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AN EQUIVALENCE FOR CATEGORIES OF MODULES OVER A COMPLETE DISCRETE VALUATION DOMAIN

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ABSTRACT. An equivalence is presented between two categories closely related to the category of valuated vector spaces. This equivalence is then recast in terms of short exact sequences of modules. A new class of mixed modules is introduced, whose members we call $\mathcal{A}$-*Warfield*, and, in particular, it is shown that an isomorphism between the endomorphism rings of two $\mathcal{A}$-Warfield modules is induced by an isomorphism of their underlying modules.

1. Introduction. Valuated vector spaces have proven to be of great importance in the study of abelian groups, see [**2**]. For example, if G is any abelian p-group, its socle $G[p]$ inherits a valuation from the height function on G. The main purpose of this note is to explore two closely related constructions using other well-behaved classes of algebraic objects. Let R be a fixed complete discrete valuation domain with quotient ring Q and $p \in R$ prime.

Define a category $\mathcal{A}$ as follows. An object $A \in \mathcal{A}$ is a reduced torsion-free algebraically compact R-module, together with a smoothly descending chain of summands $A[\alpha]$ indexed by the ordinals (where we consider the symbol ∞ to be an ordinal greater than all other conventional ordinals), starting with $A[0] = A$. It follows that, for every α, $A[\alpha]$ is also reduced, torsion-free and algebraically compact. Notice that, if $A[\alpha]$ is defined only for isolated ordinals and for each limit ordinal λ we define $A[\lambda] = \cap_{\alpha<\lambda} A[\alpha]$, then $A[\lambda]$ is pure (since A is torsion-free) and p-adically closed, hence it is also a summand of A.

Define a second category $\mathcal{D}$ as follows. An object $D \in \mathcal{D}$ is a divisible torsion R-module, together with a smoothly descending chain of submodules $D[\alpha]$ indexed by the ordinals, starting with $D[0] = D$, such that $D[\alpha]$ is a summand of D whenever α is isolated. Once again, to specify the chain of submodules, it is only necessary to describe

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them for isolated ordinals. As we shall see, $D[\lambda]$ is not necessarily a summand of D when λ is a limit ordinal.

The morphisms in $\mathcal{A}$ and $\mathcal{D}$ are the R-module homomorphisms which preserve these distinguished chains of submodules. There is a well-known equivalence between the category of reduced torsion-free algebraically compact R-modules and the category of divisible torsion R-modules, see, for example, [**8**, Theorem 6], and we extend this to an equivalence between $\mathcal{A}$ and $\mathcal{D}$ (Proposition 1). This equivalence can be approached from another viewpoint. For a torsion module H, we set up a one-to-one correspondence between the sequences $0 \to A \to T \to H \to 0$, where $A \in \mathcal{A}$ is a nice submodule of T with the submodules $A[\alpha]$ determined by the height function on T, and the sequences $0 \to M \to H \to D \to 0$, where M is an isotype submodule of H with the submodules $D[\alpha] \subseteq D \in \mathcal{D}$ determined by the height function on H (Theorem 1). In essence, this means that we can represent objects in $\mathcal{A}$ as valuated modules and their corresponding objects in $\mathcal{D}$ as c-valuated modules (see [**4**] for a discussion of the latter term).

When $0 \to A \to T \to H \to 0$ is one of these extensions and H is totally projective, we say T is $\mathcal{A}$-Warfield. We classify a couple of collections of $\mathcal{A}$-Warfield modules by cardinal invariants (Theorem 2). In addition, we show that any isomorphism between the endomorphism rings of $\mathcal{A}$-Warfield modules is induced by an isomorphism of their underlying modules (Theorem 3). It follows that any automorphism of the endomorphism ring of an $\mathcal{A}$-Warfield module is inner.

2. The equivalence. We begin with a quick review. Any terms not explicitly defined can be found in [**1**] or [**3**].

If $a \in A \in \mathcal{A}$, let $\nu_{\mathcal{A}}(a) = \max\{\alpha : a \in A[\alpha]\}$; similar notation will be employed in $\mathcal{D}$. The *rank* of $A \in \mathcal{A}$ is the dimension of A/pA as a vector space over R/pR; similarly, the rank of $D \in \mathcal{D}$ is the dimension of $D[p]$.

If G is a module and λ is a limit ordinal, then the λ-*topology* is the linear topology utilizing $\{p^\beta G : \beta < \lambda\}$ as a neighborhood base of 0. We will use without comment some standard facts regarding this topology that are available in [**10**] and [**11**].

Proposition 1. *There is a categorical equivalence between $\mathcal{A}$ and $\mathcal{D}$.*

Proof. If $A \in \mathcal{A}$, let $FA = Q/R \otimes_R A$, where $FA[\alpha] = Q/R \otimes_R A[\alpha]$ whenever α is isolated. If $D \in \mathcal{D}$, let $GD = \mathrm{Hom}_R(Q/R, D)$, where $GD[\alpha] = \mathrm{Hom}_R(Q/R, D[\alpha])$. As in Theorem 4 of [**8**], the evaluation map $FGD = Q/R \otimes_R \mathrm{Hom}_R(Q/R, D) \to D$ is an isomorphism for all $D \in \mathcal{D}$ (since D is divisible, so that $h(D) = D$). In addition, as in Theorem 5 of [**8**], the map $A \to \mathrm{Hom}_R(Q/R, Q/R \otimes_R A)$ given by $a \mapsto \phi_a$, where $\phi_a(x) = x \otimes a$, is an isomorphism for all $A \in \mathcal{A}$ (since A is cotorsion, so $\mathrm{Ext}\,(Q, A) = 0$). The result follows. □

If $X \subseteq A \in \mathcal{A}$, let $\langle\langle X\rangle\rangle$ denote the p-adic closure of the purification of the submodule generated by X. Note that $\langle\langle X\rangle\rangle$ will be a summand of A, so that $A/\langle\langle X\rangle\rangle$ will also be reduced, torsion-free and algebraically compact.

Proposition 2. *Both $\mathcal{A}$ and $\mathcal{D}$ have kernels and cokernels.*

Proof. Suppose $f : A \to B$ is a morphism in $\mathcal{A}$ and K is the usual module-theoretic kernel, so that A/K is isomorphic to a submodule of B. Since B is reduced, so is A/K, so by 54(B) of [**1**], K is cotorsion, and hence algebraically compact. In addition, since B is torsion-free, so is A/K so that K is a summand of A. Similarly, for each ordinal α, $K[\alpha] \stackrel{\mathrm{def}}{=} K \cap A[\alpha]$ is a summand of $A[\alpha]$, hence a summand of A, and finally a summand of K. It can be checked that this makes K into a kernel of f in $\mathcal{A}$. Turning to cokernels, let $C = B/\langle\langle f(A)\rangle\rangle$, and for each isolated ordinal α let $C[\alpha] = \langle\langle B[\alpha] + f(a)\rangle\rangle/\langle\langle f(A)\rangle\rangle$. If $X \in \mathcal{A}$, and $g : C \to X$ is a morphism in $\mathcal{A}$, then $(g \circ f)(A) = 0$ if and only if $g(\langle\langle f(A)\rangle\rangle) = 0$ if and only if g factors through C. It follows that C forms a cokernel of f in $\mathcal{A}$.

Even though our equivalence implies that it is only necessary to verify the result for $\mathcal{A}$, as an example we will also illustrate the constructions in $\mathcal{D}$. Suppose $g : D \to E$ is a morphism in $\mathcal{D}$. If J is the divisible part of the kernel of g and for every isolated ordinal α, $J[\alpha]$ is the divisible part of $J \cap D[\alpha]$, then J provides us with a kernel for g in $\mathcal{D}$. If $G = E/g(D)$ and for every isolated ordinal α, $G[\alpha] = (E[\alpha] + g(D))/g(D)$, then G is a cokernel of g in $\mathcal{D}$. □

Proposition 3. *Both $\mathcal{A}$ and $\mathcal{D}$ have arbitrary products and coproducts.*

Proof. Once again, though it is technically sufficient to consider only one category, we include a discussion of both. If $\{A_i\}_{i\in I}$ is a collection of objects in $\mathcal{A}$, then their usual direct product $\prod_I A_i$, where for each α, $(\prod_I A_i)[\alpha] = \prod_I (A_i[\alpha])$ is also a product in $\mathcal{A}$. To construct their coproduct, consider $\langle\langle\oplus\rangle\rangle_I A_i$, by which we mean the p-adic completion of the usual sum $\oplus_I A_i$, where for each ordinal α, $(\langle\langle\oplus\rangle\rangle_I A_i)[\alpha] = \langle\langle\oplus\rangle\rangle_I (A_i[\alpha])$. It follows easily that $\langle\langle\oplus\rangle\rangle_I A_i$ is in $\mathcal{A}$. Given any collection of morphisms $A_i \to B \in \mathcal{A}$, these first extend uniquely to a module homomorphism $\oplus_I A_i \to B$, and this extends uniquely to a morphism $\langle\langle\oplus\rangle\rangle_I A_i \to B$.

If $\{D_i\}_{i\in I}$ is a collection of objects in $\mathcal{D}$, then $\prod_I^t D_i$, i.e., the torsion submodule of their direct product, where for each α, $(\prod_I^t D_i)[\alpha] = \prod_I^t (D_i[\alpha])$, is their product, while the usual direct sum $\oplus_I D_i$, where for each α, $(\oplus_I D_i)[\alpha] = \oplus_I (D_i[\alpha])$, will provide a coproduct. $\square$

As mentioned above, these categories are closely related to the theory of valuated vector spaces. In particular, many constructions from that subject can be applied to our two categories. An object in $\mathcal{A}$ will be called $\mathcal{A}$-*cyclic* if it has rank 1, and $\mathcal{A}$-free if it is the coproduct (in $\mathcal{A}$) of a collection of $\mathcal{A}$-cyclics. On the other hand, we will say A is $\mathcal{A}$-*injective* if it is isomorphic in $\mathcal{A}$ to $\prod_\alpha A_\alpha$, where A_α is homogeneous in the sense that $v_{\mathcal{A}}(x) = \alpha$, for all nonzero $x \in A_\alpha$ (it can be seen that each A_α is $\mathcal{A}$-free). As in the case of valuated vector spaces, the $\mathcal{A}$-free modules are precisely those which are projective with respect to the class of sequences $E : 0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ such that, for all α, $E[\alpha] : 0 \to X[\alpha] \to Y[\alpha] \to Z[\alpha] \to 0$ is exact, and the $\mathcal{A}$-injectives are precisely those which are injective with respect to the class of sequences E such that for all α, $E[\alpha]$ is left exact. It follows that both of these classes are closed with respect to the formation of summands (in $\mathcal{A}$). Of course, by our equivalence, we obtain corresponding concepts in $\mathcal{D}$.

Proposition 4. *The assignment $A \mapsto A/pA$ is a surjective functor from $\mathcal{A}$ to the category of valuated vector spaces, over R/pR. Similarly,* *the assignment $D \mapsto D[p]$ is a surjective functor from $\mathcal{D}$ to the category of valuated vector spaces.*

Proof. It is obvious that these assignments are functors. To show they are surjective, suppose V is a valuated vector space. There is a free valuated vector space $W = \oplus_I \langle v_i \rangle$, and a homomorphism $f : W \to V$ with $f(W[\alpha]) = V[\alpha]$ for every ordinal α. Let $B \subseteq W$ be a basis for the kernel of f. Now, for each $i \in I$, let $\langle a_i \rangle$ be $\mathcal{A}$-cyclic in $\mathcal{A}$ such that $\nu_{\mathcal{A}}(a_i) = \nu(v_i)$. Each element of B is a linear combination of various v's, and we let B' be the set of corresponding linear combinations of the a's. If $P = \langle\langle \oplus \rangle\rangle_I \langle a_i \rangle$, $Q = \langle\langle B' \rangle\rangle \subseteq P$, $A = P/Q$ and for every isolated α, $A[\alpha] = \langle\langle P[\alpha] + Q \rangle\rangle / Q$, then there are isomorphisms $A/pA \cong (P/Q)/((pP+Q)/Q) \cong P/(pP+Q) \cong W/\langle B \rangle \cong V$ which are readily seen to preserve values. In addition, if $D = Q/R \otimes_R A$, then $D[p]$ is also isomorphic to V as a valuated vector space. □

Although from a superficial viewpoint these categories may seem straightforward (since both the reduced torsion-free algebraically compact modules and the divisible torsion modules are classified by their ranks), the following shows that their structure is at least as complicated as that of the category of torsion R-modules.

Proposition 5. *The assignment $D \to D[\omega]$ is a surjective functor from $\mathcal{D}$ to the category of torsion modules.*

Proof. Suppose G is an arbitrary torsion module and for each $i < \omega$, X_i is a copy of a divisible hull for G. Let $D = (\oplus_{i<\omega} X_i)/L$, where $L = \{(x_0, x_1, x_2, \dots) : x_i \in G, \text{ and } x_0 + x_1 + x_2 + \cdots = 0\}$. For each $n < \omega$, let $D[n] = (\oplus_{n \le i < \omega} X_i + L)/L$. It follows that $D[\omega] = \cap_{n<\omega} D[n] = (\oplus_{i<\omega} G)/L \cong G$. □

We now show that not every result which is valid for valuated vector spaces translates directly into a result for our two new categories.

Proposition 6. *There is an object $A \in \mathcal{A}$ of countable rank which is not $\mathcal{A}$-free.*

Proof. If, in the notation of the last proof, $G = R/pR$, then the object $D \in \mathcal{D}$ has countable rank. It follows that $A = \mathrm{Hom}_R(Q/R, D)$ also has countable rank. However, if A were $\mathcal{A}$-free, then it would follow that $D \cong Q/R \otimes_R A$ is $\mathcal{D}$-free, which contradicts the fact that $D[\omega] \cong R/pR$. □

The next result presents the equivalence of $\mathcal{A}$ and $\mathcal{D}$ from another perspective; it points out a correspondence between representations of objects in $\mathcal{A}$ using valuations and representations of elements of $\mathcal{D}$ using c-valuations.

Recall that a submodule $X \subseteq G$ is *full-rank* if G/X is torsion.

Theorem 1. *Suppose that H is a torsion module. There is a one-to-one correspondence between exact sequences of the form*

$$0 \longrightarrow A \longrightarrow T \longrightarrow H \longrightarrow 0, \tag{1}$$

where $A \in \mathcal{A}$ is a nice submodule of T, $A[1] = A$, and for every ordinal $\beta = \omega\alpha + n$, $A \cap p^\beta T \overset{\text{def}}{=} A(\beta) = p^n A[\alpha] + A[\alpha+1]$, and exact sequences of the form

$$0 \longrightarrow M \longrightarrow H \xrightarrow{\pi} D \longrightarrow 0, \tag{2}$$

where M is an isotype submodule of H, $D \in \mathcal{D}$ satisfies $D[1] = D$ and, for every $\beta = \omega\alpha + n$, $\pi(p^\beta H) = D[\alpha + 1]$.

Proof. If (1) is given, apply $- \otimes_R A$ to $0 \to R \to Q \to Q/R \to 0$. Letting $Q/R \otimes_R A = D$; then, since A is torsion-free, we obtain an exact sequence $0 \to A \to QA \xrightarrow{\gamma} D \to 0$. Let M be the torsion submodule of T so that T/M is torsion-free. Now $A(\omega) = A[1] = A$, so that every element of A has infinite height in T. Since A is a full-rank submodule of T, it follows that T/M is divisible. Therefore, the identity $A \to A$ extends to a homomorphism $T/M \to QA$. Letting $H = T/A$, we can, therefore, construct a commutative diagram, as follows:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A & = & A & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & T & \longrightarrow & QA & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & M & \longrightarrow & H & \xrightarrow{\pi} & D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3}
$$

For an ordinal α, there is a decomposition $A[\alpha] = B \oplus A[\alpha+1]$ so that, for every $j < \omega$, we have $A(\omega\alpha+j) = p^j A[\alpha] + A[\alpha+1] = p^j B \oplus A[\alpha+1]$. In other words, B is a pure submodule of $p^{\omega\alpha}T$, and hence $p^{\omega\alpha}T = B \oplus T'$, where $A[\alpha+1] = p^{\omega}T'$ is full-rank. It follows that if $\beta = \omega\alpha + n$, then the image of $p^{\beta}T$ in QA is $p^n B \oplus QA[\alpha+1]$, which therefore implies that $\pi(p^{\beta}H) = \pi(\{p^{\beta}T + A\}/A) = \gamma(p^n B \oplus QA[\alpha+1]) = D[\alpha+1]$. We now show that M is isotype in H. If this failed, by niceness there would be an $x \in M$ such that $ht_T(x) = ht_M(x) < ht_H(x + A)$. By the niceness of A in T, this implies that for some $a \in A$, $ht_T(x) = ht_T(a) < ht_T(x + a)$. If $p^n x = 0$, we would then have $ht_T(p^n a) = ht_T(a) + n < ht_T(x + a) + n \le ht_T(p^n(x + a)) = ht_T(p^n a)$, which cannot happen.

Conversely, if (2) is given, apply $\mathrm{Hom}_R(-, D)$ to $0 \to R \to Q \to Q/R \to 0$ to obtain a sequence $0 \to \mathrm{Hom}_R(Q/R, D) \to \mathrm{Hom}_R(Q, D) \to D \to 0$. Using our categorical equivalence, we let $A = \mathrm{Hom}_R(Q/R, D) \in \mathcal{A}$. Since D is torsion divisible, it follows that QA can be identified with $\mathrm{Hom}_R(Q, D)$ and there is a short exact sequence $0 \to A \to QA \to D \to 0$. Construct (3) using a pull-back of π along γ so that $T = \{(x, y) : x \in H, y \in QA, \pi(x) = \gamma(y)\}$. Our aim is to derive an explicit formula for the height function on T. Define a function σ on QA as follows: $\sigma(0) = \infty$,

$$\sigma((QA[\alpha] - pA[\alpha]) + QA[\alpha+1]) = \omega\alpha \tag{4}$$

$$\sigma((p^n A[\alpha] - p^{n+1}A[\alpha]) + QA[\alpha+1]) = \omega\alpha + n. \tag{5}$$

Claim. $h_T(x,y) = \mu(x,y) \overset{\text{def}}{=} \min\{ht_H(x), \sigma(y)\}$.

Proof. We first show that $\mu(px,py) > \mu(x,y)$. To begin, note that $ht_H(px) > ht_H(x)$. If $\sigma(py) > \sigma(y)$, then the inequality is clear. If $\sigma(py) = \sigma(y)$, it follows that $y \in (QA[\alpha] - A[\alpha]) + QA[\alpha+1]$ for some α. Since $\pi(p^{\omega\alpha}H) = D[\alpha+1] = \gamma(QA[\alpha+1])$, it follows that $ht_H(x) < \omega\alpha = \sigma(y)$, and hence $\mu(x,y) = ht_H(x) < ht_H(px) \le \mu(px,py)$.

We next show that if $\delta < \mu(x,y)$, there exists $(x',y') \in T$ such that $(px',py') = (x,y)$ and $\delta \le \mu(x',y')$. We must let $y' = p^{-1}y$, and clearly $\delta \le \sigma(y')$. Since $\delta < ht_H(x)$, there is an x_0 such that $px_0 = x$ and $\delta \le ht_H(x_0)$. Now assume $\delta = \omega\nu + m$. Since $\sigma(y) > \delta$, it follows that $y \in pA[\nu] + QA[\nu+1]$, and hence that $y' \in A[\nu] + QA[\nu+1]$. This means that $\gamma(y') \in D[\nu+1] = \pi(p^\delta H)$, so that $\gamma(y') = \pi(x_1)$, for some $x_1 \in p^\delta H$. Next, $\pi(p(x_0 - x_1)) = \pi(px_0) - p\pi(x_1) = \pi(x) - p\gamma(y') = 0$, so that $p(x_0 - x_1) \in M$. Since M is an isotype in H, there is an $x_2 \in p^\delta M$ such that $px_2 = p(x_0 - x_1)$. Let $x' = x_1 + x_2$, so that $ht_H(x') \ge \delta$. It follows that $px' = px_1 + px_2 = px_0 = x$ and $\pi(x') = \pi(x_1) + \pi(x_2) = \gamma(y') + 0 = \gamma(y')$, and these facts indicate that $(x',y') \in T$ has the desired properties.

It follows from the last two paragraphs that $\mu(x,y) = \min\{\mu(x',y') : p(x',y') = (x,y)\}$, but as this agrees with the inductive definition of ht_T, the claim follows.

Identifying A with the ordered pairs for the form $(0,y)$, then $A(\beta) = p^\beta T \cap A$ follows from (4). In addition, if $x \in H$ has height β, then there is an $(x,y) \in T$ with $\beta \le \sigma(y)$. It follows that $ht_T(x,y) = \beta$, so that A is a nice submodule of T. □

The situation in Theorem 1(1) will be summarized by saying that T is a nice extension of $A \in \mathcal{A}$. We also apply this terminology when $A[1] \ne A$ and, in this case, if $A = B \oplus A[1]$, then B is pure in T so that $T \cong B \oplus T'$, where T' is a nice extension of $A[1] \in \mathcal{A}$. We point out another consequence of the last result: every $a \in A$ will have no gaps in its height sequence, i.e., for each k, $ht_T(p^k a) = ht_T(a) + k$. The terminology in the next result is due to Harrison (for a discussion, see [**6**] and [**7**]).

Corollary 1. *The sequence* $0 \to M \to H \to D \to 0$ *satisfies Theorem* 1(2) *if and only if* M *is an immediate submodule of* H*, in the sense that for every* α*, the natural map* $p^\alpha M/p^{\alpha+1}M \to p^\alpha H/p^{\alpha+1}H$ *is an isomorphism.*

Proof. By [**6**, 1.1], M is an immediate submodule of H if and only if it is isotype and for every limit ordinal $\omega\alpha$, $p^{\omega\alpha}H/p^{\omega\alpha}M$ is divisible, but since this is isomorphic to $D[\alpha+1]$, the result follows. □

The modules G_i, $i = 1, 2$, are WARF-*isomorphic* if and only if there is a height preserving isomorphism $X_1 \to X_2$, where X_i is a full-rank submodule of G_i. In other words, X_1 and X_2 are isomorphic as *valuated groups*, see [**12**]. If $A \in \mathcal{A}$ and α is an ordinal, then $A[\alpha]/A[\alpha+1]$ will be homogeneous, and we let $g(A, \alpha)$ denote its rank.

Lemma 1. *Suppose for* $i = 1, 2$*,* T_i *is a nice extension of* A_i*. If* T_1 *and* T_2 *are* WARF*-isomorphic, then* $g(A_1, \alpha) = g(A_2, \alpha)$ *for every ordinal* α*.*

Proof. We borrow a construction from [**5**, Section 5]. If μ is an ordinal and G is a valuated module, let

$$G(\mu)^* = \{g \in G(\mu) : v(p^k g) > \mu + k \text{ for some } k < \omega\}.$$

Multiplication by p induces injective homomorphisms,

$$\frac{G(\mu)}{G(\mu)^*} \longrightarrow \frac{G(\mu+1)}{G(\mu+1)^*} \longrightarrow \frac{G(\mu+2)}{G(\mu+2)^*} \longrightarrow \cdots,$$

and we denote the direct limit of these by $\mathbf{w}_G(\mu)$. By Lemma 4 of [**5**], if $Z \subseteq G$ is an embedding of valuated modules and Z is full-rank, then $\mathbf{w}_G(\mu) = \mathbf{w}_Z(\mu)$.

Suppose $X_i \subseteq T_i$ are height isomorphic full-rank submodules. So A_i and X_i are valuated modules when given the valuation induced by the height function on T_i. For each $j < \omega$,

$$\begin{aligned}\frac{A_i(\omega\alpha + j)}{A_i(\omega\alpha + j)^*} &= \frac{p^j A_i[\alpha] + A_i[\alpha+1]}{p^{j+1}A_i[\alpha] + A_i[\alpha+1]} \\ &\cong \frac{A_i[\alpha] + A_i[\alpha+1]}{pA_i[\alpha] + A_i[\alpha+1]}.\end{aligned}$$

It follows that $\mathbf{w}_{A_i}(\omega\alpha) \cong (A_i[\alpha]/A_i[\alpha+1])/p(A_i[\alpha]/A_i[\alpha+1])$, so that

$$\begin{aligned}g(A_1,\alpha) &= \operatorname{rank}\{\mathbf{w}_{A_1}(\omega\alpha)\} \\ &= \operatorname{rank}\{\mathbf{w}_{T_1}(\omega\alpha)\} \\ &= \operatorname{rank}\{\mathbf{w}_{X_1}(\omega\alpha)\} \\ &= \operatorname{rank}\{\mathbf{w}_{X_2}(\omega\alpha)\} \\ &= \operatorname{rank}\{\mathbf{w}_{T_2}(\omega\alpha)\} \\ &= \operatorname{rank}\{\mathbf{w}_{A_2}(\omega\alpha)\} \\ &= g(A_2,\alpha). \qquad \square\end{aligned}$$

One direct consequence of this result is that $g(A, \alpha)$ is independent of how T is represented as a nice extension of some $A \in \mathcal{A}$. As such, we denote this by $g_T(\alpha)$, and we let f_T denote the Ulm function of T.

3. $\mathcal{A}$-Warfield modules.

We say T is an $\mathcal{A}$-*Warfield* if it is a nice extension of some $A \in \mathcal{A}$ such that T/A is totally projective. In addition, we say T is $\mathcal{A}$-free or $\mathcal{A}$-injective if A can be chosen to be $\mathcal{A}$-free or $\mathcal{A}$-injective, respectively. Though we will not do so, it is possible to characterize the Ulm functions of $\mathcal{A}$-Warfield modules.

Theorem 2. *Suppose for $i = 1, 2$, T_i is $\mathcal{A}$-Warfield. if each T_i is $\mathcal{A}$-free, or each T_i is $\mathcal{A}$-injective, then $T_1 \cong T_2$ if and only if $f_{T_1} = f_{T_2}$ and $g_{T_1} = g_{T_2}$.*

Proof. Necessity being transparent, suppose T_i is a nice extension of $A_i \in \mathcal{A}$ and these functions agree for $i = 1, 2$. Now A_1 and A_2 are both either the product or coproduct of a collection of homogeneous objects in $\mathcal{A}$, and since $g(A_1, \alpha) = g(A_2, \alpha)$ for each ordinal α, these homogeneous terms are isomorphic in $\mathcal{A}$. It follows that there is a height preserving isomorphism $k : A_1 \to A_2$. For any ordinal β, since the height sequences of elements of A_i have no gaps, it follows that

$$\frac{p^\beta T_i[p]}{(p^{\beta+1} T_i + A_i) \cap p^\beta T_i[p]} = \frac{p^\beta T_i[p]}{p^{\beta+1} T_i[p]},$$

so that the βth relative Ulm invariant of A_i in T_i is given by $f_{T_i}(\beta)$. Since $f_{T_1} = f_{T_2}$, it follows that k can be extended to an isomorphism $T_1 \cong T_2$. □

Corollary 2. *In Theorem* 2*, the condition that* $g_{T_1} = g_{T_2}$ *can be replaced by the requirement that* T_1 *and* T_2 *are* WARF*-isomorphic.*

It is logical to wonder whether we can use some broader classes of modules in the last result. However, we have the following:

Proposition 7. *There are nonisomorphic* $\mathcal{A}$*-Warfield modules* T_i*,* $i = 1, 2$*, which are* WARF*-isomorphic and have identical Ulm functions.*

Proof. We assume that R is the p-adic integers, though this could be avoided if desired. The main property we use is that for an infinite, reduced torsion module, rank agrees with cardinality.

Let S_i, $i = 1, 2$, be copies of the coproduct (in $\mathcal{A}$) of $\langle x_j \rangle$, $j < \omega$, where $\nu_{\mathcal{A}}(x_j) = j$, and let P_i be their corresponding products. We let $A_1 = P_1$ and $A_2 = pP_2 + S_2 \in \mathcal{A}$. By a result of [**12**], there are totally projective modules H_i of length $\omega\omega$ and nice extensions T_i of A_i with $T_i/A_i \cong H_i$. Adding on totally projective direct summands, we may assume $H_1 \cong H_2 \stackrel{\text{def}}{=} H$. There is clearly a height preserving isomorphism between the full-rank submodules pP_i of T_i and $f_{T_1} = f_H = f_{T_2}$. If $T_1 \cong T_2$, we can identify these, calling the result T, and then identify the A_i with two nice submodules of T. Note that A_2/pP_2 is a countable nice submodule of T/pP_2, and the corresponding quotient is isomorphic to $T/A_2 \cong H$. It follows that T/pP_2 is totally projective of length $\omega\omega$. We claim that there is a

$j < \omega$ such that $A_1[j] \subseteq pP_2$. If this failed, then we could construct elements $y_j \in A_1[j]$ such that the heights of $y_j + pP_2$ in T/pP_2 form a strictly increasing sequence. Let N be a countable nice submodule of T/pP_2 containing each y_j. Note that N is a closed submodule of T/pP_2 in the $\omega\omega$-topology. For any sequence $\sigma = \{k_n\}$, where each $k \in \{0, 1\}$, $y_\sigma = k_1 y_1 + \cdots + k_j y_j + \cdots + pP_2$ is a well-defined element of T/pP_2 in the closure of N, so $y_\sigma \in N$. But since $\sigma_1 \neq \sigma_2$ implies $y_{\sigma_1} \neq y_{\sigma_2}$, it follows that N has uncountable cardinality, which is a contradiction.

Fixing a j such that $A_1[j] \subseteq pP_2$, A_1 has elements of height exactly ωj, but pP_2 does not, and this contradiction proves the result. □

Recall that M is an *IT-module* if it can be embedded as an isotype submodule of a totally projective module. This class is quite general, and in a certain sense, the structure of this class is just as complicated as the structure of the class of all torsion modules, see [**4**]. For a module G and an ordinal α, let $U_\alpha(G) = p^\alpha G[p]/p^{\alpha+1}G[p]$ denote the αth Ulm invariant of G.

Proposition 8. *If M is any* IT*-module, then there is an $\mathcal{A}$-Warfield module T whose torsion submodule is isomorphic to M.*

Proof. We need to show that there is a totally projective module H containing M as an immediate submodule. Let G be a totally projective module containing M as an isotype submodule. A standard property of totally projective modules is that they are *fully starred*, i.e., if X is any submodule of G, then X has the same rank as one of its basic submodules (this can be seen, for example, by considering an axiom three family of nice submodules of G). It follows easily that the Ulm function of M is admissible, see [**1**], so that there is a totally projective module H such that $f_M = f_H$. For each α, let $k_\alpha : U_\alpha(M) \to U_\alpha(H)$ be an isomorphism. In the obvious way we can identify $U_\alpha(M)$ with a submodule of $U_\alpha(G)$, and so we can extend k_α to a homomorphism $U_\alpha(G) \to U_\alpha(H)$. By [**15**], there is a homomorphism $k : G \to H$ which induces k_α on $U_\alpha(G)$ for every α. If x is a nonzero element of $M[p]$ of height α, then $[x]$ is a nonzero element of $U_\alpha(M)$. So $[k(x)]$ is a nonzero element of $U_\alpha(H)$ and $k(x)$ has height α in H also. This implies that the kernel of k is zero, and hence that k is injective. We can now use

k to view M as a submodule of H, and we may assume that the maps k_α are induced by the inclusion. Therefore, by [**1**], M is an isotype submodule of H. Finally, since the maps k_α are all isomorphisms, the relative Ulm invariants of M in H are all zero, so that by [**6**], M is an immediate submodule of H.

If λ is a limit ordinal, then G is a λ-elementary KT-module if $p^\lambda G \cong R$ and $G/p^\lambda G$ is totally projective, see [**14**]. A KT-module is, then, a sum of λ-elementary KT-modules, for various limit ordinals λ (sometimes these modules are referred to as the balanced projectives). An S-module is a torsion-submodule of a KT-module.

Proposition 9. *The torsion submodule of an $\mathcal{A}$-free $\mathcal{A}$-Warfield module is an S-module.*

Proof. Suppose T is a nice extension of $A \in \mathcal{A}$ and A is $\mathcal{A}$-free. In addition, suppose that for each limit ordinal λ we fix a λ-elementary KT-module T_λ. We let $W = \oplus_\lambda(\oplus_{g(T)} T_\lambda)$. Notice also that, for each λ, $p^\lambda T_\lambda \cong R$ can be considered an $\mathcal{A}$-cyclic object in $\mathcal{A}$. Let $A' = \langle\langle\oplus\rangle\rangle_\lambda(\langle\langle\oplus\rangle\rangle_{g(T)} p^\lambda T_\lambda)$ and $Z = \oplus_\lambda(\oplus_{g(T)} p^\lambda T_\lambda)$. Finally, let T' be the amalgamation of W and A' along Z (so $T' = W + A'$, $Z = W \cap A'$). It can be checked that $A' \in \mathcal{A}$ is a nice submodule of T' which is $\mathcal{A}$-isomorphic to A. In addition, $T'/A' \cong W/Z$ is totally projective, so that T' is $\mathcal{A}$-Warfield. Choosing totally projective modules X and X' with sufficiently large Ulm functions, $T \oplus X$ and $T' \oplus X'$ will have the same Ulm functions. This means that our isomorphism $A \to A'$ extends to an isomorphism $T \oplus X \to T' \oplus X'$. If M and M' are the torsion submodules of T and T', then $M \oplus X \cong M' \oplus X'$, and since M' is an S-module, $M \oplus X$ is an S-module, and hence M is an S-module. □

On the other hand, the structure of the torsion submodules of $\mathcal{A}$-injective $\mathcal{A}$-Warfield modules is not as clear.

When considering a class of modules, it is usually important to consider how the class behaves with respect to extensions of the form $0 \to p^\beta T \to T \to T/p^\beta T \to 0$. We address this in the following.

Proposition 10. *If T is an $\mathcal{A}$-Warfield module and β is an ordinal, then $p^\beta T$ and $T/p^\beta T$ are $\mathcal{A}$-Warfield modules. The converse fails.*

Proof. Suppose T is a nice extension of $A \in \mathcal{A}$ with T/A totally projective and $\beta = \omega\alpha + n$. If $A' \stackrel{\text{def}}{=} A(\beta) \subseteq p^\beta T$, where for any ordinal $\gamma > 0$, $A'[\gamma] = A[\alpha + \gamma]$, then it can be verified that $p^\beta T$ is a nice extension of A' with a totally-projective quotient, so that it is $\mathcal{A}$-Warfield. Next, if $A = A-1 \oplus A[\alpha]$, then $T/p^\beta T$ can be shown to be a nice extension of $(A_1 + p^\beta T)/p^\beta T$ with a totally-projective quotient, so that it too is $\mathcal{A}$-Warfield.

To observe that the converse is not valid, we begin with a short exact sequence $0 \to B \to C \to L \to 0$ where B is $\sum$-cyclic, i.e., a direct sum of cyclic torsion modules, L is a reduced torsion-free algebraically compact module of infinite rank and C contains no copies of L. To construct such a sequence, let X be a reduced module whose torsion submodule, Y, is $\sum$-cyclic, satisfying $X/Y \cong Q$. If the index set I is chosen large enough, there is an embedding $L \subseteq \oplus_I X/Y$. Let $B = \oplus_I Y$ and $C \subseteq \oplus_I X$ satisfy $C/B \cong L$. Since every torsion-free submodule of X is cyclic, by [**3**], every torsion-free submodule of $\oplus_I Y$ is free, and hence C has no infinite rank algebraically compact submodules.

Let K be a module containing $B = p^\omega K$ such that K/B is $\sum$-cyclic, and let T be the amalgamation of C and K, so $T = K + C$, $B = K \cap C$. Since $B = p^\omega K \subseteq p^\omega T$ and $T/B \cong K/B \oplus C/B \cong K/B \oplus L$, it follows that $B = p^\omega T$ and $T/p^\omega T$ are $\mathcal{A}$-Warfield. If T were $\mathcal{A}$-Warfield, there would be a reduced, torsion-free, algebraically compact submodule $L' \subseteq T$ of infinite rank such that T/L' is totally projective. Since $T/C \cong K/B$ is torsion, there is an integer n such that $p^n L' \subseteq C$, but this latter module was selected specifically to make this fail. □

For a module G, let $G^\diamondsuit$ denote $\operatorname{Ext}(Q/R, G)$ so there is a natural homomorphism $G \to G^\diamondsuit$ whose kernel is the maximal divisible submodule of G, and whose cokernel is torsion-free divisible. The next result, and the structure of its proof, owes much to [**9**].

Theorem 3. *Suppose T_1 and T_2 are reduced $\mathcal{A}$-Warfield modules with endomorphism rings E_1 and E_2. Then any ring isomorphism $\phi : E_1 \to E_2$ is induced by a module isomorphism $f : T_1 \to T_2$.*

Proof. If T_1 had a summand isomorphic to R with a corresponding idempotent ρ, it would follows that $\phi(\rho)$ would be an idempotent onto an isomorphic summand of T_2. Suppose x_1 and x_2 are generators of these summands. If $a \in T_1$, then there is a unique homomorphism $\sigma_a : \langle x_1 \rangle \to T_1$ such that $\sigma_a(x_1) = a$. The assignment $a \mapsto \phi(\sigma_a \circ \rho)$ can be seen to produce an isomorphism $T_1 \to T_2$. We may assume, therefore, that $A_i = A_i[1]$.

Let M_i be the torsion submodule of T_i. The usual proof of the Baer-Kaplansky theorem, see [**1**], shows that the ring isomorphism ϕ implies the existence of a module isomorphism $M_1 \to M_2$. If we identify these modules and denote the result by M, then T_i embeds in $T_i^\diamondsuit \cong M^\diamondsuit$ (since T_i is reduced and T_i/M is divisible).

The endomorphism rings of M and $M^\diamondsuit$ can be identified, the result contains E_1 and E_2 as subrings, and our hypothesis can be restated as the condition that $E_1 = E_2 \stackrel{\text{def}}{=} E$. In other words, if $g : M^\diamondsuit \to M^\diamondsuit$ is a homomorphism, $g \in E$ if and only if $g(T_1) \subseteq T - 1$ if and only if $g(T_2) \subseteq T_2$. We show that $T_2 \subseteq T_1$ by proving that, for any $x \in M^\diamondsuit - T_1$ there is a $g \in E$ such that $g(x) \notin T_2$. By symmetry, we will be able to conclude that $T_1 \subseteq T_2$, so that $T_1 = T_2$, as desired.

Let T_i be a nice extension of $A_i \in \mathcal{A}$ with totally projective quotient $H_i = T_i/A_i$. Since A_1 is reduced and algebraically compact, $A_1^\diamondsuit = A_1$, so that $M^\diamondsuit/A_1 \cong T_1^\diamondsuit/A_1^\diamondsuit \cong (T_1/A_1)^\diamondsuit \cong H_1^\diamondsuit$. Let $\bar{x}$ be the image of x under the composite $\pi : M^\diamondsuit \to M^\diamondsuit/A_1 \cong H_1^\diamondsuit$. Its purification $\langle \bar{x} \rangle_*$ has torsion-free rank 1 and a totally projective torsion submodule, H_1.

Claim. *There is a $y \in M^\diamondsuit - T_2$ such that, for all k, $ht_{M^\diamondsuit}(p^k y) \geq ht_{H_1^\diamondsuit}(p^k \bar{x})$.*

Suppose that we have constructed such a y. The assignment $\bar{x} \mapsto y$ determines a nonheight decreasing homomorphism $\langle \bar{x} \rangle \to M^\diamondsuit$. As in [**13**] or [**1**], $\langle \bar{x} \rangle$ is a nice submodule of $\langle \bar{x} \rangle_*$ and $\langle \bar{x} \rangle_*/\langle \bar{x} \rangle$ totally projective, so that we can extend this to a homomorphism $\langle \bar{x} \rangle_* \to M^\diamondsuit$. Since $H_1^\diamondsuit/\langle \bar{x} \rangle_*$ is torsion-free and $M^\diamondsuit$ is cotorsion, we may extend this to a homomorphism $h : H_1^\diamondsuit \to M^\diamondsuit$. We then let $g = h \circ \pi$. Since $g(T_1) = h(H_1) \subseteq M \subseteq T_1$, $g \in E$, and since $g(x) = h(\bar{x}) = y \notin T_2$, the result follows. We now turn to the problem of producing such a y.

Let $\sigma = \sup\{ht_{H_1^\diamond}(p^k\bar{x})\}$, which is a limit ordinal of countable cofinality, and λ be the length of M. Since M can be thought of as an immediate submodule of H_i, H_i also has length λ.

Case 1. $\lambda \geq \sigma$. We begin by showing that there is a p-bounded sequence in $M/p^\sigma M$ which is Cauchy in the σ-topology, but does not converge in $H_2/p^\sigma H_2$. If $\{\alpha_i\}_{i<\omega}$ is a strictly ascending chain of ordinals with limit σ, then $H_2/p^\sigma H_2 \cong \oplus_{i<\omega} B_i$, where $p^{\alpha_i}B_i = 0$. Inductively choose a strictly ascending sequence $i_j < \omega$ and nonzero elements

$$z_j \in (\oplus_{\alpha_{i_j} < i < \alpha_{i_{j+1}}} B_i) \cap p^{\alpha_{i_j}}(M/p^\sigma M)[p].$$

If $w_j = z_1 + \cdots + z_j \in M/p^\sigma M$, then w_j is p-bounded and Cauchy in the σ-topology. If w_j converged to $w \in H_2/p^\sigma H_2$, then an easy argument would show that w has an infinite number of nonzero coordinates in the decomposition $\oplus_{i<\omega} B_i$.

Let $L_\sigma M$ be the completion of M in the σ-topology. Since σ has countable cofinality, $E_\sigma M = (L_\sigma M)/(M/p^\sigma M)$ is divisible. We have just shown that there is an element of $w \in L_\sigma M$ of order p such that, under the map $L_\sigma M \to L_\sigma H_2 \to E_\sigma H_2$, w does not go to zero. Hence, there is a summand of $E_\sigma M$ isomorphic to Q/R which maps injectively under the natural homomorphism $E_\sigma M \to E_\sigma H_2$. By the R-module version of [**1**, 56.7], there is a summand $J \cong R$ of $\mathrm{Hom}_R(Q/R, E_\sigma M) \oplus (p^\sigma M)^\diamond \cong p^\sigma(M^\diamond)$ which maps isomorphically to a summand J' of $\mathrm{Hom}_R(Q/R, E_\sigma H_2) \oplus (p^\sigma H_2)^\diamond \cong p^\sigma(H_2^\diamond)$. Since $J' \cap (p^\sigma H_2)^\diamond = 0$, it follows that $J' \cap H_2 = 0$, so that in $M^\diamond$, $J \cap T_2 = 0$. Any $y \in J$ can be seen to satisfy the claim.

Case 2. $\lambda < \sigma$. If $\lambda = \beta + n$, where β is a limit ordinal, then $p^\beta H_1^\diamond \cong \mathrm{Hom}_R(Q/R, E_\beta H_1) \oplus (p^\beta H_1)^\diamond$, and in this decomposition the first term is torsion-free and the second is bounded. If β has uncountable cofinality, by a result of [**10**], $H_1/p^\beta H_1$ is complete in the β-topology, so that $E_\beta H_1 =$ and $p^\lambda H_1^\diamond = 0$, which contradicts the definition of σ. So we may assume that β has countable cofinality and $\sigma = \beta + \omega$. This also implies that there are integers $m \geq n$, $j \geq 0$, such that $ht_{H_1^\diamond}(p^j\bar{x}) = \beta + m$. By the argument in Case 1 (replacing σ by β), there is an element $y_0 \in p^\beta M^\diamond$ such that $\langle y_0\rangle \cap T_2 = 0$. It is easily checked that, setting $y = p^m y_0$ finishes Case 2, hence the claim, and hence the result. □

Corollary 3. *Any automorphism of the endomorphism rings of an $\mathcal{A}$-Warfield module is inner.*

We close with a couple of questions. The equivalence between the category of divisible torsion modules and the category of reduced torsion-free algebraically compact modules can be extended to an equivalence between the category of torsion modules and the category of cotorsion modules. We have shown that there is an equivalence between the members of $\mathcal{D}$, thought of as c-valuated modules, and the members of $\mathcal{A}$, thought of as valuated modules. Is there a natural equivalence between c-valuated torsion modules and valuated cotorsion modules? Finally, is the class of $\mathcal{A}$-Warfield modules closed under summands?

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