\varphi_n$ restricted to $K_n$ is one-to-one. For such a sequence $\{\varphi_n\}_{n\geq 0}$, every subsequence of $\{\varphi_n\}_{n\geq 0}$ is also run-away. In this case it is interesting to observe that $\{\varphi_n(z)\}_{n\geq 0}$ has no accumulation point in $R$ for any $z \in R$. The converse is not true. Consider, for instance, $R = \mathbf{C}$ and $\{\varphi_n\}_{n\geq 0}$ defined by $\varphi_n(z) = n^2 z + n$. The sequence $\{\varphi_n\}_{n\geq 0}$ is not run-away, even though $\{\varphi_n(z)\}$ is a discrete subset of $\mathbf{C}$ for every $z \in \mathbf{C}$.

Definition 2.1 generalizes another definition given in [**2**] for a sequence of automorphisms on a region $\Omega \subset \mathbf{C}$. Definition 2.1 makes no sense for compact Riemann surfaces. If $R$ itself is compact, then $R \cap \varphi(R) = R$ for all nonconstant $\varphi$ of $\mathcal{O}(R; R)$, since nonconstant holomorphic mappings between compact Riemann surfaces are surjective, see [**15**, page 11], for example.

The following easy theorem begins to show the topological character of Definition 2.1.

**Theorem 2.4.** *If $R$ has a run-away sequence of holomorphic self-mappings, and $R$ is not of genus zero, then $R$ is of infinite genus. That is, a Riemann surface with finite and positive genus cannot support a run-away sequence.*

*Proof.* Suppose that the genus of $R$ is finite and greater than 0, say $g > 0$. Then, by definition of the surface of genus $g$, there exists a compact bordered Riemann surface $K \subset R$ whose genus is precisely $g$ and for which each connected component of $R \backslash K$ is a plane surface. If $\{\varphi_n\}_{n\geq 0}$ is a run-away sequence in $R$, then there exists a natural number $n_0$ such that $K \cap \varphi_{n_0}(K) = \varnothing$ and $\varphi_{n_0}$ is one-to-one on

$K$. Since the genus is a topological invariant, $\varphi_{n_0}(K)$ is a compact bordered surface of genus $g$ which is contained in some of the connected components of $R\backslash K$. This is a contradiction because any plane surface is of genus 0.    □

The following theorem relates the property of a sequence being run-away with the fixed points of its elements.

**Theorem 2.5.** *Let $\{\varphi_n\}_{n\geq 0} \subset \mathcal{O}(R;R)$ be a run-away sequence, and suppose that there exists a sequence $\{z_n\}_{n\geq 0} \subset R$ such that $\varphi_n(z_n) = z_n$, $n \geq 0$; that is, $z_n$ is a fixed point for $\varphi_n$, $n \geq 0$. Then there exist a subsequence $\{z_{n_k}\}_{k\geq 0}$ and an end $e \in \mathcal{F}(R)$ such that $\lim_{k\to\infty} z_{n_k} = e$.*

*Proof.* All we have to prove is that there is a subsequence $\{z_{n_k}\}_{k\geq 0}$ of fixed points which diverges in $R$. So, by the compactness of $\hat{R}$, we can extract from this subsequence a subsequence which converges in $\hat{R}$. Therefore, this extracted subsequence must converge to an end $e \in \mathcal{F}(R)$. If there is no subsequence which diverges in $R$, then there is a compact subset $K \subset R$ such that $\{z_n\}_{n\geq 0}$ is contained in $K$. Hence, we have $K \cap \varphi_n(K) \supset \{z_n\} \neq \varnothing$ for every positive integer $n$, which contradicts the fact that $\{\varphi_n\}_{n\geq 0}$ is run-away.    □

An important special case is that in which the sequence $\{\varphi_n\}_{n\geq 0}$ is given by the sequence of iterates of a holomorphic self-mapping $\varphi$. In this case universal functions for $\{\varphi_n\}_{n\geq 0}$ are generally called hyper-cyclic functions. Let us denote $\varphi^0 =$ identity on $R$ and $\varphi^n = \varphi^{n-1} \circ \varphi$. The following theorem gives an almost complete and easy characterization of run-away sequences of iterates for noncompact Riemann surfaces.

**Theorem 2.6.** *Let $R$ be a noncompact Riemann surface, and let $\varphi : R \to R$ be a holomorphic self-mapping of $R$. If $R$ is simply connected, then $\{\varphi^n\}_{n\geq 0}$ is run-away if and only if $\varphi$ is univalent and has no fixed point in $\overline{R}$. The same is true even if $R$ is not simply connected, provided that $\varphi$ is not an automorphism.*

*Proof.* First we prove that if $\{\varphi^n\}_{n\geq 0}$ is run-away, then $\varphi$ is a univalent holomorphic mapping without fixed points. Now, if $\varphi$ has a fixed point in $R$, so has $\varphi^n$ for every $n$. Therefore, by Theorem 2.4, $\{\varphi^n\}_{n\geq 0}$ cannot be run-away. If $\varphi$ is not univalent, then there are $z_0, z_1 \in R$, $z_0 \neq z_1$ with $\varphi(z_0) = \varphi(z_1)$ and, thus, giving $\varphi^n(z_0) = \varphi^n(z_1)$ for every $n$. Hence, there is no natural number $n$ such that $\varphi^n$ can be one-to-one on the compact subset $\{z_0, z_1\}$.

In the opposite direction, the Riemann mapping theorem states that, if $R$ is simply connected, then $R$ is either isomorphic to the complex plane $\mathbf{C}$ or $R$ is isomorphic to the unit disk $\mathbf{D}$. If $R = \mathbf{C}$ and $\varphi$ is univalent, then $\varphi$ is an automorphism of the complex plane, and since $\varphi$ has no fixed points, $\varphi$ is a nontrivial translation of the complex plane. It is now straightforward to verify that $\{\varphi^n\}_{n\geq 0}$ is run-away.

If $R$ is isomorphic to $\mathbf{D}$, without loss of generality we may assume that $R = \mathbf{D}$. From the Denjoy-Wolff theorem, see [**10**], for instance, if $\varphi$ has no fixed point in $\mathbf{D}$, then $\{\varphi^n\}_{n\geq 0}$ tends uniformly on compact subsets to a constant $\alpha$ of modulus 1. If $K$ is a compact subset of the unit disk, then for every $z \in K$ we have $|z| \leq r$ for some $r$, $0 \leq r < 1$. So there exists a positive integer $n_0$ such that, for all $n \geq n_0$, we have $|\varphi^n(z) - \alpha| \leq 1 - r$ for all $z$, $|z| \leq r$. Then $\min_K |\varphi^n(z)| \geq r$ and, consequently, $K \cap \varphi_n(K) = \varnothing$ for all natural numbers $n \geq n_0$. Hence, $\varphi$ being univalent, $\{\varphi^n\}_{n\geq 0}$ is run-away. We have proved the first statement of the theorem.

If $R$ is not simply connected, by applying the Riemann mapping theorem again, we know that the universal covering surface of a noncompact Riemann surface is either the complex plane $\mathbf{C}$ or the unit disk $\mathbf{D}$. If it is the complex plane, then $R$ is either isomorphic to $\mathbf{C}$ or isomorphic to $\mathbf{C}^*$. The former has been treated before and for the latter there is nothing to prove, since the only univalent holomorphic self-mappings of $\mathbf{C}^*$ are the automorphisms and these are excluded by hypothesis. If the universal covering of $R$ is the unit disk, there is a generalization of the Denjoy-Wolff theorem due to M.H. Heins, see [**22**]. As $\varphi$ is not an automorphism of $R$ and it has no fixed points, the sequence $\{\varphi^n\}_{n\geq 0}$ converges uniformly to the ideal boundary $\mathcal{F}(R)$ of $R$ and, in a similar way, we may do as in the paragraph above.     □

*Remarks* 1. In general, it is easy to construct non-compact Riemann

surfaces $R$ and $\varphi \in \mathcal{O}(R; R)$ such that the sequence of iterates $\{\varphi^n\}_{n \geq 0}$ is run-away in $R$. For a Riemann surface $R'$ whose universal covering surface is the unit disk, we find that if $\varphi \in \mathcal{O}(R'; R')$ is a univalent holomorphic self-mapping which is not an automorphism of $R'$ and if $\varphi$ has a fixed point in $R'$, say $z_0$, then $\{\varphi^n\}_{n \geq 0}$ tends uniformly on compact subsets of $R'$ to $z_0$, see [26]. Hence, as in Theorem 2.6, $\{\varphi^n\}_{n \geq 0}$ is run-away in $R = R' \backslash \{z_0\}$.

2. It is easy to find examples of multiply connected regions in which there are automorphisms $\varphi$ without fixed points for which $\{\varphi^n\}_{n \geq 0}$ may or may not be run-away. For instance, $R = \mathbf{C}^*$ and $\varphi(z) = cz$, $c \neq 0$. If $|c| = 1$, the sequence of iterates is not run-away, and if $|c| \neq 1$, the sequence of iterates is run-away.

Let us examine the Riemann surfaces which have a run-away sequence of automorphisms. It is well known that there are five types of noncompact Riemann surfaces which fail to have a discrete group of automorphisms, see [30, pages 243–244], for instance. These are the complex plane $\mathbf{C}$, the punctured complex plane $\mathbf{C}^*$, the unit disk $\mathbf{D}$, the punctured unit disk and the annuli. In [2], the run-away sequences of automorphisms of $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{C}^*$ are characterized. For the punctured disk and the annuli it is easy to show, by seeing their groups of automorphisms, that there is no run-away sequence of automorphisms in any of them. The rest of noncompact Riemann surfaces have a discrete group of automorphisms. If a Riemann surface has a finite discrete group of automorphisms, say $\{\varphi_1, \ldots, \varphi_n\}$, then there is no run-away sequence of automorphisms in $R$. To see this, just consider $K = \cup_1^n \varphi_j(M)$, where $M$ is any nonvoid compact subset of $R$. If $R$ has an infinite discrete group of automorphisms, then by applying the definition of discrete group it is easy to show that run-away sequences of automorphisms do exist. All these results together give the following

**Proposition 2.7.** *A Riemann surface has a run-away sequence of automorphisms if and only if it is isomorphic to* $\mathbf{C}$, $\mathbf{D}$, $\mathbf{C}^*$, *or it has an infinite discrete group of automorphisms.*

A noncompact Riemann surface with a discrete group of automor-

phisms in which there is a universal function with respect to a sequence of automorphisms has, necessarily, an *infinite* group of automorphisms. This is a consequence of the following:

**Corollary 2.8.**  *If there is a universal function with respect to a sequence of automorphisms of a Riemann surface $R$, then $R$ is one of the Riemann surfaces given by Proposition 2.7.*

*Proof.* If $f \in \mathcal{O}(R)$ is a universal function with respect to a sequence of automorphisms $\{\varphi_n\}_{n \geq 0}$, then it follows from Theorem 2.2 and the fact that the one-to-one condition is automatically satisfied, that $\{\varphi_n\}_{n \geq 0}$ is a run-away sequence of automorphisms of $R$. Hence, applying Proposition 2.7, we have the desired result.     $\square$

As a consequence, Theorem 1.2 does not hold if $R$ has a finite discrete group of automorphisms.

As a consequence of Theorem 2.16 we will deduce an old result which states that, if the connectivity of a plane region is finite and greater than two, then it has a finite group of automorphisms. So, if the connectivity of a plane region is finite and is greater than or equal to two, except $\mathbf{C}^*$, there is no sequence of automorphisms which can be run-away. The same is true for a large class of regions with infinite connectivity for which the group of automorphisms is a finite set. Thus, there are good reasons for extending the definition of run-away sequences to sequences of holomorphic self-mappings and not necessarily just sequences of automorphisms. For instance, the only automorphisms of $\Omega = \mathbf{D} \backslash \{1 - 1/n : n \in \mathbf{N}\}$ is the identity map. To see this, just show that any automorphism of $\Omega$ has a removable singularity at each point of the sequence $\{1 - 1/n\}$, then apply the open mapping theorem to prove that any automorphism of $\Omega$ extends to an automorphism of $\mathbf{D}$. This automorphism has to take the whole sequence $\{1 - 1/n\}$ onto itself and this is impossible for an automorphism of the unit disk which is not the identity. However, it is easy to show that $\{\varphi_n(z) = z/n + 1/n - 1\}_{n \geq 1}$ is a run-away sequence in $\Omega$.

**Examples 2.9.** Due to the "rigidity" of automorphisms, there are relatively few Riemann surfaces $R$ with sequences of automorphisms

which can be run-away in $R$. Here are a few examples that show the less restrictive nature of Definition 2.1. The following examples also show that the mappings involved need not be globally one-to-one.

*Example* 1. Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequences of complex numbers such that $\lim_{n\to\infty} a_n = 0$, with $a_n \neq 0$ for all natural numbers $n$, and $\lim_{n\to\infty} \operatorname{Re} b_n = +\infty$. Let us show that $\{\varphi_n(z) = e^{a_n z + b_n}\}_{n\geq 0}$ is run-away in $\mathbf{C}$. Let $\overline{B}(0, r)$ denote the closed disk of center $0$ an radius $r$. Set $a_n = r_n e^{i\theta_n}$. Each function $e^{a_n z + b_n}$ is one-to-one on the strip $B_n = \{z = e^{-i\theta_n} w \in \mathbf{C} : -\pi/r_n < \operatorname{Im} w < \pi/r_n\}$. Since $\{a_n\}_{n\geq 0}$ tends to zero, we have $\overline{B}(0, r) \subset B_n$ for all natural numbers $n \geq n_1$. On the other hand, $-r_n r + \operatorname{Re} b_n > r$ for all $n \geq n_2$. Now, if $r > 0$, we have
$$\min_{|z|\leq r} |e^{a_n z + b_n}| = \min_{|z|=r} |e^{a_n z + b_n}| = \min_{|z|=r} e^{\operatorname{Re}(a_n z + b_n)}$$
$$= e^{-r_n r + \operatorname{Re} b_n} \geq e^r > r.$$

With this, if $n \geq \max\{n_1, n_2\}$ we have that $\overline{B}(0, r) \cap \varphi_n(\overline{B}(0, r)) = \varnothing$ and $\varphi_n$ is one-to-one on $\overline{B}(0, r)$. Finally, if $K$ is any compact subset $K \subset \mathbf{C}$, then we can find a positive number $r > 0$ such that $K \subset \overline{B}(0, r)$. Therefore, $\{\varphi_n\}_{n\geq 0}$ is run-away.

*Example* 2. Let $\{a_n z + b_n\}_{n\geq 0}$ be a sequence of automorphisms of $\mathbf{C}$ such that $\lim_{n\to\infty} (\min\{|b_n/a_n|, |b_n|\}) = +\infty$. We also consider a sequence of natural numbers $\{c_n\}_{n\geq 0}$ such that $\lim_{n\to\infty} (1/c_n)|b_n/a_n| = \infty$. Then the sequence of functions $\{\varphi_n(z) = (a_n z + b_n)^{c_n}\}_{n\geq 0}$ is run-away in $\mathbf{C}$. To see this, we choose any closed disk $\overline{B}(0, r)$ with $r \geq 1$. Each function $\varphi_n$ is one-to-one on the angular region

$$B_n = \left\{ z = -\frac{b_n}{a_n} + re^{i\theta} : 0 < r < \infty; -\frac{\pi}{c_n} - \arg\frac{b_n}{a_n} < \theta < \frac{\pi}{c_n} + \arg\frac{b_n}{a_n} \right\}.$$

This angular region contains the disk $\overline{B}(0, r_n)$ with

$$r_n = \left|\frac{b_n}{a_n}\right| \sin\frac{\pi}{c_n}$$

and, since $\lim_{n\to\infty} r_n = +\infty$, there is a natural number $n_1$ such that $\varphi_n$ is one-to-one on $\overline{B}(0, r)$ for all $n \geq n_1$. It is also easy to verify that

there is a natural number $n_2$ such that $\min_{|z| \leq r} |a_n z + b_n| > r$ for all $n \geq n_2$. If $r \geq 1$, we obtain $\min_{|z| \leq r} |(a_n z + b_n)^{c_n}| > r^{c_n} \geq r$. As in Example 1, we find that $\{\varphi_n\}_{n \geq 0}$ is run-away in $\mathbf{C}$.

*Example* 3. As in the previous examples, it is easy to verify that if $\psi_n(z) = k_n(z - a_n)/(1 - \bar{a}_n z)$ is a sequence of automorphisms of $\mathbf{D}$ such that $\lim_{n \to \infty} |a_n| = 1$ and $p$ is any positive integer, then the sequence $\{\varphi_n(z) = (\psi_n(z))^p\}_{n \geq 0}$ is run-away on $\mathbf{D}$.

*Example* 4. Let $\{\varphi_n(z) = (z - a_n)(z - b_n)\}_{n \geq 0}$ be a sequence of polynomials of degree two. It is easy to verify that $\varphi_n(z) = \varphi_n(z')$ if and only if $z + z' = a_n + b_n$. So $\varphi_n$ is one-to-one in $\overline{B}(0, r)$ with $r < |a_n + b_n|/2$. On the other hand, if $r + \sqrt{r} < \min\{|a_n|, |b_n|\}$, we have, for all $z \in \overline{B}(0, r)$, the inequality $|(z - a_n)(z - b_n)| \geq (|a_n| - |z|)(|b_n| - |z|) > r$. So, if $\lim_{n \to \infty} \min\{|a_n|, |b_n|, |a_n + b_n|/2\} = \infty$, then $\{\varphi_n\}_{n \geq 0}$ is run-away in $\mathbf{C}$.

*Example* 5. a) Let $R$ be formed by doubling the region in the $xy$-plane bounded by the line $y = 0$ and the sequence of circles $|z - i - n| = 1/9$, $n \in \mathbf{Z}$. We represent $R$ in the three-space by the union of the region which results by removing from $\mathbf{C}$ the sequence of closed disks $|z \pm i - n| \leq 1/9$, $n \in \mathbf{Z}$, joined to the $\pi$-rotation around the $x$-axis of the same circles. The space of the ends of $R$ is a one-point set.

b) Let $R$ be the "infinite ladder" formed by doubling the region in the $xy$-plane bounded by the lines $y = \pm 1$ and the sequence of circles $|z - n| = 1/9$, $n \in \mathbf{Z}$. We represent $R$ as a surface in the three-space lying above and below the given region. The space of the ends of $R$ is a two-point set.

c) We remove from the surface in b) the points where the surface (in the three-space) meets the $x$-axis. The space of the ends of $R$ is an infinite set.

Now we consider a translation $w$ in the three-space whose modulus is a positive integer and whose direction coincides with the $x$-axis. This translation induces an automorphism of $R$ in the above three cases. It is easy to check that in all three cases the sequence $\{\varphi^n\}_{n \geq 0}$ is a run-away sequence. Part a) of Theorem 3.1 (or Theorem 1.2) in the next section is applicable in only cases a) and b) while part b) of Theorem

3.1 (or Theorem 1.1) is applicable in all three cases.

As mentioned in the introduction, in order to prove the main theorem in the next section we need some topological lemmas. To prove two of these lemmas, we need some "regular" compact subsets with connected complement. If $R$ is a noncompact Riemann surface, we set $\mathcal{K}'_1(R) = \mathcal{K}'(R) \cap \mathcal{K}_1(R)$. The following two properties are easy consequences of the definition of $\mathcal{K}'_1(R)$ and their proofs are omitted.

*Property* 1. A compact subset $K \subset R$ is in $\mathcal{K}'_1(R)$ if and only if $K$ is a compact bordered surface of $R$ which is homeomorphic to a sphere with finitely many handles attached from which *one* topological disk has been removed.

*Property* 2. If $K' \in \mathcal{K}'(R)$, then $K_1 \subset$ int $K'$ is in $\mathcal{K}_1(R)$ if and only if int $K' \backslash K_1$ is connected.

We shall make use of this property, especially when $K'$ belongs to $\mathcal{K}'_1(R)$.

**Lemma 2.10.** *For every noncompact Riemann surface, there exists a sequence of compact subsets $\{K'_n\}_{n \geq 0} \subset \mathcal{K}'_1(R)$ such that, for all $K_1 \in \mathcal{K}_1(R)$ there exists a positive integer $n_0$ such that $K_1 \subset$ int $K'_{n_0}$.*

*Proof.* Let $\{K_n\}_{n \geq 0} \subset \mathcal{K}'(R)$ be an exhaustive sequence of compact subsets of $R$. If we take any compact subset $K_1$ of $\mathcal{K}_1(R)$, then there is an $n_0$ such that $K_1 \subset$ int $K_{n_0}$. Let $m$ and $g$ be the number of connected components of $R \backslash K_{n_0}$ and the genus of $K_{n_0}$, respectively. Then $K_{n_0}$ is homeomorphic to a sphere with $g$ handles attached from which $m$ disjoint topological disks have been removed. Let $C_i$, $i = 1, \ldots, m$, be the boundary components corresponding to the $m$ disks. Since $K_1 \in \mathcal{K}_1(R)$, we have by Property 2 that int $K_{n_0} \backslash K_1$ is an open connected set, so it is arc-connected. We may join the circle $C_1$ to $C_2$ by a Jordan arc $L$ whose end points are in $C_1$ and $C_2$. Since $K_1$ and $L$ are compact subsets, we may cover $L$ by a connected, simply connected open set $V$, that is, $V$ is a narrow strip such that $\overline{V} \cap K_1 = \varnothing$. Without loss of generality, we may assume that $V$ is a finite union of topological

disks of the denumerable basis of $K_{n_0}$.

The resulting surface $M_1 = K_{n_0} \backslash V$ is homeomorphic to a sphere with $g$ handles attached from which $m - 1$ open topological disks have been removed. Since $\overline{V} \cap K_1 = \varnothing$, we find that $K_1 \subset \operatorname{int} M_1$. In this way, in finitely many steps we reach $M_{m-1} = K'_{n_0}$ which is homeomorphic to a sphere with $g$ handles attached from which one topological disk has been removed.

Finally, observe that $K'_{n_0}$ has been constructed as the difference between two kinds of sets (one of the exhaustive sequence $\{K_n\}_{n \geq 0}$ and the other as a finite union of the denumerable basis of each $\overline{K_n}$) and both kinds of sets are denumerable; therefore, we have the statement of the lemma. $\qquad \square$

The previous lemma is better, and its proof easier, than an analogous lemma for the compact subsets of $Z(R)$ proved by Zappa, see [**31**, Lemma 2].

Zappa [**31**] remarks that, in a Riemann surface $R$, if $K_1 \in \mathcal{K}_1(R)$ and $K \in \mathcal{K}(R)$, the disjoint union does not necessarily belong to $\mathcal{K}(R)$. This is true (in fact, the disjoint union of two elements of $\mathcal{K}_1(R)$ need not be in $\mathcal{K}(R)$). However, when dealing with run-away sequences of holomorphic self-mappings we have the following crucial lemma (and not only with run-away sequences of automorphisms).

**Lemma 2.11.** *Let $R$ be a noncompact Riemann surface, and let $\{\varphi_n\}_{n \geq 0} \subset \mathcal{O}(R; R)$ be a run-away sequence. Then, given $K_1 \in \mathcal{K}_1(R)$ and $K \in \mathcal{K}(R)$ there exists a positive integer $n_0$ such that $K \cap \varphi_{n_0}(K_1) = \varnothing$, $\varphi_{n_0}$ restricted to $K_1$ is one-to-one and $K \cup \varphi_{n_0}(K_1) \in \mathcal{K}(R)$.*

*Proof.* According to Lemma 2.10, we may choose a compact subset $K'_1 \in \mathcal{K}'_1(R)$ such that $K_1 \subset \operatorname{int} K'_1$. Since $\{\varphi_n\}_{n \geq 0}$ is run-away, there is a positive integer $n_0$ such that $K \cap \varphi_{n_0}(K'_1) = \varnothing$ and $\varphi_{n_0}$ restricted to $K'_1$ is one-to-one. This is the key point: the run-away property is applied on $K'_1$, not on $K_1$. Since $K'_1$ is connected, so is $\varphi_{n_0}(K'_1)$. Hence, $\varphi_{n_0}(K'_1)$ is contained in $U$, where $U$ is a connected component of $R \backslash K$ which is not relatively compact. We observe that $U$ is also a Riemann subsurface of $R$. Now, since $\varphi_{n_0}$ is a one-to-one holomorphic mapping

on $K_1'$, we know that it is a homeomorphism from $K_1'$ onto its image $\varphi_{n_0}(K_1')$. Since $\varphi_{n_0}(K_1') \subset U$, we can apply Property 1 to conclude that $\varphi_{n_0}(K)$ belongs to $\mathcal{K}_1'(U)$. Therefore, $U \backslash \varphi_{n_0}(K_1')$ is connected.

On the other hand, $R \backslash (K \cup \varphi_{n_0}(K_1'))$ has the same connected components as $R \backslash K$, except that $U$ is replaced by $U \backslash \varphi_{n_0}(K_1')$. But the latter is also not relatively compact, for if it were relatively compact then the same would be true of $U = (U \backslash \varphi_{n_0}(K_1')) \cup \varphi_{n_0}(K_1')$, a contradiction. Thus, we have proved that $K \cup \varphi_{n_0}(K_1') \in \mathcal{K}(R)$.

It remains to be proved that $K \cup \varphi_{n_0}(K_1) \in \mathcal{K}(R)$. To this end it is sufficient to show that $U \backslash \varphi_{n_0}(K_1)$ is connected (the proof that it is not relatively compact is as before). But this has to be true, because $\operatorname{int} K_1' \backslash K_1$ is connected and connectedness is a topological invariant; therefore, $\varphi_{n_0}(\operatorname{int} K_1' \backslash K_1) = \operatorname{int} \varphi_{n_0}(K_1') \backslash \varphi_{n_0}(K_1)$ is a connected set in $\varphi_{n_0}(K_1')$ and, since $\varphi_{n_0}(K_1') \subset U$, we can apply Property 2 to conclude that $U \backslash \varphi_{n_0}(K_1)$ is also connected. $\quad\square$

The statement of the previous lemma will be used only for compact subsets of $\mathcal{K}_1'(R)$. The fact that this statement is true for compact subsets of $\mathcal{K}_1(R)$ suggests that the existence of a run-away sequence in a Riemann surface $R$ imposes strong topological properties on $R$. If we add a condition to Definition 2.1 we can improve Lemma 2.11 for those Riemann surfaces whose Freudenthal space of ends is not a two-point set. This extra condition is satisfied by run-away sequences of automorphisms.

If $K \in \mathcal{K}'(R)$ and $\varphi$ is continuous and one-to-one on $K$, then it is a homeomorphism onto $\varphi(K)$. Therefore, since the genus is a topological invariant, $\varphi(K)$ has the same genus as $K$. However, each connected component of $R \backslash \varphi(K)$ may be relatively compact, so $\varphi(K)$ is not necessarily in $\mathcal{K}'(R)$. In fact, $R \backslash \varphi(K)$ may not have the same number of connected components as $R \backslash K$. To avoid these difficulties we give the following

**Definition 2.12.** A sequence $\{\varphi_n\}_{n \geq 0} \subset \mathcal{O}(R; R)$ is said to be *run-away preserving* if, for all $K \in \mathcal{K}'(R)$, there exists a positive integer $n_0$ such that $K \cap \varphi_{n_0}(K) = \varnothing$, $\varphi_{n_0}$ restricted to $K$ is one-to-one and $\varphi_{n_0}(K) \in \mathcal{K}'(R)$.

Since $K$ and $\varphi_n(K)$ are in $\mathcal{K}'(R)$, and for this kind of compact subset the number of connected components of the complement is the same as the number of the connected components of the boundary, we may conclude from the fact that $\varphi_{n_0}$ is a homeomorphism that $R\backslash K$ has as many connected components as $R\backslash\varphi_n(K)$.

Despite the extra requirement of Definition 2.12, the following proposition shows that there are still many examples of run-away preserving sequences.

**Proposition 2.13.** *Under any of the following conditions if $\{\varphi_n\}_{n\geq 0}$ is run-away, then it is also run-away preserving.*

*i) The space of ends $\mathcal{F}(R)$ is a one-point set, in particular, if $R$ is simply connected.*

*ii) The sequence $\{\varphi_n\}_{n\geq 0}$ is contained in the group of automorphisms of $R$.*

*iii) The Riemann surface $R$ is a plane region and each element of $\{\varphi_n\}_{n\geq 0}$ is a covering map.*

*Proof.* i) Since $\mathcal{F}(R)$ is a one-point set, we have that $\mathcal{K}'(R) = \mathcal{K}'_1(R)$. Let $K$ be a compact subset in $\mathcal{K}'(R)$. Since $\{\varphi_n\}_{n\geq 0}$ is run-away, then there is a natural number $n_0$ such that $K \cap \varphi_{n_0}(K) = \varnothing$, and $\varphi_{n_0}$ restricted to $K$ is one-to-one. This implies that $\varphi_{n_0}$ restricted to $K$ is a homeomorphism from $K$ onto $\varphi_{n_0}(K)$. Hence, $\varphi_{n_0}(K) \in \mathcal{K}'_1(R) = \mathcal{K}'(R)$. Therefore, $\{\varphi_n\}_{n\geq 0}$ is run-away preserving.

ii) We only have to prove that each $\varphi_n$ preserves the relatively compact connected components. But this is obvious because $\varphi_n$ is a global homeomorphism from $R$ onto itself.

iii) Let us consider a compact subset $K$ in $\mathcal{K}'(R)$. Let $C_1,\dots,C_m$ be the boundary connected components of $K$. Since $\{\varphi_n\}_{n\geq 0}$ is run-away in $R$, there is a natural number $n_0$ such that $K \cap \varphi_{n_0}(K) = \varnothing$ and $\varphi_{n_0}$ is one-to-one on $K$. Since $\varphi_{n_0}$ is one-to-one on $K$, we have that $\varphi_{n_0}(K)$ has as many boundary connected components as $K$. Since $\varphi_{n_0}(K)$ is contained in a plane surface, we have that $K$ and $\varphi_n(K)$ have the same order of connectivity. Hence, $R\backslash K$ has as many connected components as $R\backslash\varphi_{n_0}(K)$. As before, we only have to prove that none of the connected components of $R\backslash\varphi_{n_0}(K)$ is relatively compact.

To see this, we may suppose that $R \subset \mathbf{C}$. If $U$ is the nonbounded connected component in $\mathbf{C}$ of $R \backslash \varphi_{n_0}(K)$, then automatically it is not relatively compact. Now suppose that $U$ is a bounded relatively compact connected component of $R \backslash \varphi_{n_0}(K)$, and let $\varphi_{n_0}(C_i)$ be the boundary connected component of $U$; then the closure $\overline{U} = U \cup \varphi_{n_0}(C_i)$ is a closed topological disk which is contained in $R$. This implies that $\varphi_{n_0}(C_i)$ is null-homotopic in $R$. However, the connected component of $R \backslash K$ corresponding to $C_i$ is not relatively compact and from this we find that $C_i$ is not null-homotopic in $R$. This is a contradiction because any covering map between Riemann surfaces induces a one-to-one homomorphism between their fundamental groups, see [**15**, Chapter 1, Section 4].    $\square$

Consequently, all examples given in Examples 2.9 are run-away preserving. There is a standard way to construct examples of noncompact Riemann surfaces with a run-away preserving sequence of mappings. Consider, for instance, the unit disk $\mathbf{D}$ and the sequence given by $\{\varphi_n(z) = (1/n)z\}_{n \geq 1}$. For each natural number $k$ we define $F_k = \{\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_k}(1) : i_1, i_2, \ldots, i_k \in \mathbf{N}\}$, and we denote $F' = \cup_{i=1}^{\infty} F_k$ and $F = F' \cup \{0\}$. Then it is easy to show that $\{\varphi_n\}_{n \geq 0}$ is run-away preserving in the region $\Omega = \mathbf{D} \backslash F$.

As we shall see shortly, if $\mathcal{F}(R)$ is finite and has three or more elements, there is no sequence which can be run-away preserving. So, in Lemmas 2.15 and 2.17, $\mathcal{F}(R)$ will be infinite. We shall make use of the following notation. If $\hat{S} \subset \hat{R}$, then we denote $S = \hat{S} \cap R$.

**Definition 2.14.** Let $R$ be a noncompact Riemann surface. An end $e \in \mathcal{F}(R)$ is said to be *nonisolated* if it is a nonisolated point in the topological space $\mathcal{F}(R)$; that is, for every open neighborhood $\hat{U} \subset \hat{R}$ with $e \in \hat{U}$ there is another end $e' \neq e$ such that $e' \in \hat{U}$.

Observe the use of the fact that $\mathcal{F}(R)$ has at least three elements in the proof of the following lemma.

**Lemma 2.15.** *Let $R$ be a noncompact Riemann surface with an infinite space of ends $\mathcal{F}(R)$ and $\{\varphi_n\}_{n \geq 0} \subset \mathcal{O}(R; R)$ a run-away preserving sequence. Then there exist a nonisolated end $e$ and a run-*

*away preserving subsequence $\{\varphi_{n_k}\}_{k\geq 0}$ such that, for every compact subset $K \subset R$ and for every open neighborhood $\hat{U} \subset \hat{R}$ with $e \in \hat{U}$, there exists a positive integer $k_0$ such that, for every $k \geq k_0$, we have $\varphi_{n_k}(K) \subset U$ and $\varphi_{n_k}$ restricted to $K$ is one-to-one.*

*Proof.* We may choose an exhaustive sequence $\{K_n\}_{n\geq 0} \subset \mathcal{K}'(R)$ in such a way that each $\hat{R}\backslash K_n = \cup_{j\in J_n}\hat{U}^n_j$, where the union is disjoint, $J_n$ is a finite set, and each $\hat{U}^n_j$ is an open, connected subset of $\hat{R}$. Furthermore, we may choose each $\hat{U}^n_j$ such that it contains a unique, isolated end or it contains a nonisolated end. In the latter case, it must therefore contain infinitely many ends of $\mathcal{F}(R)$. Since the space of ends $\mathcal{F}(R)$ is infinite, we may suppose that $\hat{R}\backslash K_n$ has three or more components for each $n$. Since $\{\varphi_n\}_{n\geq 0}$ is run-away preserving, we may assume, by extracting a subsequence if necessary, that for each $n$, $K_n \cap \varphi_n(K_n) = \varnothing$, $\varphi_n$ restricted to $K_n$ is one-to-one and $R\backslash \varphi_n(K_n)$ has as many connected components as $R\backslash K_n$. Hence, $\varphi_n(K_n) \subset R\backslash K_n$ and, from the connectedness of $\varphi_n(K_n)$, there exists $j_0 \in J_n$ such that $\varphi_n(K_n) \subset U^n_{j_0}$ and $\hat{U}^n_{j_0}$ contains a nonisolated end of $\mathcal{F}(R)$ which we denote by $e_n$. This is so because $R\backslash \varphi_n(K_n)$ has at least three connected components, two of which (determining at least two distinct ends, since they are disjoint) are subsets of $U^n_{j_0}$, which is impossible if $U^n_{j_0}$ determines exactly one end (or equivalently, if $\hat{U}^n_{j_0}$ contains an isolated end). Hence, by the compactness of $\hat{R}$, there exists an end $e$ of $\mathcal{F}(R)$ and a subsequence $\{e_{n_k}\}_{k\geq 0}$ tending to $e$.

Clearly, $e$ is nonisolated. Let $\hat{U}$ be a neighborhood of $e$. Without loss of generality, we may suppose that $U$ is a connected component of $R\backslash K_{n_{k_1}}$ for some natural number $k_1$. Since $\{e_{n_k}\}_{k\geq 0}$ tends to $e$, we know that $e_{n_k} \in \hat{U}$ for all $k \geq k_2$. So $U$ determines $e_{n_k}$ for all $k \geq k_2$. Therefore, since $\{K_n\}_{n\geq 0}$ is exhaustive for all $k \geq \max\{k_1, k_2\}$, we have $\hat{U}^{n_k}_{j_0} \subset \hat{U}$.

Now, given any compact subset $K$, there is $k_3$ such that, for all $k \geq k_3$, we have $K \subset \text{int } K_{n_k}$. So, if $k \geq \max\{k_1, k_2, k_3\}$, we have $\varphi_{n_k}(K) \subset \varphi_{n_k}(K_{n_k}) \subset U^{n_k}_{j_0} \subset U$, which is the statement of the lemma. $\square$

In the sequel, if $R$ has an infinite space of ends $\mathcal{F}(R)$ and $\{\varphi_n\}_{n\geq 0} \subset$

$\mathcal{O}(R;R)$ is a run-away preserving sequence, we may assume by extracting a subsequence if necessary that $\{\varphi_n\}_{n\geq 0}$ satisfies the property of the previous lemma.

The proof of Lemma 2.15 gives the following topological requirement on a Riemann surface to have a run-away preserving sequence.

**Theorem 2.16.** *Let $R$ be a noncompact Riemann surface. If $\mathcal{F}(R)$ is finite and has three or more elements, then $R$ cannot support a run-away preserving sequence.*

*Proof.* Since $\mathcal{F}(R)$ is finite, all ends are isolated. Since $\mathcal{F}(R)$ has three or more elements, we can repeat step by step the first part of the proof of Lemma 2.15 to obtain a nonisolated end, a contradiction. □

Of course, the statement of the above theorem is not true if $\mathcal{F}(R)$ has one or two elements, see Examples 2.9. On the other hand, it is very easy to construct run-away (not preserving) sequences on the unit disk with a finite set of points deleted. Hence, Theorem 2.16 is false for run-away (not preserving) sequences.

Since run-away sequences of automorphisms are preserving (Proposition 2.13 ii)), we find that if $\mathcal{F}(R)$ is finite and has three or more elements, then $R$ cannot support any run-away sequence of automorphisms. Therefore, by Proposition 2.7, we find that $R$ is isomorphic to a punctured disk or to an annulus or has a finite discrete group of automorphisms. Since $\mathcal{F}(R)$ is finite and has three or more elements, the last possibility must occur. Recalling that the connectivity of a plane region coincides with the cardinal of $\mathcal{F}(R)$, we have reproved the old result that if the connectivity of a plane region $\Omega$ is finite and greater than two, the group of automorphisms of $\Omega$ is finite.

**Lemma 2.17.** *Let $R$ be an open Riemann surface with an infinite space of ends $\mathcal{F}(R)$, $\{\varphi_n\}_{n\geq 0} \subset \mathcal{O}(R;R)$ a run-away preserving sequence and $K_1, K \in \mathcal{K}'(R)$. Then there exists a positive integer $n_0$ such that $K_1 \cap \varphi_{n_0}(K) = \varnothing$, $\varphi_{n_0}$ restricted to $K$ is one-to-one and $K_1 \cup \varphi_{n_0}(K) \in \mathcal{K}(R)$.*

*Proof.* Let $l_1$ and $l$ be the number of connected components of $K_1$ and $K$ which are not relatively compact, respectively. Consider the end $e$ furnished by Lemma 2.15. Let $\hat{U}_1$ be the connected component of $\hat{R}\backslash K_1$ which contains $e$. Since $e$ is nonisolated, there are an open neighborhood $\hat{U}$ of $e$ and an end $e_0 \in \mathcal{F}(R)$ such that $e \in \hat{U} \subset \hat{U}_1$ and $e_0 \in \hat{U}_1\backslash\hat{U}$. According to Lemma 2.15, there exists a positive integer $n_0$ such that $\varphi_{n_0}(K) \subset U$ and $\varphi_{n_0}$ restricted to $K$ is one-to-one. Clearly, $K_1 \cap \varphi_{n_0}(K) = \varnothing$.

Let $U_1, U_2, \dots, U_l$ and $V_1, \dots, V_l$ be the connected components of the complements of $K_1$ and $\varphi_{n_0}(K)$, respectively. We may assume that $V_1$ is the connected component containing $K_1$. Then the connected components of $R\backslash(K_1 \cup \varphi_{n_0}(K))$ are $U_1 \cap V_1$, $U_2, \dots, U_l$, $V_2, \dots, V_l$. Clearly, $l_1 + l - 2$ of these connected components are not relatively compact. Since the remaining connected component $U_1 \cap V_1$ contains $U_1\backslash U$, and this latter set determines $e_0$, we conclude that $U_1 \cap V_1$ is not relatively compact and the proof is complete. $\square$

The statement of the previous lemma is trivial when $\mathcal{F}(R)$ is a one-point set. But it is false when $\mathcal{F}(R)$ is a two-point set. (See $\mathbf{C}^*$ or Example 2.9.5 b).)

**3. Existence of universal functions.** In this section we prove the existence of universal functions when the sequence of mappings $\{\varphi_n\}_{n\geq 0}$ is run-away.

**Theorem 3.1.** *Let $R$ be a noncompact Riemann surface. Then, given a sequence $\{\varphi_n\}_{n\geq 0} \subset \mathcal{O}(R; R)$, we have*

a) *If $\mathcal{F}(R)$ is not a two-point set and $\{\varphi_n\}_{n\geq 0}$ is run-away preserving, then there exists a residual set of functions of $\mathcal{O}(R)$ which are universal in $\mathcal{O}(R)$.*

b) *If $\{\varphi_n\}_{n\geq 0}$ is run-away, then there exists a residual set of functions of $\mathcal{O}(R)$ which are universal in $A(K)$ for all $K \in \mathcal{K}_1(R)$.*

*Proof.* First we prove a). Let $\{f_k\}_{k\geq 0} \subset \mathcal{O}(R)$ be a denumerable dense subset of $\mathcal{O}(R)$. We also consider a strictly decreasing sequence of positive real numbers $\{\varepsilon_m\}_{m\geq 0}$ with limit 0 and $\{K_n\}_{n\geq 0} \subset \mathcal{K}'(R)$

an exhaustive sequence of compact subsets in $R$. Note that, according to the hypothesis and Theorem 2.16, $\mathcal{F}(R)$ must be either a one-point set or an infinite set.

Since $\{\varphi_n\}_{n \geq 0} \subset \mathcal{O}(R; R)$, we may define its corresponding sequence of composition operators $T_n : \mathcal{O}(R) \to \mathcal{O}(R)$, $n \geq 0$, by $T_n(f) = f \circ \varphi_n$. Obviously, every $T_n$ is a continuous linear operator on $\mathcal{O}(R)$.

Let $K \subset R$ be a compact subset, $f \in \mathcal{O}(R)$ and $\varepsilon > 0$. We define the following subsets of $\mathcal{O}(R)$:

$$G(f, \varepsilon, K) = \{g \in \mathcal{O}(R) : \text{ there is an } n \in N \text{ such that}$$
$$\max_{z \in K} |T_n(g(z)) - f(z)| < \varepsilon\}$$
$$O(f, \varepsilon, K) = \{h \in \mathcal{O}(R) : \max_{z \in K} |h(z) - f(z)| < \varepsilon\}.$$

In fact, the subsets $O(f, \varepsilon, K)$ are an open basis for the topology of $\mathcal{O}(R)$. Since each $T_n$ is continuous for every positive integer $n$, we find that $G(f, \varepsilon, K)$ is an open subset of $\mathcal{O}(R)$, because it is given by

$$G(f, \varepsilon, K) = \bigcup_{n=0}^{\infty} T_n^{-1}(O(f, \varepsilon, K)).$$

We now prove that, if $K \in \mathcal{K}'(R)$, then $G(f, \varepsilon, K)$ is a dense subset in $\mathcal{O}(R)$. To see this, fix $\varepsilon' > 0$, $h \in \mathcal{O}(R)$ and $K' \in \mathcal{K}'(R)$. We must prove that $G(f, \varepsilon, K) \cap O(h, \varepsilon', K') \neq \varnothing$, that is to say, there exists $g \in \mathcal{O}(R)$ such that

$$(1) \qquad\qquad \max_{z \in K'} |h(z) - g(z)| < \varepsilon'$$

and

$$(2) \qquad\qquad \max_{z \in K} |f(z) - T_{n_0}(g(z))| < \varepsilon,$$

for some positive integer $n_0$.

Since $\{\varphi_n\}_{n \geq 0}$ is run-away preserving on $R$, from Lemma 2.17 there exists a positive integer $n_0$ such that $K' \cap \varphi_{n_0}(K) = \varnothing$, $L = K' \cup \varphi_{n_0}(K) \in \mathcal{K}(R)$ and $\varphi_{n_0}$ restricted to $K$ is a homeomorphism onto its image $\varphi_{n_0}(K)$.

We define on $L$ the function

$$h_1 = \begin{cases} h(z) & \text{if } z \in K', \\ f(\varphi_{n_0}^{-1}(z)) & \text{if } z \in \varphi_{n_0}(K), \end{cases}$$

where $\varphi_{n_0}^{-1}$ denotes the inverse mapping of $\varphi_{n_0} : K \to \varphi_{n_0}(K)$. It is clear that $h_1 \in A(L)$; then, since $L \in \mathcal{K}(R)$, we may apply Mergelyan's approximation theorem to obtain a holomorphic function $g \in \mathcal{O}(R)$ such that $\max_{z \in L} |h_1(z) - g(z)| < \varepsilon'' = \min(\varepsilon, \varepsilon')$. Therefore, we have

$$\max_{z \in K'} |h(z) - g(z)| < \varepsilon'$$

and

$$\begin{aligned} \max_K |f(z) - T_{n_0}(g(z))| &= \max_{z \in K} |f(z) - g(\varphi_{n_0}(z))| \\ &= \max_{z \in \varphi_{n_0}(K)} |f(\varphi_{n_0}^{-1}(z)) - g(z)| \\ &\leq \max_{z \in L} |h_1(z) - g(z)| < \varepsilon, \end{aligned}$$

which are (1) and (2), respectively.

On the other hand, it is easy to see that the set $G$ of universal functions on $\mathcal{O}(R)$ may be written as

$$(3) \qquad G = \bigcap_{k=0}^{\infty} \bigcap_{m=0}^{\infty} \bigcap_{n=0}^{\infty} G(f_k, \varepsilon_m, K_n).$$

Thus $G$ is a $G_\delta$ set and since $\mathcal{O}(R)$ is a Baire space we conclude that $G$ is a residual set. So, we have proved a).

In order to prove b), we replace the exhaustive sequence of compact subsets by the sequence of compact subsets furnished by Lemma 2.10. Analogously, as before, we may prove that if $K \in \mathcal{K}_1(R)$, then $G(f, \varepsilon, K)$ is a dense subset on $\mathcal{O}(R)$, the only difference being that Lemma 2.11 instead of Lemma 2.17 has to be applied. Finally, if the set of universal functions in $A(K)$ for all $K \in \mathcal{K}_1(R)$ is denoted by $G$, then this set may be written again as in (3). Since the left inclusion is obvious, we only prove the right inclusion. We have to prove that if $f$ is universal in $A(K_n)$ for each $n$, then it is also universal in $A(K)$ for all $K \in \mathcal{K}_1(R)$. Given any $K \in \mathcal{K}_1(R)$ there exists $n_0$

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such that $K \subset \operatorname{int} K_{n_0}$ and $\operatorname{int} K_{n_0} \backslash K$ being connected, we may apply Mergelyan's theorem again to show that $A(K_{n_0})$ is dense in $A(K)$. Since $f$ is universal in $A(K_{n_0})$, it is also universal in $A(K)$. This proves the right inclusion. Hence, we also have b) and the proof of the theorem is concluded. □

*Remarks* 1. Part a) of Theorem 3.1 gives sufficient conditions on a Riemann surface and on a given sequence of holomorphic self-mappings for the existence of a universal function with respect to this sequence. Observe that the Birkhoff-Seidel-Walsh theorem and Theorem 1.2 follow from part a) of Theorem 3.1, Proposition 2.7 and Proposition 2.13 ii). Analogously, Zappa's theorem and Theorem 1.1 follow from part b) of Theorem 3.1 and Proposition 2.7. Observe that, by an application of Mergelyan's theorem $\mathcal{O}(R)$ is dense in $A(K)$ for all $K \in \mathcal{K}(R)$. Consequently, in case a) the universal functions on $\mathcal{O}(R)$ are also universal on $A(K)$ for all $K \in \mathcal{K}(R)$. Since $\mathcal{K}(R) = \mathcal{K}_1(R)$ if and only if $\mathcal{F}(R)$ is a one-point set, a further improvement of part b) has been obtained.

2. The proof of Theorem 3.1 still works for sequences of continuous mappings $\{\varphi_n\}_{n \geq 0}$ such that, for every compact subset $K \in \mathcal{K}'(R)$ there exists $n_0$ such that $K \cap \varphi_{n_0}(K) = \varnothing$, $\varphi_{n_0}$ restricted to $K$ is in $A(K)$ and one-to-one on $K$ and $\varphi_{n_0}(K) \in \mathcal{K}'(R)$. In fact, in the proofs of the topological lemmas in Section 2, we have not used any analytic property, and the analytic properties needed in the proof of Theorem 3.1 are satisfied if the restriction of $\varphi_{n_0}$ to $K$ is in $A(K)$, the space of analytic functions in the interior of $K$ and continuous on $K$.

3. It is also possible to define *run-away semi-preserving* sequences of mappings $\{\varphi_n\}_{n \geq 0} \subset \mathcal{O}(R; R)$ as follows. For every compact subset $K \in \mathcal{K}'(R)$ there is a natural number $n_0$ such that $K \cap \varphi_{n_0}(K) = \varnothing$, $\varphi_{n_0}$ restricted to $K$ is one-to-one and $\varphi_{n_0}(K) \in \mathcal{K}(R)$ (instead of $\mathcal{K}'(R)$). This definition would alter the topological lemmas at the end of Section 2 and Theorem 2.16. For instance, the existence of run-away semi-preserving sequences of mappings would be possible for noncompact Riemann surfaces whose Freudenthal space is finite and has three or more elements. It is still possible to show a similar statement to that of Theorem 3.1 for this kind of sequence.

4. Finally, we examine the necessity of the topological condition on $R$ in part a) of Theorem 3.1. That is, the statement of Theorem 3.1 a) is not true if $\mathcal{F}(R)$ is a two-point set. To see this, we consider $R = \mathbf{C}^*$ and let $\{c_n\}$ be a sequence that converges to zero; thus, $\{c_n z\}_{\geq 0}$ is a run-away preserving sequence for $\mathbf{C}^*$, and suppose that there is a function $f \in \mathcal{O}(\mathbf{C}^*)$ which is universal in $\mathcal{O}(\mathbf{C}^*)$ (this forces $f$ to have an essential singularity at 0 and at $\infty$). Without loss of generality, we may assume that $\{c_n\}_{n\geq 0}$ decreases to zero. Let $f_n(z) = f(c_n z)$, $n \geq 0$. Since $f$ is universal, we may extract a subsequence $\{f_{n_k}\}_{k\geq 0}$ which converges uniformly to zero on $|z| = 1$. Consequently, the functions $\{f_{n_k}\}_{k\geq 0}$ are uniformly bounded on $|z| = 1$. This implies that $|f(z)| < M$ on $|z| = |c_{n_k}|$ for all $k \geq 0$. Applying the maximum principle, we see that $|f(z)| < M$ on $0 < |z| < |c_{n_1}|$, which contradicts the fact that $f$ has an essential singularity at the origin.

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