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A NOTE ON PRIME n-TUPLES

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Twin primes and *prime triples* are common names given to special prime numbers related to a famous conjecture of Goldbach. A twin prime is an integer p such that p and $p+2$ are both prime numbers. The so-called "twin prime conjecture" states that there exist infinitely many twin primes. Although believed to be true, it remains an intriguing open question.

Prime triples with respect to two integers $\{r, s\}$ are integers p such that $p, p+r$ and $p+s$ are all primes. The question of how many prime triples exist with respect to a given $\{r, s\}$ depends, very much so, on r and s. Only two prime triples with respect to $\{2, 4\}$ exist, namely, $p = 1$ and $p = 3$, whereas, for the case $r = 2$ and $s = 6$ it is again a widely open question. In [**3**, Problem 4, p. 177] a bound on $\pi_3(x)$, the number of all prime triples with respect to $\{2, 6\}$ that are less than x is given. Here, too, infinitely many prime triples of these are believed to exist.

Prime n-tuples with respect to $\{r_1, r_2, \dots, r_{n-1}\}$ are similarly defined and for an intelligent guess of the r_i's, i.e., where one cannot prove by elementary means that there are finitely many prime n-tuples, the conjecture is that there are infinitely many such primes. Clearly, consecutive prime n-tuples are farther apart as n gets larger. The following theorem gives a partial quantitative measure of that spread. To the best of our knowledge our method is new.

Theorem. *Let $Q = \{q_1 < q_2 < \cdots < q_n\}$ and $P = \{p_1 < p_2 < \cdots < p_n\}$ be two sets of positive integers such that each p_i is a prime $(n+1)$-tuple with respect to Q. Then there exists a positive constant c, independent of n, such that*

$$(p_n - p_1)(q_n - q_1) \geq cn^4.$$

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The proof depends on a divisibility property of the determinant of the Van der Monde matrix with integer entries. Let $d_1, d_2, \dots, d_n$ be integers, the Van der Monde matrix $V(d_1, d_2, \dots, d_n)$ is the $n \times n$-matrix whose ith row, $i = 1, \dots, n$, is the vector $(d_1^{i-1}, d_2^{i-1}, \dots, d_n^{i-1})$. A well-known theorem states that

$$\det V(d_1, d_2, \dots, d_n) = \prod_{1 \le i < j \le n} (d_i - d_j). \tag{1}$$

The following lemma is problem 270 in [**1**].

Lemma. *Let $d_1, d_2, \dots, d_n$ be distinct integers. Then*

$$1!2! \cdots (n-1)! = \det V(1, 2, \dots, n) \mid \det V(d_1, d_2, \dots, d_n).$$

Proof of Theorem. By an immediate application of the lemma, we have

$$1!2! \cdots (2n-1)! | \det V(q_1, q_2, \dots, q_n, -p_1, -p_2, \dots, -p_n)$$

and therefore by (1)

$$1!2! \cdots (2n-1)! | \det V(q_1, q_2, \dots, q_n) \det V(p_1, p_2, \dots, p_n) \prod_{i,j=1}^{n} (q_j + p_i).$$

The numbers $p_1 + q_1, p_1 + q_2, \dots, p_1 + q_n$ are primes, hence by the prime number theorem

$$p_1 + q_j \ge c_1 j \log j \quad \text{for } j = 1, 2, 3, \dots, n,$$

for some $c_1 > 0$. Here c_1 and henceforth all c_k, $k = 1, 2, \dots$, will denote positive constants. Therefore, $p_1 + q_j > p_1 + n$, i.e.,

$$q_j \ge n \quad \text{for} \quad j > c_2 \frac{(n + p_1)}{\log(n + p_1)} > c_3 \frac{n}{\log n}.$$

Also

$$p_i \ge n \quad \text{for} \quad i > c_4 \frac{n}{\log n}$$

and therefore

$$1!2!\cdots(2n\text{–}1)!|\det V(q_1,q_2,\dots,q_n)\det V(p_1,p_2,\dots,p_n)\prod_{i,j=1}^{c_5 n/\log n\;*}(q_i+p_j)$$

where the $*$ indicates that the multiplication is to be taken over all couples (i,j) such that $q_i+p_j<2n$. Hence

$$1!2!\cdots(2n-1)!\le(q_n-q_1)^{n(n-1)/2}(p_n-p_1)^{n(n-1)/2}(2n)^{c_5^2n^2/\log^2 n}.$$

Taking the $n(n-1)/2$ root of both sides, we remain with

$$(q_n-q_1)(p_n-p_1)\ge c_7\bigg(\prod_{k=1}^{2n}k!\bigg)^{2/n^2}n^{-c_6/\log^2 n}.$$

Using Stirling's formula for approximating $k!$ and noting that $n^{-1/\log^2 n}$ converges to one as n tends to infinity, we finally get

$$(q_n-q_1)(p_n-p_1)\ge c_8n^4. \qquad \square$$

Remark . All constants c_k in the above theorem can be approximated leading to the asymptotic estimate

$$c=2^8e^{-6}.$$

This follows from the asymptotic estimate

$$\det V(p_1,\dots,p_n)^{2/(n(n-1))}\le\frac{p_n-p_1}{4}$$

due to Fekete, see [**2**, Chapter 7].

Let $P=\{p_1<p_2<\cdots<p_n\}$ and $Q=\{q_1<q_2<\cdots<q_m\}$, $m\le n$, be two sets of integers such that the terms of their sumset $P+Q$ are primes. Without loss of generality we may assume that P is a set of primes and $q_1=0$. Otherwise, replace P by $P'=P+q_1$ and Q by $Q'=Q-q_1$. Hence each p_i in P is a prime m-tuple with respect to $Q\backslash\{0\}$. Our theorem can then be generalized to give

Corollary. *Let* $P = \{p_1 < p_2 < \cdots < p_n\}$ *and* $Q = \{q_1 < q_2 < \cdots < q_m\}$, $m \leq n$, *be two sets of integers such that the terms of their sumset* $P + Q$ *are primes. Then there exists a positive constant* c *such that*

$$(q_m - q_1)^{(m/n)^2}(p_n - p_1) > cn^{(1+m/n)^2}.$$

Far more reaching studies exist in the literature. For example, in [**4**] small divisors of sums of sets, not necessarily of the same size, are studied using advanced tools such as the large sieve. In connection with our theorem, the authors show the existence of arbitrary large sets P and Q such that each $p_i + q_j$ is prime. Moreover, if P and Q are in $[1, N]$, then $n^2 < c_9 N$, see [**4**, Section 6], which follows directly from our result with the advantage of being elementary.

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