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p-LAPLACIAN AND LIENARD-TYPE EQUATION

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ABSTRACT. It is shown that the generalized Liénard-type equation

$$(|u'|^{p-2}u')' + \mu f(u)|u'|^{p-2}u' + g(u) = 0$$

where $p > 1$ and μ is a small parameter has exactly one limit cycle.

1. Introduction. We will consider here a generalized Liénard-type equation of the form

$$(\phi_p(u'))' + \mu f(u)\phi_p(u') + g(u) = 0, \tag{1.1}$$

where $p > 1$ and $\mu > 0$ is a small parameter. The functions $f, g : \mathbf{R} \to \mathbf{R}$ are continuous.

The quasilinear operator

$$((\phi_p(u'))' := \frac{d}{dt}\bigg[\bigg|\frac{du}{dt}\bigg|^{p-2}\frac{du}{dt}\bigg],$$

called the one-dimensional p-Laplacian, has been dealt with in several papers, see [**4, 5, 3, 2**]. For $p = 2$ equation (1.1) reduces to the classical Liénard equation.

In this paper we are concerned with existence and uniqueness of a limit cycle for (1.1). Our method is based on an old, seldom cited, result due to Pontryagin concerning the existence of limit cycles for perturbed Hamiltonian systems. This approach has been recently used by Sędziwy in [**7**] to obtain a proof, different from the one based on

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the Poincaré-Bendixson theorem, (see, e.g., [**1**]) of the classical result on limit cycles of Liénard equations.

We note that, as a consequence of our conditions on f and g and of the method we use, our existence and uniqueness result is global.

We now state the original Pontryagin result.

Theorem P. *Consider the system*

$$\begin{cases} u' = -H_v(u,v) + p(u,v,\mu) \\ v' = H_u(u,v) + q(u,v,\mu), \end{cases} \tag{1.2}$$

where H is C^1 in an open domain $D \subset \mathbf{R}^2$, and the functions $p, q : D \times [-M, M] \to \mathbf{R}$ are continuous. Suppose that $p(u,v,0) = q(u,v,0) = 0$, and that $p_1(u,v) := (\partial/\partial\mu)p(u,v,0)$, $q_1(u,v) := (\partial/\partial\mu)q(u,v,0)$, $(\partial/\partial u)p_1$ and $(\partial/\partial v)q_1$ exist and are continuous in D. Assume also that for a certain h_0 the set $C_{h_0} = \{(u,v) : H(u,v) = h_0\}$ is a simple closed curve and $|H_u| + |H_v| > 0$ for every $(u,v) \in C_{h_0}$.

For $|h - h_0|$ sufficiently small, define

$$\psi(h) = \int\int_{D_h} \left(\frac{\partial p_1}{\partial u} + \frac{\partial q_1}{\partial v}\right) du\,dv, \tag{1.3}$$

where D_h is a bounded domain with boundary $C_h = \{(u,v) : H(u,v) = h\}$. If

$$\psi'(h_0) = e \neq 0, \quad \text{and} \quad \psi(h_0) = 0, \tag{1.4}$$

then for small $|\mu|$ the system has exactly one limit cycle K_μ, which is positively orbitally asymptotically stable provided that μe is negative. Moreover, $\lim_{\mu\to 0} K_\mu = C_{h_0}$.

In Section 2 we will prove our main theorem. In the proof of this result we will need the following auxiliary lemma.

Lemma. *Assume $U, V : (0, v_0) \to \mathbf{R}$ are two continuous functions and that for a certain $\alpha \leq v_0$ the following conditions are satisfied:*

$$V(v) > 0 \quad \text{for } v \in (0, \alpha) \tag{i}$$

(ii) $$\int_0^t V(v)U(v)\,dv > 0 \quad \text{for all } v \in (0, v_0];$$

(iii) $$U(0) = 0 \quad \text{and, for all } v \in (0, v_0], \quad U'(v) > 0.$$

Then

$$U(t)\int_0^t V(y)\,dy > \int_0^t U(y)V(y)\,dy, \tag{1.4}$$

for all $t \in (0, v_0]$.

This lemma has been proved in [**7**]. For the sake of completeness we next sketch the proof.

Proof. Set

$$R(t) = U(t)\int_0^t V(y)\,dy - \int_0^t U(y)V(y)\,dy.$$

Then R satisfies the following differential equation

$$R' = (U'/U)\biggl(R + \int_0^t V(y)U(y)\,dy\biggr). \tag{1.5}$$

Moreover, from (i) $R(t_0)$ is positive for all small $t_0 > 0$. Now, since the solution to the initial value problem given by (1.5) and $R(t_0) = R_0 > 0$ has the form

$$R(t) = R_0\frac{U(t)}{U(t_0)} + U(t)\int_{t_0}^t \frac{U'(y)}{U^2(y)}\biggl(\int_0^y V(s)U(s)\,ds\biggr)\,dy,$$

it is clear that $R(t) > 0$ for all $t \in (0, v_0]$. □

2. Main theorem. In this section we will prove the following

Theorem. *Suppose* $f, g : \mathbf{R} \to \mathbf{R}$ *are two continuous functions that satisfy the following conditions*

(i) $$g(x) = -g(-x), \quad g(x)x > 0 \quad \text{for } x \neq 0;$$

(ii) $$f(x) = f(-x);$$

also, if we set $F(x) = \int_0^x f(s)\,ds$*, then we assume*
(iii)
$F(x) < 0$ *for* $0 < x < a$, *and* $F(x)$ *is positive and increasing for* $a < x$;

(iv) $$\lim_{x\to+\infty} F(x) = +\infty.$$

Under these conditions and for sufficiently small μ*, the equation* (1.1) *has exactly one limit cycle which is orbitally asymptotically stable.*

We note that our assumptions on f and g are independent of p.

Proof. Let p^* be the conjugate exponent of p, i.e., $1/p + 1/p^* = 1$. Then, using the substitution $v = -\phi_p(u')$ and the identity $\phi_{p^*} \circ \phi_p(s) = s$, we have that equation (1.1) can be replaced by the following equivalent first order system

$$u' = -\phi_{p^*}(v), \qquad v' = -\mu f(u)v + g(u). \tag{2.1}$$

System (2.1) is a particular case of system (1.2) for a Hamiltonian given by

$$H(u,v) = G(u) + \Phi_{p^*}(v)$$

with

$$G(u) = \int_0^u g(s)\,ds, \qquad \Phi_{p^*}(v) = \int_0^v \phi_{p^*}(s)\,ds = \frac{|v|^{p^*}}{p^*}$$

and

$$p(u,v,\mu) \equiv 0, \qquad q(u,v,\mu) \equiv -\mu f(u)v.$$

Observe that since both functions G and Φ_{p^*} are convex, nonnegative and unbounded from above, it follows that for any positive number h, the set

$$D_h = \{(u,v) : H(u,v) \le h\}$$

is convex and bounded. This, in turn, implies that the curve $C_h = \partial D_h$ is closed and star-shaped with respect to the origin.

Let us set as before

$$\psi(h) = -\int\int_{D_h} f(u)\,du\,dv.$$

Then, from the symmetry of D_h with respect to the u and v axes, and performing the integration with respect to u, we find that

$$\psi(h) = -4\int_0^{v_0(h)} F(\rho(v,h))\,dv \tag{2.2}$$

where the positive number $v_0 \equiv v_0(h)$ and the function $\rho \equiv \rho(v,h)$ are implicitly defined by

$$h = H(0,v_0) = \Phi_{p^*}(v_0), \qquad h = H(\rho,v) = G(\rho) + \Phi_{p^*}(v).$$

Thus, $v_0 = (p^*h)^{1/p^*}$, and

$$\rho = \gamma(h - \Phi_{p^*}(v))$$

where $\gamma : [0,+\infty) \to \mathbf{R}$ is the inverse function of $G_{|[0,+\infty)}$. It is clear that, for $v \in (0,v_0)$, we have

$$0 = \rho(v_0,h) < \rho(v,h) < \rho(0,h) = \gamma(h) \tag{2.3}$$

and

$$\begin{aligned} \frac{\partial}{\partial h}\rho(v,h) &= \frac{1}{g(\rho(v,h))} > 0, \\ \frac{\partial}{\partial v}\rho(v,h) &= -\frac{\phi_{p^*}(v)}{g(\rho(v,h))} < 0. \end{aligned} \tag{2.4}$$

By (iii), $\psi(h) > 0$ for small h. Since $\lim_{h\to 0} v_0(h) = +\infty$, condition (iv) implies that $\psi(h) < 0$ for large enough h. Thus equation $\psi(h) = 0$ is solvable. Let h_0 be its smallest solution. Condition (iii) implies that $\gamma(h_0) > a$. We will prove that $\psi'(h) < 0$ provided $\gamma(h) \geq a$. Thus, ψ is strictly decreasing in the interval $[G(a), +\infty)$, and therefore h_0 is the unique root of $\psi(h) = 0$. Clearly this will complete the proof of the theorem.

Let us set $U(v) = \phi_{p^*}(v)$ and $V(v) = f(\rho(v,h))/g(\rho(v,h))$ in the lemma of the first section. Then conditions (i) and (iii) of this lemma are obviously satisfied. To show (ii) observe that, from assumption (iii) of the theorem and conditions (2.3) and (2.4) we obtain that

$$\begin{aligned}\int_0^{v_0} V(v)U(v)\,dv &= \int_0^{v_0} \frac{f(\rho(v,h))}{g(\rho(v,h))}\phi_{p^*}(v)\,dv \\ &= -\int_0^t \frac{\partial}{\partial v} F(\rho(v,h))\,dv \\ &= F(\gamma(h)) - F(\rho(t,h)) > 0.\end{aligned}$$

On the other hand, (2.2), (2.3) and (2.4) yield

$$\int_0^{v_0} V(v)\,dv = \int_0^{v_0} \frac{f(\rho(v,h))}{g(\rho(v,h))}\,dv = -\frac{1}{4}\psi'(h). \tag{2.5}$$

Also, from (1.4) with $t = v_0$, we obtain that

$$-(1/4)\psi'(h)\phi_{p^*}(v_0) > F(\gamma(h)).$$

Thus, $\psi'(h)$ is negative whenever $\gamma(h) > a$. From Theorem P we then conclude that system (2.1) has a periodic solution. Since h_0 is the unique root of ψ, system (2.1) possesses exactly one periodic solution which is globally asymptotically stable. □

Remarks. 1) The integral in (2.5) is convergent. This is due to the fact that the integral

$$4\int_0^{v_0} \frac{dv}{g(\rho(v,h))}$$

is equal to the period of the solution to (2.1) with $\mu = 0$ and the initial conditions $u(0) = 0$, $v(0) = v_0$, and therefore is finite.

2) It is an interesting question to prove existence and uniqueness of a limit cycle for "large" values of μ, i.e., $\mu = 1$.

3) Our main theorem will still hold true if the function ϕ_p appearing in (1.1) is replaced by an odd strictly increasing function $\phi : \mathbf{R} \to \mathbf{R}$.

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