ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1998

ON THE ABSOLUTE HARDY-BOHR CRITERIA

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ABSTRACT. We consider the general question of when the series to sequence variational summability domain, of an absolutely regular matrix method, is a sum space. In particular, for Nörlund polynomial methods it is shown to never be the case unless the method is equivalent to convergence.

1. Introduction. In most instances where the summability factors for a matrix method or pair of matrix methods have been determined, they have been characterized by the classical Hardy-Bohr conditions, see [**1, 3**]. In sequence space theory the notion of a sum space was introduced in [**13**], and in [**3**] it is shown that these classical conditions characterizing the summability factors are equivalent to the series to sequence convergence domain being a sum space. The situation is similar in the case of absolute summability factors, see [**2, 3**]. In [**6**] and [**7**], for example, certain classes of Nörlund methods are studied and results on when the series to sequence convergence domain is a sum space are given. Here we begin the study of similar questions in the context of absolute summability. In Section 2 we use the notion of a T-solid sequence space introduced in [**10**] to show that for any nontrivial Nörlund polynomial method N_p the summability domain $bv_{N_p\Sigma}$ is never a sum space. In Section 3 we note that, for an absolutely regular matrix method A, a necessary condnition for the series to sequence summability domain $bv_{A\Sigma}$ to be a sum space is that the method be of type $M(bv_0)$. This coincides with the situation for regular matrix methods given in [**3**]. That is, if $c_{A\Sigma}$ is a sum space, then the matrix A is of type M. In the final section we make several observations concerning certain absolutely regular Nörlund methods and pose the open question as to whether their series to sequence summability domains are in fact sum spaces.

Received by the editors on July 25, 1995, and in revised form on December 18, 1995.
1991 AMS *Mathematics Subject Classification.* Primary 40C05, 40G99; Secondary 40H05, 46A45.

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2. Notation and terminology. Throughout we use the notation and results given by Wilansky [**17**] and by Zeller and Beekmann [**19**]. Let ω denote the space of all sequences, m the space of bounded sequences, c the convergent sequences, c_0 sequences that converge to 0, $cs = \{x \in \omega : \Sigma_n x_n \text{ is convergent}\}$, $bv = \{x \in \omega : \Sigma_n |x_n - x_{n-1}| < \infty\}$, $bv_0 = bv \cap c_0$, $l_1 = \{x \in \omega : \Sigma_n |x_n| < \infty\}$ and φ all finitely nonzero sequences. If $A = (a_{nk})$ is an infinite matrix, the method A defines a seqeunce to sequence transformation mapping a sequence s (real or complex) to t where

$$t_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, \dots .$$

The convergence domain c_A for a matrix A consists of those sequences s for which $t = As$ exists and belongs to c. The A-limit is defined for $s \in c_A$ by $A - \lim s_k = \lim t_n$. The method A is conservative provided $c \subset c_A$ and strictly stronger than convergence if, moreover, $c \neq c_A$. A conservative method A is regular provided $A - \lim s_k = \lim s_k$ for all $s \in c$. The summability domain bv_A consists of those sequences s for which $t = As$ exists and belongs to bv. The regular matrix method A is absolutely regular provided, in addition, $bv \subset bv_A$. The matrix A is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all n. Let Σ denote the triangle of ones, so that $A\Sigma$ is the series to sequence method associated with A.

An FK-space containing φ densely is said to be an AD space. If E is an FK-space containing φ, the multipliers on E are defined as $M(E) = \{x \in \omega : xy \in E, \text{ for all } y \in E\}$, where xy denotes the coordinatewise product. The f-dual of E is $E^f = \{(f(e_i))_{i=1}^{\infty} : f \in E'\}$, where E' is the topological dual of E and e_i denotes the ith coordinate sequence. Then E is said to be a sum space provided $E^f = M(E)$. The notion of a sum space was defined and studied by Ruckle in [**13, 14, 15**] and [**16**]. A regular method A is Hardy-Bohr if and only if $c_{A\Sigma}$ is a sum space. In [**3**] it is noted that the summability factors for A are characterized by the classical Hardy-Bohr conditions if and only if A is Hardy-Bohr. That is, for these methods the summability factors represent the continuous linear functionals on $c_{A\Sigma}$. An absolutely regular triangle A is absolute Hardy-Bohr if and only if $bv_{A\Sigma}$ is a sum space. For an absolutely regular matrix method A, the absolute summability factors, see for example [**2**], are characterized by the classical Hardy-Bohr conditions if and only if $bv_{A\Sigma}$ is a sum space with AD (see Theorem 4.1 of [**8**] and Theorem 3.7 of [**3**]).

3. T-solid sequence spaces. In Theorem 4.1 of [**8**] equivalent conditions are given for $bv_{A\Sigma}$ to be a sum space. In particular, [**8**] covers the case in which A is an absolutely regular triangle and $T = A\Sigma$. However, it is difficult to determine if one of these conditions is satisfied for specific matrix methods. In [**10**], the notion of a T-solid sequence space is introduced, and it is shown that bv_T is a sum space if and only if bv_T is T-solid. For T bv-reversible, the sequence space E is T-solid if and only if for each $x \in E$ and $y \in m$

$$\left(\sum_{n=1}^{\infty} y_n(t_{nk} - t_{n-1,k})x_k\right)_k \in E$$

[**10**, Theorem 4.2]. It is this condition that we now use to show that, for any Nörlund polynomial method N_p, $bv_{N_p\Sigma}$ is not $N_p\Sigma$-solid and hence $bv_{N_p\Sigma}$ is not a sum space.

The Nörlund polynomial method associated with the complex polynomial $p(z) = \sum_{k=0}^{N} p_k z^k$ is defined by $N_p = (a_{nk})$ where

$$a_{nk} = \begin{cases} p_{n-k} & \text{if } k \le N, \\ 0 & \text{if } k > N. \end{cases}$$

To ensure that N_p is a regular triangle, we need $p(1) = 1$ and $p_0 \neq 0$. Any regular Nörlund polynomial method is absolutely regular [**12**, Theorem 1].

Theorem 3.1. *If N_p is an absolutely regular Nörlund polynomial method with $bv_{N_p} \neq bv$, then $bv_{N_p\Sigma}$ is not a sum space.*

Proof. Let $p(z) = \sum_{k=0}^{N} p_k z^k$ generate an absolutely regular Nörlund polynomial method such that $bv_{N_p} \neq bv$. Let $P_j = \sum_{k=0}^{j} p_k$, $T = N_p\Sigma$ and $u_k = x_k \sum_{n=1}^{\infty} y_n(t_{nk} - t_{n-1,k})$, where $x \in bv_T$ and $y \in m$. We need to show that there exists an $x \in bv_T$ and $y \in m$ so that $u \notin bv_T$. After some simplification,

$$u_k = x_k \sum_{j=0}^{N} p_j y_{k+j}$$

and for $n > N$,

$$
\begin{aligned}
(Tu)_n - (Tu)_{n-1} &= \sum_{j=0}^{N} p_{N-k} u_{n-N+k} \\
&= \sum_{j=0}^{N} p_j \sum_{k=0}^{N} p_{N-k} x_{n-N+k} y_{n-N+k+j} \\
&= \left\{ \sum_{j=0}^{N} p_j y_{n-N+j} \right\} x_n - n p_N \\
&\quad + \sum_{j=0}^{N} p_j \sum_{k=1}^{N} p_{N-k} x_{n-N+k} y_{n-N+k+j} \\
&= \sum_{j=0}^{N} p_j y_{n-N+j} \left\{ \sum_{k=0}^{N} p_{N-k} x_{n-N+k} - \sum_{k=1}^{N} p_{N-k} x_{n-N+k} \right\} \\
&\quad + \sum_{j=0}^{N} p_j \sum_{k=1}^{N} p_{N-k} x_{n-N+k} y_{n-N+k+j} \\
&= \sum_{j=0}^{N} p_j y_{n-N+j} ((Tx)_n - (Tx)_{n-1}) \\
&\quad - \sum_{k=1}^{N} p_{N-k} x_{n-N+k} \sum_{j=0}^{N} p_j y_{n-N+j} \\
&\quad + \sum_{j=0}^{N} p_j \sum_{k=1}^{N-1} p_{N-k} x_{n-N+k} y_{n-N+k+j} + \sum_{j=0}^{N} p_j p_0 y_{n-N+j} \\
&= \sum_{j=0}^{N} p_j y_{n-N+j} ((Tx)_n - (Tx)_{n-1}) \\
&\quad - \sum_{k=1}^{N} p_{N-k} x_{n-N+k} \sum_{j=0}^{N} p_j y_{n-N+j} \\
&\quad + \sum_{j=0}^{N} p_j \sum_{k=1}^{N-1} p_{N-k} x_{n-N+k} y_{n-N+k+j}
\end{aligned}
$$

$$+\sum_{j=0}^{N-1} p_j p_0 x_n y_{n+j} + p_N p_0 x_n y_{n+N}.$$

In the last expression for $(Tu)_n - (Tu)_{n-1}$, the second term is a linear combination of $y_{n-N}, y_{n-N+1}, \dots, y_n$, the third term is a linear combination of $y_{n-N+1}, \dots, y_{n+N-1}$, and the fourth term is a linear combination of $y_n, \dots, y_{n+N-1}$. Denote the coefficients of the terms $y_{n-N}, \dots, y_{n+N-1}$ by $b_{n-N}, \dots, b_{n+N-1}$. Then

$$\begin{aligned}(Tu)_n - (Tu)_{n-1} = &\sum_{j=0}^{N} p_j y_{n-N+j}((Tx)_n - (Tx)_{n-1})\\ &+ \sum_{k=0}^{2N-1} b_{n-N+k} y_{n-N+k}\\ &+ p_N p_0 x_n y_{n+N}.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{n=N+1}^{\infty} |(Tu)_n - (Tu)_{n-1}|\\ &\qquad \geq \sum_{n=N+1}^{\infty} \left| \sum_{k=0}^{2N-1} b_{n-N+k} y_{n-N+k} + p_N p_0 x_n y_{n+N} \right|\\ &\qquad\quad - \sum_{n=N+1}^{\infty} \sum_{j=0}^{N} |p_j||y_{n-N+j}||(Tu)_n - (Tu)_{n-1}|.\end{aligned}$$

For any $x \in bv_T$ and $y \in m$, the second term on the right side of the previous inequality is bounded. It suffices to show that there exists an $x \in bv_T$ and $y \in m$ so that the first term of the inequality is unbounded. Since $l_1 \neq bv_T$, there exists an $x \in bv_T \backslash l_1$, i.e., $bv_A = bv$ implies $bv_{A\Sigma} = l_1$. Consider the following disjoint subsequences of the sequence x : $(x_{(2N+1)n})_{n=1}^{\infty}, (x_{(2N+1)n+1})_{n=1}^{\infty}, \dots, (x_{(2N+1)n+2N})_{n=1}^{\infty}$. Then there exists an index j, $0 \leq j \leq 2N$ such that $\sum_{n=1}^{\infty} |x_{(2N+1)n+j}| = \infty$, and

$$\sum_{n=N+1}^{\infty}\left|\sum_{k=0}^{2N-1} b_{n-N+k}y_{n-N+k} + p_N p_0 x_n y_{n+N}\right| > \sum_{n=N+1}^{\infty}\left|\sum_{k=0}^{2N-1} b_{n-N+k}y_{(2N+1)n+j-N+k} + p_N p_0 x_{(2N+1)n+j} y_{(2N+1)n+j+N}\right|.$$

Let $a = \sum_{k=0}^{2N-1} b_{n-N+k}$, $b = p_N p_0 x_{(2N+1)n+j}$, and set $y_{(2N+1)n+j-N+k} = 1$ for each $k = 0, \dots, 2N-1$. If $|a+b| > |a-b|$, then set $y_{(2N+1)n+j+N} = 1$ and if $|a-b| > |a+b|$, then set $y_{(2N+1)n+j+N} = -1$. Then we have chosen the $y_{(2N+1)n+j+N}$ so that

$$\left|\sum_{k=0}^{2N-1} b_{(2N+1)n+j-N+k}y_{(2N+1)n+j-N+k} + p_N p_0 x_{(2N+1)n+j} y_{(2N+1)n+j+N}\right|$$

is as large as possible. If L is the larger of $|a+b|$ and $|a-b|$, then

$$L \geq \frac{|a+b|+|a-b|}{2} \geq \max\{|a|,|b|\} \geq |b| = |p_N p_0 x_{(2N+1)n+j}|.$$

Consequently,

$$\sum_{n=N+1}^{\infty}\left|\sum_{k=0}^{2N-1} b_{n-N+k}y_{n-N+k} + p_N p_0 y_{n+N}\right| \geq \sum_{n=N+1}^{\infty} |p_N||p_0||x_{(2N+1)n+j}| = \infty$$

(since $p_N p_0 \neq 0$) and bv_T is not T-solid. ◻

4. Type $M(bv_0)$. Let E be an FK-space containing φ and A a matrix method which maps E into E. In [**4**] the method A is defined to be of type $M(E)$ if, whenever $tA = 0$ for $t \in E^{\beta}$ (the β-dual of E), then $t = 0$. This definition agrees with the usual definition of type M and M^* when $E = c$ or l_1, respectively. It also agrees with the definition given in [**9**] for E with basis (e_i, E_i) since, in this case, $E^{\beta} = E^{\delta}$ (the δ-dual of E). In Theorem 2 of [**9**], it is shown that for A a reversible $E-E$ method, A is perfect if and only if A is of type $M(E)$. In the present setting we have A is bv_0-perfect, i.e., $\overline{bv_0} = (bv_0)_A$, if and only if A is of type $M(bv_0)$. The next proposition gives a little more. If A is a regular matrix method with row sums one, then $Ae = e$.

Proposition 4.1. *Let A be an absolutely regular triangle with row sums* 1. *Then A is of type $M(bv_0)$ if and only if A is bv-perfect, i.e., $\overline{bv} = bv_A$.*

Proof. $\Rightarrow$. Since A is absolutely regular, A maps bv_0 into bv_0, so that assuming A is of type $M(bv_0)$ gives A is bv_0-perfect [**9**]. That is, $\overline{bv_0} = (bv_0)_A$. Let $x \in bv_A$. If $x \in (bv_0)_A \subset bv_A$, then choose a sequence $(x^{(n)}) \subset bv_0$ such that $x^{(n)} \to x$ in $(bv_0)_A$. But then $x^{(n)} \to x$ in bv_A so that $x \in \overline{bv}$ in bv_A. Now suppose $x \in bv_A \backslash (bv_0)_A$. Then $Ax \in bv \backslash bv_0$ and there exist some $y \in bv_0$ and scalar $k \neq 0$ such that $Ax = y + ke$. Then $x = A^{-1}y + kA^{-1}e = A^{-1}y + ke$, where $A^{-1}y \in (bv_0)_A$. Choose $(z^{(n)}) \subset bv_0$ such that $z^{(n)} \to A^{-1}y$ in $(bv_0)_A$. Let $\omega^{(n)} = z^{(n)} + ke$. Then $\omega^{(n)} \in bv$ and

$$A\omega^{(n)} = Az^{(n)} + ke \longrightarrow y + ke = Ax$$

in bv. Therefore, $\omega^{(n)} \to x$ in bv_A and $\overline{bv} = bv_A$.

$\Leftarrow$. Let $x \in (bv_0)_A \subset bv_A$. Since $\overline{bv} = bv_A$, choose a sequence $(x^{(n)}) \in bv$ such that $x^{(n)} \to x$ in bv_A. For each n write $x^{(n)} = y^{(n)} + k_n e$ where $y^{(n)} \in bv_0$ and the k_n are scalars. Then $y^{(n)} + k_n e \to x$ in bv_A which implies

$$\lim_{n\to\infty} \|Ay^{(n)} + k_n Ae - Ax\|_{bv} = 0.$$

Let $z^{(n)} = Ay^{(n)} + k_n Ae - Ax$. Then, for all n,

$$\|z^{(n)}\|_{bv} = |\lim_k z_k^{(n)}| + \sum_k |z_k^{(n)} - z_{k+1}^{(n)}|.$$

But since $Ay^{(n)}$, $Ax \in bv_0$, $\lim_k (Ay^{(n)})_k = \lim_k (Ax)_k = 0$ and, since A has row sums one, $(Ae)_k = 1$ for each k. Therefore,

$$\|z^{(n)}\|_{bv} = |k_n| + \sum_k |z_k^{(n)} - z_{k+1}^{(n)}|$$

and hence $\|z^{(n)}\|_{bv} \to 0$ implies $k_n \to 0$. Finally,

$$\begin{aligned}\|Ay^{(n)} - Ax\|_{bv_0} &= \|Ay^{(n)} - Ax\|_{bv} \\ &\leq \|Ay^{(n)} - Ax + k_n Ae\|_{bv} + |k_n|\|Ae\|_{bv} \longrightarrow 0,\end{aligned}$$

and hence

$$y^{(n)} \longrightarrow x \quad \text{in } (bv_0)_A,$$

which implies $\overline{bv_0} = (bv_0)_A$. □

Proposition 4.2. *Let A be an absolutely regular triangle and $T = A\Sigma$. Then bv_T has AD if and only if $\overline{bv} = bv_A$.*

Proof. $\Rightarrow$. Suppose bv_T has AD and $x \in bv_A$. Then $x \in bv_A \Rightarrow Ax \in bv \Rightarrow T^{-1}(Ax) \in bv_T$. Hence there exists a sequence $(z^{(n)}) \subset \varphi$ such that

$$\|T^{-1}(Ax) - z^{(n)}\|_{bv_T} \longrightarrow 0 \Longrightarrow \|Ax - Tz^{(n)}\|_{bv} \longrightarrow 0.$$

But $Tz^{(n)} = A\Sigma z^{(n)} = A(\Sigma z^{(n)})$ and $(z^{(n)}) \subset \varphi \subset l_1$ implies that $(\sum z^{(n)}) \in bv$. Thus we have

$$\|Ax - A(\Sigma z^{(n)})\|_{bv} \longrightarrow 0,$$

which implies

$$\left\| x - \Sigma z^{(n)} \right\|_{bv_A} \longrightarrow 0.$$

This gives $\overline{bv} = bv_A$.

$\Leftarrow$. Suppose $\overline{bv} = bv_A$, and let $x \in bv_T$. Now $x \in bv_T \Rightarrow Tx \in bv \Rightarrow A^{-1}(Tx) \in bv_A$. So there exists a sequence $(y^{(n)}) \subset bv$ such that

$$\begin{aligned}\|y^{(n)} - A^{-1}(Tx)\|_{bv_A} \longrightarrow 0 &\Longrightarrow \|Ay^{(n)} - Tx\|_{bv} \longrightarrow 0 \\ &\Longrightarrow \|T^{-1}(Ay^{(n)}) - T^{-1}(Tx)\|_{bv_T} \longrightarrow 0 \\ &\Longrightarrow \|T^{-1}(Ay^{(n)}) - x\|_{bv_T} \longrightarrow 0 \\ &\Longrightarrow \|\Sigma^{-1}A^{-1}Ay^{(n)} - x\|_{bv_T} \longrightarrow 0 \\ &\Longrightarrow \|\Sigma^{-1}y^{(n)} - x\|_{bv_T} \longrightarrow 0.\end{aligned}$$

But $\Sigma : l_1 \to bv$ and $bv_\Sigma = l_1$ so that $\Sigma^{-1} : bv \to l_1$ which gives $\Sigma^{-1}y^{(n)} \in l_1$ for each n. Then $l_1 \subset bv \subset bv_T$ and φ dense in l_1 implies, given $\varepsilon > 0$, we can choose an n_0 such that

$$\|z^{(n_0)} - x\|_{bv_T} < \varepsilon.$$

Thus $\bar{\varphi} = bv_T$ and bv_T has AD. □

In Proposition 3.2 of [**3**] it is shown that, for a regular triangle A, if $c_{A\Sigma}$ is a sum space, then A is of type M. We have the analogous result in this setting.

Proposition 4.3. *Let A be an absolutely regular triangle. If $bv_{A\Sigma}$ is a sum space, then A is of type $M(bv_0)$.*

Proof.

$bv_{A\Sigma}$ a sum space

$$\begin{aligned} &\Longrightarrow bv_{A\Sigma} \text{ is a sum space with } AD \quad (\text{see [8, Theorem 4.1]})\\ &\Longrightarrow \overline{bv} = bv_A \quad \text{by Proposition 4.2}\\ &\Longrightarrow A \text{ is of type } M(bv_0) \quad \text{by Proposition 4.1.} \end{aligned}$$ □

Proposition 4.3 can be used to give another proof that for any Nörlund polynomial method N_p which is absolutely regular with $bv \neq bv_{N_p}$, $bv_{N_p\Sigma}$ is not a sum space. If it were a sum space, then by Proposition 4.3, N_p would be of type $M(bv_0)$. However, by Theorem 4.4 of [**4**] and Theorem 4.6 of [**5**], no coregular Nörlund polynomial method stronger than convergence can be of type $M(bv_0)$.

5. Nörlund methods. The Cesàro methods C_α for $\alpha > 0$ satisfy both $c_{C_\alpha\Sigma}$ is a sum space, see [**1, 18, 3**], and $bv_{C_\alpha\Sigma}$ is a sum space, see [**2, 3**]. In [**6**], the Nörlund method N_p defined by $p(z) = (1+z)/(1-z)$ is shown to satisfy $c_{N_p\Sigma}$ is a sum space. In [**7**], other, non-Cesàro like, Nörlund methods are shown to have series to sequence convergence domains that are sum spaces. For example, if $p(z) = (1+z)/(1-z)^\alpha$ where $\alpha \geq 1$, then $c_{N_p\Sigma}$ is a sum space. If $p(z) = (1+z)/(1-z)$, then N_p is absolutely regular, see [**12**, Theorem I], with $bv \neq bv_{N_p\Sigma}$.

Moreover, if $p(z) = (1+z)/(1-z)^r$ and $q(z) = (1+z)/(1-z)^{r+1}$ where r is a positive integer, then $bv_{N_p\Sigma} \subset bv_{N_q\Sigma}$, see [**11**, Theorem 2.19], so that all these Nörlund methods are absolutely regular. If the series to sequence summability domains $bv_{N_q\Sigma}$ are to be sum spaces by Proposition 4.3, the methods must be of type $M(bv_0)$.

Proposition 5.1. *If*

$$p(z) = \frac{1+z}{(1-z)^r} = \sum_{k=0}^{\infty} p_k^{(r)} z^k,$$

where r is a positive integer, then the Nörlund method N_p is of type $M(bv_0)$.

Proof. If $p(z) = (1+z)/(1-z)^r$, denote the nkth term of the matrix method by $p_{n-k}^{(r)}/P_n^{(r)}$ for $k \le n$, 0 otherwise and wehre $P_n^{(r)} = \sum_{j=0}^{n} p_j^{(r)}$. Let $t \in bs$, the space of bounded series, be such that $tN_p = 0$. Then

$$H_k = \langle t, N_p e_k \rangle = \sum_{j=k}^{\infty} \frac{p_{j-k}^{(r)} t_j}{P_j^{(r)}} = 0 \quad \text{for all } k \ge 0.$$

For all positive integers m, $p_{j-k}^{(m)} = P_{j-k}^{(m-1)}$, so that if $m > 1$, $\Delta_k p_{j-k}^{(m)} = p_{j-k}^{(m-1)}$, where Δ denotes the first forward difference. This gives

$$\begin{aligned}
\Delta H_k &= \sum_{j=k}^{\infty} \frac{p_{j-k}^{(r)} t_j}{P_j^{(r)}} - \sum_{j=k+1}^{\infty} \frac{p_{j-k-1}^{(r)} t_j}{P_j^{(r)}} \\
&= \frac{p_0^{(r)} t_k}{P_k^{(r)}} + \sum_{j=k+1}^{\infty} \frac{\Delta_k p_{j-k}^{(r)} t_j}{P_j^{(r)}} \\
&= \frac{t_k}{P_k^{(r)}} + \sum_{j=k+1}^{\infty} \frac{p_{j-k}^{(r-1)} t_j}{P_j^{(r)}} \\
&= \sum_{j=k}^{\infty} \frac{p_{j-k}^{(r-1)} t_j}{P_j^{(r)}}
\end{aligned}$$

since each $p_0^{(r)} = 1$. It then follows that

$$\Delta^{r-1} H_k = \sum_{j=k}^{\infty} \frac{p_{j-k}^{(1)} t_j}{P_j^{(r)}} = \frac{t_k}{P_k^{(r)}} + 2 \sum_{j=k+1}^{\infty} \frac{t_j}{P_j^{(r)}} = 0 \quad \text{for all } k \geq 0,$$

and hence,

$$\begin{aligned} \Delta^r H_k &= \frac{t_k}{P_k^{(r)}} + 2 \sum_{j=k+1}^{\infty} \frac{t_j}{P_j^{(r)}} - \frac{t_{k+1}}{P_{k+1}^{(r)}} - 2 \sum_{j=k+2}^{\infty} \frac{t_j}{P_j^{(r)}} \\ &= \frac{t_k}{P_k^{(r)}} + \frac{t_{k+1}}{P_{k+1}^{(r)}} = 0 \quad \text{for all } k \geq 0. \end{aligned}$$

Thus, for all $k \geq 0$, $t_k = (-1)^k P_k^{(r)} t_0$, and for each j,

$$\sum_{k=1}^{2j} t_k = \sum_{k=1}^{2j} (-1)^k P_k^{(r)} t_0 = -t_0 \sum_{k=1}^{j} \Delta P_{2k-1}^{(r)} = t_0 \sum_{k=1}^{j} p_{2k}^{(r)}.$$

Since $t \in bs$, the lefthand side of the previous equality is bounded for all j. Since $p_k^{(r)} \geq 1$ for all k it follows that $t_0 = 0$ and hence $t_k = 0$ for all k. Therefore, N_p is of type $M(bv_0)$. □

An interesting open question is whether the summability domains $bv_{N_p\Sigma}$ are sum spaces for the Nörlund methods of the previous proposition.

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