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Q-REFLEXIVE BANACH SPACES

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Let E be a Banach space. There are several natural ways in which any polynomial $P \in \mathcal{P}(^n E)$ can be extended to $\tilde{P} \in \mathcal{P}(^n E'')$, in such a way that the extension mapping is continuous and linear (see, for example, [**6**]). Taking the double transpose of the extension mapping $P \to \tilde{P}$ yields a linear, continuous mapping from $\mathcal{P}(^n E)''$ into $\mathcal{P}(^n E'')''$. Further, since $\mathcal{P}(^n E'')$ is a dual space, it follows that there is a natural projection of $\mathcal{P}(^n E'')''$ onto $\mathcal{P}(^n E'')$, and thus we have a mapping of $\mathcal{P}(^n E)''$ into $\mathcal{P}(^n E'')$. If all polynomials on a Banach space E are weakly continuous on bounded sets, then these mappings from $\mathcal{P}(^n E)''$ into $\mathcal{P}(^n E'')$ coincide and have a particularly simple description. We discuss this in some detail below.

In this article we restrict ourselves to the situation in which all polynomials on E are weakly continuous on bounded sets, and we study when this mapping is an isomorphism. As we will see, if three "ingredients" are present, then the mapping will be an isomorphism: (1) E'' has the Radon-Nikodym property [**18**], (2) E'' has the approximation property [**30**], and (3) every polynomial on E is weakly continuous on bounded sets. In addition, we will construct an example of a quasi-reflexive (nonreflexive) Banach space E for which the extension mapping is an isomorphism.

It is well known that $\mathcal{P}(^n E)$ is isomorphic to $(\oplus_{n,s} E)'$, the dual of the n-fold *symmetric* tensor product of E endowed with the projective topology. In fact, our results carry over to the space $(\otimes_n E)'$. However, since our interest is in polynomials and holomorphic functions on E, we have preferred to concentrate on symmetric tensor products.

By [**9**, Theorem 2.9] if all continuous polynomials are weakly continuous on bounded sets, then they are in fact uniformly weakly continuous on bounded sets and so have a unique extension to polynomials

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on E'' which are weak* continuous on bounded sets. The mapping $P \in \mathcal{P}(^nE) \to \tilde{P}(x)$, where $x \in E''$ and $\tilde{P}$ is the weak* continuous extension of P, is continuous when spaces of polynomials are given their norm topology. We define the canonical mapping of the symmetric tensor product of E'' into the dual of $\mathcal{P}(^nE)$ in the following fashion

$$\begin{aligned} J_n : \bigotimes_{n,s} E'' &\longrightarrow \mathcal{P}(^nE)' \\ \bigotimes_n x &\longrightarrow [P \in \mathcal{P}(^nE) \longrightarrow \tilde{P}(x)]. \end{aligned}$$

Since $\|P\| = \|\tilde{P}\|$ it follows that

$$\begin{aligned} \left\| J_n\left(\bigotimes_n x\right)\right\| &= \sup_{\substack{P\in\mathcal{P}(^nE)\\ \|P\|\le 1}} |\tilde{P}(x)| \\ &\le \sup_{\substack{P\in\mathcal{P}(^nE)\\ \|P\|\le 1}} \|\tilde{P}\| \cdot \|x\|^n \\ &\le \|x\|^n. \end{aligned}$$

Hence

$$\begin{aligned} \|J_n(z)\| &\le \inf\left\{\sum_{i=1}^m \|x_i\|^n ; z = \sum_{i=1}^m \bigotimes_n x_i\right\} \\ &= \|z\|_\pi, \end{aligned}$$

and J_n can be extended in a unique fashion to a continuous linear mapping

$$J_n : \widehat{\bigotimes}_{n,s,\pi} E'' \longrightarrow \mathcal{P}(^nE)'.$$

The transpose ${}^tJ_n$ is the required canonical mapping from

$$\mathcal{P}(^nE)'' \quad \text{into} \quad \mathcal{P}(^nE'') \simeq \left(\bigotimes_{n,s,\pi} E''\right)'.$$

Various attempts at defining a canonical mapping from $\mathcal{P}(^nE)''$ into $\mathcal{P}(^nE'')$ have convinced us that the class of Banach spaces we are considering is the natural class[1] for such a mapping. Note that if J denotes the canonical mapping from E into E'', then $J_1 \circ J = J$.

Definition 1. We shall say that a Banach space E is Q-reflexive if the *canonical mapping* ${}^tJ_n$ is an *isomorphism* from $\mathcal{P}({}^nE)''$ onto $\mathcal{P}({}^nE'')$ for all n.

It is easily seen that a reflexive Banach space is Q-reflexive if and only if $\mathcal{P}({}^nE)$ is reflexive for all n. This collection of spaces has been studied in [**1, 2, 3, 4, 20, 21, 23, 25**] and so we confine ourselves here to nonreflexive Banach spaces which are Q-reflexive. If E is Q-reflexive, then, since $\mathcal{P}({}^nE)$ is a dual space it follows that $\mathcal{P}({}^nE)$ is 1-complemented in $\mathcal{P}({}^nE'')$, and we have the decomposition

$$\mathcal{P}({}^nE'') = \mathcal{P}_{\omega^*}({}^nE'') \oplus \mathcal{P}_{E^\perp}({}^nE'')$$

where $\mathcal{P}_{\omega^*}({}^nE'') \approx \mathcal{P}({}^nE)$ denotes the set of all polynomials on E'' which are weak* continuous on bounded sets and

$$\mathcal{P}_{E^\perp}({}^nE'') = \{P \in \mathcal{P}({}^nE''); P|_E \equiv 0\}.$$

The following result of Gutierrez, which improves a result in [**7**], reduces the study of Q-reflexive spaces to a more manageable collection of Banach spaces in which we find a more practical characterization of polynomials which are weakly continuous on bounded sets.

Proposition 2 [**25**]. *If E is a Banach space which contains a copy of l_1, then E admits a* $\mathbf{C}$*-valued homogeneous polynomial which is not weakly continuous on bounded sets.*

Proposition 3. *If E is a Banach space and $l_1 \not\hookrightarrow E$, then the following are equivalent*:

(a) *all continuous polynomials on E are weakly continuous on bounded sets*,

(b) *all continuous polynomials on E are weakly sequentially continuous at the origin.*

Proof. By [**9**, Proposition 2.12] the continuous polynomials on E which are weakly sequentially continuous are weakly (uniformly) continuous on bounded sets. By [**3**] it is sufficient to check weak sequential continuity at the origin. $\square$

To obtain examples of nonreflexive Q-reflexive Banach spaces, we note that the proof of Theorem 1 in [**20**] does not require reflexivity and consequently the following is true.

Proposition 4. *If E is a Banach space such that no spreading model built on a normalized weakly null sequence has a lower q-estimate for any $q < \infty$, then any continuous polynomial on any subspace of E is weakly sequentially continuous at the origin.*

We recall that a sequence of vectors $(u_j)_j$ in a Banach space is said to have a lower q-estimate if there exists $c > 0$ such that

$$\left\| \sum_{j=n}^{m} a_i u_j \right\| \geq c \left(\sum_{j=n}^{m} |a_j|^q \right)^{1/q}$$

for any positive integers n and m. Also, the dual of a Banach space E has the Radon-Nikodym property (RNP) if and only if each separable subspace of E has a separable dual [**18**]. Such spaces are also called Asplund spaces. If E is an Asplund space, then $l_1 \not\hookrightarrow E$ since $(l_1)' = l_\infty$ is nonseparable. Also, if E' is an Asplund space, then $l_1 \not\hookrightarrow E$. Otherwise, the image l_∞ of the transpose mapping would be an Asplund space and this is clearly false. Hence, if either E or E' is Asplund, it follows that $l_1 \not\hookrightarrow E$.

Theorem 5. *Let E denote a Banach space such that no spreading model built on a normalized weakly null sequence has a lower q-estimate for any $q < \infty$.*

(a) *If E' has* RNP *and the approximation property, then $\mathcal{P}(^n E)$ has* RNP *for all n.*

(b) *If E'' has* RNP *and the approximation property, then E is a Q-reflexive Banach space.*

Proof. By the above remarks it follows that in both cases $l_1 \not\hookrightarrow E$. By Propositions 3 and 4 we see that all continuous polynomials on any subspace of E are weakly continuous on bounded sets. By [**30**, Proposition 1.e.7] and our hypothesis in (a) it follows that E' has the approximation property in both cases. By [**9**, Corollary 2.11] it follows that $\mathcal{P}(^nG) \simeq (\hat{\otimes}_{n,s,\varepsilon} G')$, for any subspace G of E such that G' has the approximation property. We now complete the proof of (a).

By [**18**, p. 218], it suffices to show that any separable subspace H of $\mathcal{P}(^nE)$ is isomorphic to a subspace of a separable dual space. Suppose that $\{\phi_j^n\}_{j=1}^{\infty}$, $\phi_j \in E'$, spans a dense subspace of H. Let F denote the closed subspace of E' generated by $\{\phi_j\}_{j=1}^{\infty}$. By [**32**, Proposition 2], there exists a separable subspace E_1 of E and a complemented subspace F_1 of E' such that $F \subset F_1$ and $E_1' \simeq F_1$. Since E' has the RNP and the approximation property and F_1 is complemented in E', it follows that $E_1' \simeq F_1$ also has both of these properties. Hence,

$$H \subset \left(\widehat{\bigotimes}_{n,s,\varepsilon} F_1\right) \simeq \left(\widehat{\bigotimes}_{n,s,\varepsilon} E_1'\right) \simeq \mathcal{P}(^nE_1).$$

This implies that $\mathcal{P}(^nE_1)$ is a separable dual space and that H is isomorphic to a subspace of a separable dual space. Thus, $\mathcal{P}(^nE)$ has RNP. This completes the proof of (a).

We now complete the proof of (b). By the above, $\mathcal{P}(^nE) = (\hat{\otimes}_{n,\varepsilon,s} E')$. Since E'' has the RNP and the approximation property, we have by [**14, 24**],

$$\mathcal{P}(^nE)' \simeq \left(\widehat{\bigotimes}_{n,\varepsilon,s} E'\right)' \simeq \widehat{\bigotimes}_{n,\pi,s} E''$$

where the isomorphism I_n between these spaces satisfies

$$\left(I_n\left(\bigotimes_{n,s} x''\right)\right)(\phi^n) = (x''(\phi))^n$$

for all $x'' \in E''$ and $\phi \in E'$.

Hence, $I_n = J_n$ and ${}^tJ_n$ is an isomorphism, i.e., E is Q-reflexive. This completes the proof. $\square$

Corollary 6. *If E is a Banach space such that no spreading model built on a normalized weakly null sequence has a lower q-estimate for any $q < \infty$ and E' has the approximation property, then $\otimes_{n,\pi,s} E$ is Asplund for all n if and only if E is Asplund.*

Example 7. Since all spreading models built on a normalized weakly null sequence in c_0 are isomorphic to c_0 [**9**, p. 72] and $c_0' = l_1$ has the approximation property and RNP, it follows that $\mathcal{P}(^n c_0)$ has RNP for all n. This, however, also follows immediately from the fact, proved in [**5**], that $\mathcal{P}(^n c_0)$ is separable and thus as a separable dual space $\mathcal{P}(^n c_0)$ has RNP. We now show that c_0 is not Q-reflexive. In this case the canonical mapping is $J_n : \mathcal{P}_N(^n l_1) \to \mathcal{P}_I(^n l_1)$ where $\mathcal{P}_N(^n l_1)$ and $\mathcal{P}_I(^n l_1)$ are respectively the n-homogeneous nuclear and integral polynomials on l_1. There are various ways in which one can show that J_n is not an isomorphism and hence that c_0 is not Q-reflexive. For instance, $(\overline{B_{l_\infty}}, \sigma(l_\infty, l_1)) \simeq \Delta^N$ where Δ is the closed unit disc in $\mathbf{C}$. Let μ denote a Borel probability measure on Δ such that $\int_\Delta z\, d\mu(z) = 0$ and $\int_\Delta z^2\, d\mu(z) = 1$. If $v = \prod_{n=1}^\infty \mu_n$ on Δ^N, where $\mu_n = \mu$ all n, then the mapping

$$(*)\qquad \begin{aligned} (x_n)_n \in l_1 &\longrightarrow \int_{\Delta^N} [(y_n)_n((x_n)_n)]^2\, dv((y_n)_n) \\ &= \sum_{n,m} x_n x_m \int_{\Delta^N} y_n y_m\, dv((y_n)_n) \\ &= \sum_{n=1}^\infty x_n^2 \end{aligned}$$

defines an element of $\mathcal{P}(^n c_0)' = \mathcal{P}_I(^n l_1)$. The associated integral mapping from l_1 into l_∞ is not compact and hence not nuclear. This proves our claim.

Remarks. (a) The action of the polynomial represented by $(*)$ on $\mathcal{P}(^2 c_0)$ is given by

$$P \in \mathcal{P}(^2 c_0) \longrightarrow \sum_{n=1}^\infty P(e_n)$$

where $(e_n)_n$ is the standard unit vector basis in c_0. Clearly, replacing μ_n by a point mass at the origin for all $x \notin M$, M some subset of N, we see that $\sum_{n\in M} P(e_n) < \infty$ for all $P \in \mathcal{P}(^2 c_0)$. This provides another proof of a result in [**8, 13, 34**], namely, that $\sum_{n=1}^\infty |P(e_n)| < \infty$ for all $P \in \mathcal{P}(^2 c_0)$.

(b) The above shows that there is a one-to-one correspondence between $\mathcal{P}_I(^2 l_1)$ and the covariances of signed Borel measures on Δ^N.

By using Grothendieck's inequality, it follows that $l_\infty \hat{\otimes}_\pi l_\infty$ can be represented at the set of all functions of the form

$$f : N \times N \to \mathbf{C}$$

where $f(n,m) = \langle g(n), h(m) \rangle$ and $(g(n))_n$ and $(h(m))_m$ are relatively compact sequences in l_2. (See, for example, A. Defant and K. Floret [**15**, p. 171].)

The space $l_1 \hat{\otimes}_\varepsilon l_1$ has a representation as the set of all series of the form $\sum_{n=1}^\infty e_n^* \otimes x_n$ where $(e_n^*)_n$ is the standard unit vector basis in l_1 and $(x_n)_n$ is an unconditionally convergent series in l_1 [**2, 22**].

(c) Example 7 shows that the hypothesis "E'' has RNP" cannot be removed in Theorem 5 (b).

Our next step is to produce an example of a nonreflexive Q-reflexive Banach space. Theorem 5 clearly suggests that we should consider a space like Tsirelson's space but not a reflexive space and we were thus led to the (quasi-reflexive) James space modeled on the original Tsirelson space. It is clear, however, that this is representative of a class of examples which can be found using the appropriate properties of the James and Tsirelson space. We refer to [**30**] for details concerning the James spaces and the dual of the original Tsirelson space. Following standard notation, we let T denote the dual of the original Tsirelson space normed as in [**30**, p. 95], and we denote its dual by T^*. The James space modelled on T is discussed in [**11**] and [**29**].

Let $(t_n)_{n=1}^\infty$ denote the standard unconditional basis for T^* which is dual to the basis given in [**30**, p. 95]. The following two properties of T^* play an essential role in our construction. For the sake of completeness, we include a proof of (2).

For any positive integer n, we have

$$\left\| \sum_{j=n}^{2n} a_j t_j \right\|_{T^*} \leq 2 \sup_{n \leq i \leq 2n} |a_i| \tag{1}$$

[**12**, Proposition 1.7].

If $(k_j)_{j=1}^\infty$ is an increasing sequence of integers, $k_1 = 0$, then for any $\sum_{j=1}^\infty a_j t_j \in T^*$, we have

$$\Big\|\sum_{j=1}^\infty a_j t_j\Big\|_{T^*} \le \Big\|\sum_{j=1}^\infty \Big\|\sum_{i=k_j+1}^{k_j+1} a_i t_i\Big\|_{T^*} t_j\Big\|_{T^*} \tag{2}$$

[**12**, Lemma II.1 and notes and remarks, p. 24].

Proof of inequality (2). Let

$$u_j = \frac{\sum_{i=k_j+1}^{k_{j+1}} a_i t_i}{\|\sum_{i=k_{j+1}}^{k_{j+1}} a_i t_i\|_{T^*}}.$$

Then

$$\begin{aligned}\Big\|\sum_{j=1}^\infty a_j t_j\Big\|_{T^*} &= \Big\|\sum_{j=1}^\infty \Big(\sum_{i=k_j+1}^{k_{j+1}} a_i t_i\Big)\Big\|_{T^*}\\ &= \Big\|\sum_{j=1}^\infty \Big\|\sum_{i=k_j+1}^{k_{j+1}} a_i t_i\|_{T^*} u_j\Big\|_{T^*}\\ &\le \Big\|\sum_{j=1}^\infty \Big\|\sum_{i=k_j+1}^{k_{j+1}} a_i t_i\Big\|_{T^*} t_{k_j+1}\Big\|_{T^*}\\ &\le \Big\|\sum_{j=1}^\infty \Big\|\sum_{i=k_j+1}^{k_{j+1}} a_i t_i\Big\|_{T^*} t_j\Big\|_{T^*}\end{aligned}$$

where we use u_j to get the normalized sequence in [**12**] and we have reversed the inequality since we are dealing with T^* in place of T. Since $j \le k_j + 1$ and moving the support to the right in T increases the norm [**12**, Proposition 1.9(3)], it follows that in T^* moving the support to the left increases the norm and this is what we have done here.

For $(a_n)_{n=1}^\infty \in c_{00}$, the space of all sequences which are eventually zero, let

$$\|(a_n)_{n=1}^\infty\|_{T_J^*} = \sup_{\substack{p_1<p_2<\cdots<p_{2k}\\ k}} \Big\|\sum_{j=1}^k (a_{p_{2j-1}} - a_{p_{2j}}) t_j\Big\|_{T^*}. \tag{3}$$

The completion of c_{00} with respect to the norm $\|\cdot\|_{T_J^*}$ is denoted by T_J^* and is called the Tsirelson*-James space.

Proposition 8. *T_J^* has a monotone Schauder basis.*

Proof. Let $(e_j)_{j=1}^\infty$ denote the canonical unit vector basis for $c_{00} \subset T_J^*$. By (3) it is clear that

$$\left\| \sum_{j=1}^{n} a_j e_j \right\|_{T_J^*} \leq \left\| \sum_{j=1}^{n+1} a_j e_j \right\|_{T_J^*}$$

for any sequence of scalars $(a_j)_{j=1}^{n+1}$ and hence, by [**30**, Proposition 1.a.3], the sequence $(e_j)_{j=1}^\infty$ is a monotone Schauder basis for T_J^*.

Our next proposition shows that the norms on T^* and T_J^* behave in the same way with respect to normalized block basic sequences.

Proposition 9. *Let $(u_n)_{n=1}^\infty$ denote a normalized block basic sequence in T_J^*. For any sequence of scalars $(a_j)_{j=1}^\infty$ and for any positive integer n, we have*

$$\left\| \sum_{j=1}^{n} a_j u_j \right\|_{T_J^*} \leq \left\| \sum_{j=1}^{n} (|a_j| + |a_{j+1}|) t_j \right\|_{T^*}.$$

Proof. We first fix n and let

$$\sum_{l=1}^{\infty} b_l e_l = \sum_{j=1}^{n} a_j u_j \quad \text{where} \quad u_j = \sum_{i=k_j+1}^{k_{j+1}} a_{ij} e_i$$

and $(k_j)_{j=1}^\infty$ is a strictly increasing sequence of integers with $k_1 = 0$. Let $1 \leq p_1 < p_2 < \cdots < p_{2k} \leq k_{s+1}$ denote an increasing sequence of positive integers. By (2),

$$\begin{aligned} \left\| \sum_{j=1}^{k} (b_{p_{2j-1}} - b_{p_{2j}}) t_j \right\|_{T^*} \leq \Bigg\| \Bigg\| \sum_{\substack{j \\ p_{2j-1} \leq k_2}} (b_{p_{2j-1}} - b_{p_{2j}}) t_j \Bigg\|_{T^*} t_1 \\ + \sum_{l=2}^{s} \Bigg\| \sum_{\substack{j \\ k_l < p_{2j-1} \leq k_{l+1}}} (b_{p_{2j-1}} - b_{p_{2j}}) t_j \Bigg\|_{T^*} t_l \Bigg\|_{T^*}. \end{aligned}$$

Let $a_0 = a_{s+1} = 0$. Then

$$\begin{aligned}\Big\| \sum_{j=1}^{k} (b_{p_{2j-1}} - b_{p_{2j}}) t_j \Big\|_{T^*} &\le \Big\| \sum_{l=1}^{s} (\|a_l u_l\|_{T_J^*} + |a_{l+1}|) t_l \Big\|_{T^*} \\ &\le \Big\| \sum_{l=1}^{s} (|a_l| + |a_{l+1}|) t_l \Big\|_{T^*}.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{\substack{p_1 < \cdots < p_{2k} \\ k}} \Big\| \sum_{j=1}^{k} (b_{p_{2j-1}} - b_{p_{2j}}) t_j \Big\|_{T^*} &= \Big\| \sum_{j=1}^{n} a_j u_j \Big\|_{T_J^*} \\ &\le \Big\| \sum_{j=1}^{n} (|a_j| + |a_{j+1}|) t_j \Big\|_{T^*}. \qquad \square\end{aligned}$$

Corollary 10. *If $(u_j)_{j=1}^{\infty}$ is a normalized block basic sequence in T_J^* then*

$$\Big\| \sum_{j=n}^{2n} a_j u_j \Big\|_{T_j^*} \le 4 \sup_{n \le j \le 2n} |a_j|.$$

Proof. It suffices to apply (1) and Proposition 9. $\square$

Corollary 11. *No normalized block basic sequence in T_j^* satisfies a lower q estimate for any $q < \infty$.*

Proof. Otherwise, by Corollary 10, there would exist $q < \infty$ and $c > 0$ such that

$$\Big(\sum_{j=n}^{2n} |a_j|^q \Big)^{1/q} \le c \sup_{n \le j \le 2n} |a_j|$$

for all sequences of scalars $(a_j)_{j=1}^{\infty}$.

If $n = 2^j$, $m = 2^{j+1}$, and $a_j = 1$ for all j, this would imply

$$2^{j/q} \le c$$

for all j, and this is impossible. $\square$

Corollary 12. *The sequence $(e_j)_{j=1}^{\infty}$ is a shrinking basis for T_J^*.*

Proof. Otherwise there would exist $\phi \in (T_J^*)'$, $(u_j)_{j=1}^{\infty}$ a normalized block basic sequence in T_J^* and $\delta > 0$ such that $\phi(u_j) \geq \delta$ for all j.

Let $\alpha_j = 1/2^n$ for $2^n < j < 2^{n+1}$, $n = 1, \dots$. We have $\sum_{j=1}^{\infty} \alpha_j = \infty$ and $\alpha_j > 0$ for all j. On the other hand, by Corollary 10,

$$\left\| \sum_{j=2^n+1}^{2^{n+1}} \alpha_j u_j \right\|_{T_J^*} \leq 4 \sup_{2^n < j \leq 2^{n+1}} |\alpha_j| \leq \frac{4}{2^n}$$

and

$$\begin{aligned} \sum_{j=1}^{\infty} \left\| \sum_{j=2^n+1}^{2^{n+1}} \alpha_j u_j \right\|_{T_J^*} &\leq 4 \sum_{n=1}^{\infty} \sup_{2^n < j \leq 2^{n+1}} |\alpha_j| \\ &\leq 4 \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty. \end{aligned}$$

Hence $\sum_{j=1}^{\infty} \alpha_j u_j \in T_J^*$.

However,

$$\lim_{n\to\infty} \phi\left(\sum_{j=1}^{n} \alpha_j u_j \right) \geq \delta \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j = \infty.$$

This is impossible and shows that the basis is shrinking. $\square$

By Corollary 12 and [**30**, Proposition 1.b.1 and 1.b.2] the biorthogonal functions $(e_j^*)_{j=1}^{\infty}$ form a Schauder basis for $(T_J^*)'$ and, moreover, $(T_J^*)''$ can be identified with the space of all sequences $(a_j)_j$ such that

$$\sup_n \left\| \sum_{j=1}^{n} a_j e_j \right\|_{T_J^*} < \infty.$$

The correspondence is given by

$$x^{**} \in (T_J^*)'' \longleftrightarrow (x^{**}(e_j^*))_{j=1}^{\infty}$$

and we have

$$\|x^{**}\| = \sup_n \left\| \sum_{j=1}^n x^{**}(e_j^*)e_j \right\|_{T_J^*}.$$

Proposition 13. *T_J^* is not reflexive.*

Proof. Let $w_n = \sum_{j=1}^n e_j \in T_J^*$. Let $b_j = 1$ for $j \leq n$ and $b_j = 0$ for $j > n$. Since

$$\sum_{\substack{j \\ p_1<p_2<\cdots<p_{2k}}} (b_{p_{2j-1}} - b_{2j})e_j = \begin{cases} 0 & \text{if } p_1 > n \text{ or } p_{2k} \leq n \\ e_j & \text{if } p_{2j-1} \leq n \leq p_{2j} \text{ for some } j, \\ 0 & \text{if } p_{2j} \leq n < p_{2j+1} \text{ for some } j, \end{cases}$$

it follows that $\|w_n\|_{T_J^*} = 1$ for all n. If T_J^* were reflexive, then the sequence $\{w_n\}_{n=1}^\infty$ would contain a subsequence which was weakly convergent to some $w \in T_J^*$. Since $e_m^*(w_n) = 1$ for all $n \geq m$, it follows that $e_m^*(w) = 1$ for all m. If $w = \sum_{m=1}^\infty \beta_m e_m \in T_J^*$, then $\|\beta_m e_m\|_{T_J^*} \to 0$ and hence $|\beta_m| \to 0$ as $m \to \infty$. Since $\beta_m = 1$ for all m, this is impossible and completes the proof. $\square$

We now describe $(T_J^*)''$ and in so doing note the analogy of T_J^* with the classical James space J [**26**]. If $(a_j)_{j=1}^\infty$ is a sequence of scalars and $\sup_n \| \sum_{j=1}^n a_j e_j\|_{T_J^*} = M < \infty$, then we claim that $\lim_{j\to\infty} a_j$ exists. Otherwise, there exists $\delta > 0$ and a strictly increasing sequence of positive integers $(p_j)_j$ such that $|a_{p_{2j-1}} - a_{p_{2j}}| \geq \delta > 0$ for all j. Hence,

$$\sup_k \left\| \sum_{j=1}^k (a_{p_{2j-1}} - a_{p_{2j}})t_j \right\|_{T^*} \leq M.$$

Since the basis in T^* is 1-unconditional, we have

$$\sup_k \left\| \sum_{j=1}^k \delta t_j \right\| \leq \sup_k \left\| \sum_{j=1}^k |a_{p_{2j-1}} - a_{p_{2j}}| t_j \right\| \leq M.$$

Since T^* is reflexive, this implies that the sequence $\{\sum_{j=1}^k t_j\}_{k=1}^\infty$ has a weak Cauchy subsequence and the proof of the previous proposition can now be adapted to show that this is impossible. Hence, we have established our claim.

In proving Proposition 13, we showed that $\sup_n \|\sum_{j=1}^n e_j\|_{T_J^*} < \infty$. Let $x_0^{**} \in (T_J^*)''$ be given by $x_0^{**}(e_n^*) = 1$ for all n. If x^{**} is an arbitrary vector in $(T_J^*)''$, then $\lim_{j\to\infty} x^{**}(e_j^*) = \alpha(x^{**})$ exists. Let $y^{**} = x^{**} - \alpha(x^{**})x_0^{**}$. It follows that $y^{**} \in T_J^*$ and $(T_J^*)'' \cong T_J^* \oplus \mathbf{C}x_0^{**}$. Hence $(T_J^*)''$ is separable and, by [**30**, Theorem 1.c.12], the basis $(e_j)_{j=1}^\infty$ in T_J^* is not unconditional. The above also shows that T_J^* and all its duals are quasi-reflexive and hence all have the Radon-Nikodym property [**18**, p. 219]. Moreover, T_J^* and all its higher duals have a basis and hence the approximation property.

The following proposition may also be proved by using the method used for Tsirelson's space in [**3**] and Corollary 10.

Proposition 14. *Continuous polynomials on T_J^* are weakly continuous on bounded sets.*

Proof. Since T_J^* and $(T_J^*)''$ are both separable it follows by [**30**, Theorem 2.e.7] that $l_1 \not\hookrightarrow T_J^*$. Hence by Proposition 3, it suffices to show that each continuous polynomial on T_J^* is weakly sequentially continuous at the origin. By Proposition 4 it suffices to show that no spreading model built on a normalized weakly null sequence in T_J^* has a lower q estimate for any $q < \infty$.

Suppose $(u_j)_j$ is a normalized weakly null sequence in T_J^* which has a spreading model having a lower q estimate for some $q < \infty$. This means that there exists a Banach space X with an unconditional basis, $(f_j)_j$, the *spreading model*, and a subsequence of $(u_j)_j$, $(u_{n_j})_{j=1}^\infty$, such that for all $\varepsilon > 0$ and all k there exists $N = N(\varepsilon, k)$ such that for all $N < n_1 < n_2 < \cdots < n_k$ and for all scalars with $\sup_i |a_i| \le 1$ we have

$$\left| \left\| \sum_{j=1}^k a_j f_j \right\|_X - \left\| \sum_{j=1}^k a_j u_{n_j} \right\|_{T_J^*} \right| < \varepsilon,$$

see, for instance, [**10, 20, 21, 28**], and that there exists $c > 0$ such that

$$\left\| \sum_{j=1}^k a_j f_j \right\|_X \ge c \left(\sum_{j=1}^k |a_j|^q \right)^{1/q}$$

for any sequence of scalars $(a_j)_j$ and for all k.

By choosing, if necessary, a further subsequence of $(n_j)_j$ we may suppose that

$$u_{n_j} = x_{n_j} + y_{n_j}$$

where $(x_{n_j})_j$ is a block basic sequence in T_J^* and $\sum_{j=1}^{\infty} \|y_{n_j}\|_{T_J^*} \leq 1/4$.

Since $\|u_{n_j}\| = 1$ for all j this implies that $\|x_{n_j}\| \geq 3/4$ for all j. By Corollary 10 we have for k sufficiently large and $|a_j| \leq 1$.

$$\begin{aligned} 4 \sup_{k+1 \leq j \leq 2k} |a_j| &\geq \Big\| \sum_{j=k+1}^{2k} a_j x_{n_j} \Big\|_{T_J^*} \\ &\geq \Big\| \sum_{j=k+1}^{2k} a_j u_{n_j} \Big\|_{T_J^*} - \Big\| \sum_{j=k+1}^{2k} a_j y_{n_j} \Big\|_{T_J^*} \\ &\geq \Big\| \sum_{j=k+1}^{2k} a_j f_j \Big\|_X - \sup_{1 \leq j \leq k} |a_j| \sum_{j=k+1}^{2k} \|y_{n_j}\|_{T_J^*} - \varepsilon \\ &\geq c \bigg(\sum_{j=k+1}^{2k} |a_j|^q \bigg)^{1/q} - \frac{1}{4} \sup_{k+1 \leq j \leq 2k} |a_j| - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary this implies that for any k and any sequence $(a_j)_j$ with $|a_j| \leq 1$, we have

$$c \bigg(\sum_{j=1}^{k} |a_j|^q \bigg)^{1/q} \leq 5 \sup_{1 \leq j \leq k} |a_j|.$$

Letting $a_j = 1$ for all j this implies $ck^{1/q} \leq 5$ for all positive integers k. This contradiction shows that no normalized weakly null sequence in T_J^* has a spreading model with a lower q estimate for some $q < \infty$ and completes the proof. $\square$

Proposition 15. *T_J^* is a Q-reflexive Banach space and $\mathcal{P}({}^n T_J^*)$ has* RNP *for all n.*

Proof. Since T_J^* is a quasi-reflexive space with basis, it follows that $(T_J^*)'$ and $(T_J^*)''$ have the approximation property and RNP. The remaining hypothesis required for the application of Theorem 5 is satisfied by Proposition 14. $\square$

The hypotheses of Theorem 5 are satisfied by any subspace of $(T_J^*)^{(2n)}$, the $2n$th dual of T_J^*, which has the approximation property.

The mapping

$$U : (T_J^*)'' \longrightarrow T_J^*$$
$$U(x^{**}) = (-\lambda, x^{**}(e_1^*) - \lambda, x^{**}(e_2^*) - \lambda, \dots)$$

where $\lambda = \alpha(x^{**})$ is a linear isomorphism from $(T_J^*)''$ onto T_J^*. Consequently, since T_J^* is Q-reflexive, $\mathcal{P}(^nT_J^*)$ and $\mathcal{P}(^nT_J^*)''$ are isomorphic for all n, (but not under the canonical mapping since T_J^* is not reflexive). This property, which is shared by many, but possibly not all, quasi-reflexive spaces, suggested the terminology Q-reflexive spaces.

If E is a Banach space we let $\mathcal{H}_b(E)$ denote the space of $\mathbf{C}$-valued holomorphic functions on E which are bounded on bounded sets and endowed with the topology τ_b of uniform convergence on bounded sets.

Proposition 16. $(\mathcal{H}_b(T_J^*), \tau_b)'' \cong (\mathcal{H}_b((T_J^*)''), \tau_b)$.

Proof. It suffices to apply Theorem 12 of [**31**] and Proposition 15. $\square$

Further applications to spaces of holomorphic functions are also possible, and we will discuss these in a further paper. In addition, the recent book by R. Deville, G. Godefroy and V. Zizler [**16**, Chapter 4] contains applications of polynomial techniques to the study of structural properties of Banach spaces. Finally, we have recently obtained preprints by J.A. Jaramillo, A. Prieto, I. Zalduendo [**27**] and by M. Valdivia [**33**], in which the bidual of $\mathcal{P}(^nE)$ is discussed.

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ENDNOTES

1. M. Gonzalez has recently obtained a characterization of Q-reflexive spaces which justifies our choice.

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