ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 2, Spring 1997

REGULARITY OF MEASURES INDUCED BY SOLUTIONS OF INFINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

FARIBORZ ASADIAN

ABSTRACT. This paper is concerned with measures induced by solutions of infinite dimensional stochastic differential equations. Necessary and sufficient conditions are obtained for Fomin differentiability and Skorokhod differentiability of a σ-additive set function defined on the Borel field of an abstract Wiener space. Fomin differentiability or Skorokhod differentiability is established for measures associated with large classes of Ito processes. It is shown that under certain assumptions measures induced by such processes satisfy the Kolmogorov forward equation.

1. Introduction. The concept of differentiable measures was introduced by Fomin in 1968. The initial motivation was to extend the theory of generalized functions to infinite dimensional spaces. In this context differentiable measures emerged to be utilized as elements of the space of test functions as well as solutions of the equations corresponding to differential and pseudo-differential operators. Introduction of smooth measures was a successful attempt to bypass obstacles that one encounters in extending the theory of generalized functions to infinite dimensional spaces. As is shown in [**13**], in infinite dimensional spaces, certain distributions which may not be representable by point functions can be represented by set functions which possess certain regularity properties. The difficulties encountered in infinite dimensional spaces arise partly because of the fact that the fundamental volume measure (Lebesgue measure) is not available in such spaces.

In a Euclidean space, with every bounded σ-additive measure is associated its (possibly generalized) density which is a point function.

Received by the editors on January 17, 1995, and in revised form on September 11, 1995.
Key words and phrases. Abstract Wiener space, Fomin differentiable measure, Skorokhod differentiable measure, stochastic integral equation, Girsanov theorem, Kolmogorov forward equation.
AMS *Classification Numbers.* Primary 28C15, 28C20, 60H20, 60H30; Secondary 60H05.

In the context of infinite dimensional spaces, a notion of differentiability for measures is introduced in such a way that, when taken in the context of a Euclidean space, it reduces to some regularity property for the point functions associated with these measures. Consider, for example, for a bounded Borel measure μ in a Euclidean space the property of having (to within equivalence) a smooth density whose derivatives are integrable with respect to the Lebesgue measure. It can be shown that this property is equivalent to the smoothness of the function $h \mapsto \mu(A+h)$ for each Borel set A [**1**, Theorem 3.3.1], a property that can be generalized to an infinite dimensional setting. The example given illustrates one possible way of extending results concerning properties of densities of measures in Euclidean spaces to infinite dimensional settings, namely by defining notions of differentiability directly on measures. A natural step to take after defining smooth measures is to develop a differential calculus for them. This makes it possible, among other things, to apply differential operators to such measures, and hence consider differential equations involving measures. It also makes it possible to represent solutions of differential equations by measures. It is obvious that in a Euclidean space, the differential calculus developed for measures reduces to ordinary differential calculus for point functions.

An area in which differentiable measures and their calculus can be utilized is infinite dimensional stochastic differential equations (see [**11**] and [**12**] for a development of the theory of stochastic integrals and stochastic differential equations in Banach spaces). To give an example, let (H, B) be an abstract Wiener space. Consider the stochastic differential equation

$$d\xi(t) = A(t, \xi(t))\, dW(t) + \sigma(t, \xi(t))\, dt.$$

It is well known that if B is finite dimensional and the coefficients A and σ satisfy the requirements of the existence and uniqueness theorem and are smooth in the second variable with bounded derivatives, then the transition probability (at each time t) associated with the solution of this equation has a smooth density with respect to the Lebesgue measure. Furthermore, this density satisfies the Kolmogorov forward equation. Our main purpose in this paper is to extend these results to the case in which B is infinite dimensional. To this end, necessary elements of the theory of differentiable measures and infinite dimensional stochastic integration are produced in Sections 3 and 4. The method used in Section 5 to obtain regularity of measures induced by solutions of certain stochastic integral equations is similar to that used in [**2**], where stochastic differential equations in finite dimensional spaces is considered.

2. Preliminaries. Let H be a real separable Hilbert space with norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$. Let $\mathcal{R}$ be the field generated by cylindrical subsets of H. For each $t > 0$, define the Gauss measure μ_t on $\mathcal{R}$ in the following way:

$$\mu_t(C) = \frac{1}{(\sqrt{2\pi t})^{\dim\, PH}} \int_A e^{-|x|^2/(2t)}\, dx,$$

where $C = \{x \in H; PH \in A\}$, P is a finite dimensional orthogonal projection of H, A is a Borel subset of PH, and dx is the Lebesgue measure on PH. It can be shown that if H is infinite dimensional, then μ_t is not σ-additive.

A norm $\|\cdot\|$ on H is called measurable if, for each $\varepsilon > 0$, there is a finite dimensional orthogonal projection P_ε such that

$$\mu_t\{x \in H; \|Px\| > \varepsilon\} < \varepsilon,$$

for each finite dimensional orthogonal projection P that is orthogonal to P_ε. If H is infinite dimensional, then a measurable norm on H is strictly weaker than the norm of H. let B be the completion of H with respect to $\|\cdot\|$. We call the pair (H, B) an abstract Wiener space. Gross proved in [**8**] that the measure $\mu_t \circ i^{-1}$ (where i is the continuous inclusion from H to B) is σ-additive on the field of cylindrical subsets of B and hence, by Hahn extension theorem, has a σ-additive extension (denoted by p_t and called the Wiener measure with variance parameter t) to the σ-field generated by the cylindrical subsets of B. This σ-field turns out to be the Borel σ-field of B.

The natural pairing of each $x^* \in B^*$ with each $x \in B$ will be denoted by (x^*, x). B^* is a subset of $H^* = H$. It should be clear that if $h \in H$, then $(x^*, h) = \langle x^*, y \rangle$, where x^* is considered as an element of H.

In what follows, $\mathcal{L}_1(H)$ and $\mathcal{L}_2(H)$ will denote the Banach space of trace class operators of H (with norm $|\cdot|_1$) and the Hilbert space of Hilbert Schmidt operators of H, respectively. If X and Y are Banach spaces, $\mathcal{L}(X;Y)$ will denote the Banach space (with norm $|\cdot|_{XY}$) of bounded linear functions from X to Y. If X and Y are Hilbert spaces, $\mathcal{L}_2(X;Y)$ will denote the Hilbert space of Hilbert Schmidt operators from X to Y with norm $|\cdot|_{HS} = \sqrt{\langle\cdot,\cdot\rangle_2}$. The collections of n-linear maps on H and n-linear maps of Hilbert-Schmidt type on H will be denoted by $\mathcal{L}^n(H)$ and $\mathcal{L}_2^n(H)$, respectively.

We shall say that a function f from B to a Banach space X is differentiable at $x \in B$ in the directions of H if there is some element $Df(x) \in \mathcal{L}(H;X)$ such that

$$|f(x+h) - f(x) - Df(x)(h)|_X = o(|h|)$$

as $h \to 0$. We use induction to define n-times differentiability in the directions of H, and denote the nth derivative of f by $D^n f(x)$. It is shown in [**14**] that each bounded real valued uniformly continuous function f on B can be approximated uniformly by elements of the collection $\mathcal{D}$ consisting of the functions u that satisfy the following properties:

(i) u is bounded and infinitely many times differentiable in the directions of H with bounded derivatives.

(ii) $Du(x) \in B^*$ for every $x \in B$ and $\|Du(x)\|_{B^*}$ is bounded on B. There is a constant c such that $|Du(x) - Du(y)| \le c\|x-y\|$ for all $x, y \in B$.

(iii) $D^2u(x)$ is a trace class operator for each $x \in B$ and D^2u is bounded and uniformly continuous from B into $\mathcal{L}_1(H)$.

For our purpose it is suitable to take the collection $\mathcal{D}$ as our space of test functions. We equip $\mathcal{D}$ with the topology according to which a net f_α converges to a function $f \in \mathcal{D}$ if and only if $D^n f_\alpha$ converges to $D^n f$ pointwise and boundedly in $\mathcal{L}^n(H)$, and $D^2 f_\alpha$ converges to $D^2 f$ in $\mathcal{L}_1(H)$. A continuous linear functional on $\mathcal{D}$ is called a distribution. We denote the collection of distributions by $\mathcal{D}'$ and supply this space with the topology according to which a net ϕ_α in $\mathcal{D}'$ converges to $\phi \in \mathcal{D}'$ if $\phi_\alpha(f) \to \phi(f)$ for each $f \in \mathcal{D}$.

3. Differentiable measures. Throughout, (H,B) will be a fixed abstract Wiener space, and $\mathcal{M}$ will denote the collection of bounded σ-additive Borel measures on B. The total variation of $\nu \in \mathcal{M}$ will be denoted by $\|\nu\|$, and for each $y \in B$, $\nu_y(dx)$ is defined to be the measure $\nu(dx+y)$. The topology, on $\mathcal{M}$, of convergence in variation will be denoted by τ. $C_b(B)$ will denote the Banach space of bounded uniformly continuous functions from B to R supplied with the sup norm $\|\cdot\|_\infty$.

Definition 3.1. A measure $\nu \in \mathcal{M}$ is said to be *continuous in the directions of* H (or is H*-continuous*) if the function $h \mapsto \nu_h$ from H to $(\mathcal{M}, \tau)$ is continuous.

The collection of H-continuous measures will be denoted by C_H^0. It can be shown that any measure that is absolutely continuous with respect to an H-continuous measure is itself H-continuous (see [**5**]). The Wiener measure p_t, and hence any gp_t with $g \in L^1(p_t)$, is H-continuous.

Definition 3.2. A measure $\mu \in \mathcal{M}$ is said to be *n-times S-differentiable* (or *n-times differentiable in the sense of Skorokhod*) in the directions of H if, for each $f \in C_b(B)$, the function $\phi_f : y \mapsto \int f(x+y)\mu(dx)$ from B to R is n-times differentiable at 0 in the directions of H.

The collection of measures in $\mathcal{M}$ that are n-times S-differentiable in the directions of H is denoted by $C_{S,H}^n$. The Wiener measure clearly belongs to $C_{S;H}^n$ for every n. We list the following consequences of Definition 3.2:

(A) For each i, $1 \leq i \leq n$, and each i-tuple $(h_1, h_2, \dots, h_i) \in H \times \cdots \times H$, there is a measure $d_{S;h_1,\dots,h_i}^{(i)}\mu \in \mathcal{M}$ such that

$$D^{(i)}\phi_f(0)(h_1, \dots, h_i) = (-1)^i \int f d_{S;h_1,\dots,h_i}^{(i)}\mu$$

for each bounded uniformly continuous function f. For the proof of this result, for the case $n = 1$, see [**4**].

(B) For each $f \in C_b(B)$, the function

$$(h_1, \dots, h_i) \mapsto D^{(i)}\phi_f(0)(h_1, \dots, h_i)$$

is continuous and linear. Since the collection $\{D^{(i)}\phi_f(0)(h_1, \dots, h_i)\|f\|_\infty \leq 1\}$ is bounded by the total variation of the measure $d_{S;h_1,\dots,h_i}^{(i)}$, we may invoke the principle of uniform boundedness to infer that

$$\sup_{f\in C(B),|f|_\infty\le 1}\int f d_{S;h_1,\dots,h_i}\mu \longrightarrow 0$$

as

$$(h_1,\dots,h_i)\longrightarrow(0,\dots,0).$$

But this implies that

$$d_{S;h_1,\dots,h_i}\mu(O)\longrightarrow 0 \quad\text{as}\quad (h_1,\dots,h_i)\longrightarrow(0,\dots,0)$$

for each open set O, which in turn implies that the function $(h_1,\dots,h_i)$ $\mapsto \int f d_{S;h_1,\dots,h_i}\mu$ is continuous if f is only bounded and continuous. Invoking the principle of uniform boundedness a second time, we infer that the linear map $(h_1,\dots,h_i)\mapsto d^{(i)}_{S;h_1,\dots,h_i}\mu$ from $H\times\cdots\times H$ to $(\mathcal{M},\tau)$ is continuous. By a theorem of Pettis [**6**], there is a vector valued σ-additive Borel measure $\mu^{(i)}(\cdot)$ with values in the adjoint space of $\mathcal{L}(H\times\cdots\times H;R)$ such that

$$d^{(i)}_{S;h_1,\dots,h_i}(A)=\mu^{(i)}(A)(h_1,\dots,h_i).$$

(C) For each $h\in H$, $f\in C_b(B)$, and $1\le i<n$, an application of the mean value theorem to the function $t\mapsto\int f(d^{(i)}_{S;h_1,\dots,h_i}\mu)_{th}$ gives us some number $\lambda\in(0,1)$ such that

$$\int f[(d^{(i)}_{S;h_1,\dots,h_i}\mu)_h-d^{(i)}_{S;h_1,\dots,h_i}\mu]=-\int f(d^{(i+1)}_{S;h_1,\dots,h_i,h}\mu)_{\lambda h}.$$

The equality holds for each bounded continuous function. Therefore,

$$\|(d^{(i)}_{S;h_1,\dots,h_i}\mu)_h-d^{(i)}_{S;h_1,\dots,h_i}\mu\|\le\|d^{(i+1)}_{S;h_1,\dots,h_i,h}\mu\|.$$

This inequality together with (B) above implies that for each $0\le i<n$, and each i-tuple $(h_1,\dots,h_i)\in H\times\cdots\times H$, the measure $d^{(i)}_{S;h_1,\dots,h_i}\mu$ is continuous in the directions of H.

(D) For each $f\in C_b(B)$, for each $0\le i<n$, for each i-tuple $(h_1,\dots,h_i)\in H\times\cdots\times H$, and for each $h\in H$, the mean value theorem applied to the map

$$t\longmapsto\int f[(d^{(i)}_{S;h_1,\dots,h_i}\mu)_{th}-td^{(i+1)}_{S;h_1,\dots,h_i,h}\mu]$$

on the interval $[0,1]$ gives us a number $\sigma \in (0,1)$ such that

$$\int f[(d^{(i)}_{S;h_1,\dots,h_i}\mu)_h - d^{(i)}_{S;h_1,\dots,h_i}\mu - d^{(i+1)}_{S;h_1,\dots,h_i,h}\mu] = \int f[(d^{(i+1)}_{S;h_1,\dots,h_i,h}\mu)_{\sigma h} - d^{(i+1)}_{S;h_1,\dots,h_i,h}\mu].$$

Therefore,

$$\begin{aligned} \|(d^{(i)}_{S;h_1,\dots,h_i}\mu)_h - d^{(i)}_{S;h_1,\dots,h_i}\mu - d^{(i+1)}_{S;h_1,\dots,h_i,h}\mu\| &\le \sup_{0\le\sigma\le 1} \|(d^{(i+1)}_{S;h_1,\dots,h_i,h}\mu)_{\sigma h} - d^{(i+1)}_{S;h_1,\dots,h_i,h}\mu\| \\ &= \sup_{0\le\sigma\le 1} \|(\mu^{(i+1)}_{\sigma h} - \mu^{(i+1)})(h_1,\dots,h_i,h)\| \\ &\le \sup_{0\le\sigma\le 1} \|\mu^{(i+1)}_{\sigma h} - \mu^{(i+1)}\|\,|h_1|\cdots|h_i||h|, \end{aligned}$$

where $\|\mu^{(i+1)}_{\sigma h} - \mu^{(i+1)}\|$ is the total variation of the vector valued measure $\mu^{(i+1)}_{\sigma h} - \mu^{(i+1)}$.

(E) If $\mu \in C^n_{S;H}$, then for each $i = 1,\dots,n$ we have

$$\int D^{(i)} f(x)(h_1, h_2 m, \dots, h_i)\mu(dx) = (-1)^i \int f(x) d^{(i)}_{S;h_1,\dots,h_i}\mu(dx)$$

for every bounded function f that is n-times differentiable in the directions of H with bounded derivatives. This is an obvious consequence of the definition.

(F) If $\mu \in C^n_{S;H}$ and $\nu \in \mathcal{M}$, then $\mu * \nu \in C^n_{S;H}$ and

$$d^{(n)}_{S;h_1,\dots,h_n}(\mu * \nu) = (d^{(n)}_{S;h_1,\dots,h_n}\mu) * \nu.$$

For a proof see [**10**].

(G) The Wiener measure p_1 belongs to $C^n_{S;H}$ for every $n = 1,2,\dots$ with

$$d^{(n)}_{S;h_1,\dots,h_n} p_1 = \theta_n(y; h_1,\dots,h_n) p_1,$$

where $\theta_n(y; h_1,\dots,h_n)$ is defined by the following recursion formula:

$$\begin{aligned} \theta_1(y; h) &= (h, y), \\ \theta_n(y; h_1,\dots,h_n) &= (h_n, y)\theta_{n-1}(y; h_1,\dots,h_{n-1}) \\ &\quad - \partial_y \theta_{n-1}(y; h_1,\dots,h_{n-1})(h_n), \end{aligned}$$

where ∂_y is differentiation with respect to y, in the directions of H.

(H) If $\mu \in C^1_{S;H}$, then for each test function f and each μ-integrable function $\sigma : B \to H$, which is differentiable in the directions of H with μ-integrable derivative $D\sigma : B \to \mathcal{L}(H)$ we have

$$\begin{aligned}\int \langle \sigma(x), Df(x)\rangle \mu(dx) &= \lim_n \sum_{i=1}^n \int Df(x)(e_i)\langle \sigma(x), e_i\rangle \mu(dx)\\ &= -\lim_n \int f(x) \sum_{i=1}^n d_{S;e_i}(\langle \sigma(x), e_i\rangle \mu(dx)).\end{aligned}$$

The weak limit of $\sum_{i=1}^n d_{S;e_i}(\langle \sigma(x), e_i\rangle \mu)$ is denoted by trace $(\sigma\mu)'$ and is an element of $\mathcal{D}'$.

(I) The same type of argument used in (H) shows that if $\mu \in C^2_{S;H}$ and if $A : B \to \mathcal{L}(H)$ is a bounded μ-integrable map that is twice differentiable in the directions of H with μ-integrable derivatives, then there is an element TRACE $(A(\cdot)^2\mu)''$ (which is the weak limit of the series $\sum_{i,j=1}^\infty d^{(2)}_{S;e_i,e_j}(\langle A^2(x)(e_i, e_j)\mu)$ in $\mathcal{D}'$) such that

$$\int \operatorname{trace} D^2 f(x)(A(x)(\cdot), A(x)(\cdot))\mu(dx) = \text{TRACE}\,(A(\cdot)^2\mu)''(f).$$

Definition 3.3. A measure $\mu \in \mathcal{M}$ is said to be *n-times F-differentiable* (or *n-times differentiable in the sense of Fomin*) in the directions of H if the function $h \mapsto \mu_h$ from H into $(\mathcal{M}, \tau)$ is n-times Frechet differentiable.

The collection of n-times F-differentiable measures in the directions of H will be denoted by $C^n_{F;H}$. It follows from consequences (C) and (D) of Definition 3.2 that if $\mu \in C^n_{S;H}$, then $\mu \in C^{n-1}_{F;H}$, and if in addition the vector-valued measure $\mu^{(n)}$ is continuous in the directions of H, then $\mu \in C^n_{F;H}$.

Theorem 3.4. *Let $\mu \in \mathcal{M}$. Suppose that for each $i = 1, \dots, n$, there exists a constant $M_i > 0$ such that $|\int D^i f(x)(h_1, \dots, h_i)\mu(dx)| \le M_i |f|_\infty$ for each i-tuple $(h_1, \dots, h_i) \in H \times \cdots \times H$, and for every $f \in C_b(B)$ that is i-times differentiable in the directions of H with* *bounded derivatives that are uniformly continuous. Then μ is n-times S-differentiable in the directions of H.*

Proof. For each $f \in C_b(B)$ we have

$$\begin{aligned}
&\sup_x \left| \int f(x+\cdot)(p'_{1/n}(h) - p'_{1/m}(h)) * \mu \right| \\
&\quad = \sup_x \left| \int D_y \Big(\int f(x+y+z)(p_{1/n} - p_{1/m})(dz) \Big)(h)\mu(dy) \right| \\
&\quad \le M_1 \sup_x \left\| \int f(x+\cdot+z)(p_{1/n} - p_{1/m})(dz) \right\|_\infty \\
&\quad = M_1 \sup_{x_i, y_i} \left| \int \left[f\Big(x_i + y_i + \frac{z}{\sqrt{n}}\Big) - f\Big(x_i + y_i + \frac{z}{\sqrt{m}}\Big) \right] p_1(dz) \right| \\
&\quad \le M_1 \int \sup_{x_i, y_i} \left| f\Big(x_i + y_i + \frac{z}{\sqrt{n}}\Big) - f\Big(x_i + y_i + \frac{z}{\sqrt{m}}\Big) \right| p_1(dz) \\
&\quad \longrightarrow 0 \quad \text{as } n, m \longrightarrow \infty,
\end{aligned}$$

where $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ are dense subsets of B. Therefore we have shown that the sequence $\{\int f(x+\cdot)p'_{1/n}(h)*\mu\}_{n=1}^\infty$ converges, uniformly in x, for each $f \in C_b(B)$.

For each $f \in C_b(B)$, consider the following functions:

$$\phi_n : x \longmapsto \int f(x+\cdot)p_{1/n} * \mu, \qquad \phi : x \longmapsto \int f(x+\cdot)\mu,$$

and

$$\psi_n^h : x \longmapsto \int f(x+\cdot)p'_{1/n}(h) * \mu.$$

ϕ_n converges uniformly to ϕ, ψ_n^h converges uniformly in x, ϕ_n is H-differentiable, and $D\phi_n(x)(h) = \psi_n^h(x)$. All this implies that the map $\phi : x \mapsto \int f(x+\cdot)\mu$ is differentiable in the directions of H.

If the conclusion of the theorem holds for $i = n$, then

$$\int D^{(n+1)} f(x)(h_1, \dots, h_{n+1})\mu(dx) = (-1)^n \int Df(x)(h_{n+1}) d^{(n)}_{S;h_1,\dots,h_n}\mu,$$

and since $|\int Df(x)(h_{n+1})d^{(n)}_{S;h_1,\dots,h_n}\mu| \leq M_{n+1}\|f\|_\infty$, it follows that $d^{(n)}_{S;h_1,\dots,h_n}\mu$ is S-differentiable in the directions of H. □

4. Stochastic integration in abstract Wiener space.

Definition 4.1. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space, and let $\{F_t\}_{t\geq 0}$ be an increasing family of σ-algebras of $\mathcal{F}$. Let $\{W(t,.)\}_{t\geq 0}$ be a collection of B-valued random elements satisfying the following conditions:

a) $W(0) = 0$, $\mathcal{P}$ almost surely,

b) the sample paths of $\{W(t,.)\}_{t\geq 0}$ are continuous for almost all $\omega \in \Omega$,

c) for each $t > 0$, the σ-algebra generated by the random elements $\{W(s,.); 0 \leq s \leq t\}$ is a subset of $\mathcal{F}_t$,

d) for every $0 \leq s < t < \infty$, the random element $W(t) - W(s)$ is independent of the σ-algebra $\mathcal{F}_s$, and

e) $W(t,.) - W(s,.)$ is distributed according to the Wiener measure p_{t-s}.

The process $\{W(t,.)\}_{t\geq 0}$ is called a B-valued standard Wiener process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Existence. For each $0 = t_0 < t_1 < t_2 < \cdots < t_n$, consider the distribution $Q_{t_0,t_1,\dots,t_n}$ on the measure space $(B \times B \times \cdots \times B, \mathcal{B}(B) \times \cdots \times \mathcal{B}(B))$ defined by

$$Q_{t_0,\dots,t_n}(A_0 \times, \cdots, \times A_n) = \int_{A_n} \cdots \int_{A_1} \int_{A_0} p_{t_n - t_{n-1}}(y_{n-1}, dy_n) \cdots p_{t_1}(y_0, dy_1)\delta_0(dy_0),$$

where $A_1, \dots, A_n \in \mathcal{B}(B)$ and δ_0 is the measure concentrated at 0. The collection $\{Q_{t_0,\dots,t_n}\}_{0=t_0<t_1<\cdots<t_n}$ is a consistent family of probability measures. By Kolmogorov's extension theorem, there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a collection of B-valued random

elements satisfying (a), (c) and (d) of Definition 2.1. Now,

$$E\|W(t,.)-W(s,.)\|^{\alpha}=\int_B \|x\|^{\alpha}p_{t-s}(dx)$$
$$=(t-s)^{\alpha/2}\int_B \|x\|^{\alpha}p_1(dx).$$

One can prove (using the fact that $\int_B \|x\|^{\alpha}p_1\,dx<\infty$ (see [**9**]) and the same argument used in the finite dimensional case) that $\{W(t,.)\}_{t\geq 0}$ has a continuous version and that almost all sample paths of the continuous version of $\{W(t,.)\}_{t\geq 0}$ are Hölder continuous, with exponent β, for any $\beta<1/2$, on bounded intervals of t. The martingale property of $\{W(t,\cdot)\}_{t\geq 0}$ follows from the fact that p_t is symmetric, i.e., $p_t(-A)=p_t(A)$ for all $A\in\mathcal{B}(B)$.

Let G be a Banach space. A process $\xi:[0,\infty)\times\Omega\to G$ that is (t,ω) jointly measurable is called nonanticipating if for each $t\geq 0$, $\xi(t)$ is $\mathcal{F}_t$ measurable. For each $p\geq 1$, $0\leq\alpha<\beta$, we introduce two classes of processes:

$\mathcal{M}^p_{\alpha,\beta}[G]$ denotes the Banach space of G-valued nonanticipating processes satisfying the inequality $E\int_\alpha^\beta \|\xi(t)\|^p_G<\infty$.

$\mathcal{S}^p_{\alpha,\beta}[G]$ denotes the Banach subspace of $\mathcal{M}^p_{\alpha,\beta}[G]$ consisting of those processes ξ with $E(\sup_{\alpha\leq t\leq\beta}\|\xi(t)\|^p_G)<\infty$.

Kuo proves in [**9**] that there is a Banach subspace B_o of B and an increasing sequence $\{Q_n\}_{n=1}^{\infty}$ of finite dimensional orthogonal projections of H such that (H,B_o) is an abstract Wiener space, each Q_n takes values in B_o^*, Q_n converges strongly to identity in H, each Q_n extends to a projection (still denoted by Q_n) of B_o, Q_n converges strongly in B_o to identity, and $p_t(B_o)=1$ for each $t>0$. In the following $\{e_n\}_{n=1}^{\infty}$ will be an orthonormal basis of H such that $\{e_1,\dots,e_{l_n}\}$ is a basis for $Q_n(H)$. Every measurable function f defined on B_o can be considered as a measurable function defined on B by letting $f(x)=0$ for every $x\in B$ that does not belong to B_o.

Theorem 4.2. *Let K be a separable Hilbert space. Then the linear manifold consisting of the bounded simple processes in $\mathcal{M}^p_{\alpha,\beta}[\mathcal{L}(B_o;K)]$ is dense in $\mathcal{M}^p_{\alpha,\beta}[\mathcal{L}_2(H;K)]$.*

Let $\xi(t,\cdot) \in \mathcal{M}^p_{\alpha,\beta}[\mathcal{L}(B_o;K)]$ be a bounded simple process with jumps at $\alpha = t_0 \le t_1 < \cdots < t_n = \beta$. When $t_j \le t < t_{j+1}$, $0 < j \le n$, the K-valued random element

$$\sum_{i=0}^{j-1} \xi(t_i)(W(t_{i+1}) - W(t_i)) + \xi(t_j)(W(t) - W(t_j)),$$

is denoted by $\int_\alpha^t \xi(s)\,dW(s)$. The map $J : \xi \mapsto \int_\alpha^\cdot \xi\,dW$ is therefore a densely defined map from $\mathcal{M}^2_{\alpha,\beta}[\mathcal{L}_2(H;K)]$ into $\mathcal{M}^2_{\alpha,\beta}[K]$ which can be shown to be isometric. The value of the extension of this map at each $\xi \in \mathcal{M}^2_{\alpha,\beta}[\mathcal{L}_2(H;K)]$ is still denoted by $\int_\alpha^\cdot \xi\,dW$, and if $K = R$, by $\int_\alpha^\cdot \langle \xi, dW\rangle$. If $A(t) = I + \xi(t)$, where I is the identity map from B_o to B_o and $\xi \in \mathcal{M}^2_{\alpha,\beta}[\mathcal{L}_2(H;K)]$, then we define

$$\int_\alpha^t A(s)\,dW(s) = W(t) - W(\alpha) + \int_\alpha^t \xi(s)\,dW(s).$$

It should also be noted that, for each $k \in K$,

$$\left\langle \int_\alpha^t \xi(s)\,dW(s), k \right\rangle_K = \int_\alpha^t \langle \xi(s)^*(k), dW(s)\rangle.$$

Definition 4.3. Let K be a Hilbert subspace of a Banach space G such that the norm of G is strictly weaker than that of K, then the couple (K, G) is called a *conditional Banach space.*

Definition 4.4. Let K_1 and K_2 be Hilbert spaces. A continuous bilinear map S from $K_1 \times K_1$ into K_2 is called *trace-class type* if (i) for each $x \in K_2$, the function S_x defined by $\langle S_x y, z\rangle = \langle S(y,z), x\rangle$ belongs to $\mathcal{L}_1(K_1)$, and (ii) the linear functional $x \mapsto \operatorname{trace} S_x$ is continuous.

It follows from this definition that an element $\mathrm{TRACE}\, S$ exists in K_2 such that $\langle \mathrm{TRACE}\, S, x\rangle_{K_2} = \operatorname{trace} S_x$ for all $x \in K_2$.

Theorem 4.5 (Ito's lemma [**9**]). *Let (K_1, G_1) and (K_2, G_2) be conditional Banach spaces. Let $\rho : [0,\infty) \times G_1 \mapsto G_2$ satisfy the following conditions*:

(i) *for each* $x \in G_1$, $\rho(\cdot, x)$ *is continuously differentiable and* $\partial\rho/\partial t$ *is continuous from* $[0, \infty) \times G_1$ *into* G_2,

(ii) *for each* $t \geq 0$, $\rho(t, \cdot) : G_1 \mapsto G_2$ *is twice Frechet differentiable such that* ρ' *and* ρ'' *are* (t, x)*-jointly continuous,* $\rho'(t, x)(K_1) \subset K_2$ *and* $\rho''(t, x)(K_1 \times K_1) \subset K_2$. *If* $X(t) = x + \int_0^t \zeta(s)\, dW(s) + \int_0^t \sigma(s)\, ds$, *where* $x \in G_1$, $\zeta \in \mathcal{M}^2_{0,\infty}[\mathcal{L}_2(H, K_1)]$ *and* $\sigma \in \mathcal{M}^2_{0,\infty}[K_1]$. *Then*

$$\begin{aligned}\rho(t, X(t)) = \rho(0, x) &+ \int_0^t [\rho'(s, X(s)) \circ \zeta(s)]\, dW(s) \\ &+ \int_0^t \left\{ \frac{\partial \rho}{\partial s}(s, X(s)) + \rho'(s, X(s))(\sigma(s)) \right. \\ &\qquad \left. + \frac{1}{2} \operatorname{TRACE} (\rho''(s, X(s)) \circ [\zeta(s) \times \zeta(s)]) \right\} ds.\end{aligned}$$

We use Ito's lemma to prove two theorems which will be used in the next section.

Theorem 4.6. *Let* K *be a separable Hilbert space and* $\xi \in \mathcal{M}^{2m}_{\alpha,\beta}[\mathcal{L}_2(H; K)]$ *for some integer* $m \geq 1$; *then the process*

$$\left\{ \int_\alpha^t \xi(s)\, dW(s) \right\}_{\alpha \leq t \leq \beta}$$

belongs to the space $\mathcal{S}^{2m}_{\alpha,\beta}[K]$ *and furthermore,*

$$\begin{aligned}E\left[\sup_{\alpha \leq s \leq \beta} \left| \int_\alpha^s \xi(u)\, dW(u) \right|_K^{2m} \right] \\ \leq \left(\frac{4m^3}{2m-1} \right)^m (\beta - \alpha)^{m-1} \int_\alpha^\beta E|\xi(u)|^{2m}_{HS}\, du.\end{aligned}$$

Proof. The result for $m = 1$ follows from the fact that the process $|\int_\alpha^\cdot \xi(s)\, dW(s)|_K$ is a submartingale. So we assume that $m > 1$. First we show that for any $\xi \in \mathcal{M}^{2m}_{\alpha,\beta}[\mathcal{L}(H;K)]$ we have

$$E\left|\int_\alpha^\beta \xi(s)\,dW(s)\right|_K^{2m} \leq m^m(2m-1)^m(\beta-\alpha)^{m-1}E\int_\alpha^\beta |\xi(s)|_{HS}^{2m}\,ds. \tag{4.1}$$

If the inequality holds for bounded simple processes, then an application of Fatou's lemma proves that it must hold for each $\xi \in \mathcal{M}^{2m}_{\alpha,\beta}[\mathcal{L}_2(H;K)]$ as well. So we assume that ξ is a bounded simple process that takes values in $\mathcal{L}(B_o, K)$. In the following we use the fact that the adjoint of the restriction of $\xi(s)$ to H takes values in B_o^*. We also need the fact that the norm $\|\cdot\|_o$ (of B_o) which is defined p_t-almost surely on B is in $L^r(p_t)$ for every $r \geq 1$. To prove this statement let $\tilde{p}_t$ be the extension of the Gauss measure to the Borel σ-field of B_o. It follows from Fernique's theorem (see [**9**]) that $\|\cdot\|_o \in L^r(\tilde{p}_t)$ for every $r \geq 1$. Next we note that the sequence $\{\|Q_n x\|_o\}_{n=1}^\infty$ converges p_t-almost surely to $\|\cdot\|_o$. Hence, starting by Fatou's lemma, we have

$$\begin{aligned}\int_B \|x\|_o^r p_t(dx) &\leq \lim_n \int_B \|Q_n x\|_o^r p_t(dx)\\ &= \lim_n \int_{B_o} \|Q_n x\|_o^r \tilde{p}_t(dx)\\ &= \int_{B_o} \|x\|_o^r \tilde{p}_t(dx)\\ &< \infty.\end{aligned}$$

The first equality above follows from the fact that the distribution of $\|Q_n x\|_o$ with respect to both p_t and $\tilde{p}_t$ is the same as the distribution of $\|\cdot\|_o$ in the space $Q_n H$, with respect to the Gauss measure. A useful consequence is that, for each $t > s \geq 0$, the random variable $\|W(t) - W(s)\|_o$ belongs to $L^r(\mathcal{P})$ for every $r \geq 1$.

Now we apply Ito's formula (Theorem 4.5) to the function $f : K \mapsto R$ defined by $f(x) = |x|_K^{2m}$ and to the process $\int_\alpha^\cdot \xi(s)\,dW(s)$. Note that

$$f'(x)(\cdot) = 2m|x|_K^{2m-2}\langle x, \cdot\rangle_K$$

and

$$f''(x)(\cdot,\cdot) = 2m|x|_K^{2m-2}\langle \cdot,\cdot\rangle_K + 4m(m-1)|x|_K^{2m-4}\langle x,\cdot\rangle_K\langle x,\cdot\rangle_K.$$

This gives us

(4.2)

$$
\begin{aligned}
\Big| \int_\alpha^\beta \xi(s)\, dW(s) \Big|_K^{2m}
&= \int_\alpha^\beta \Big\langle 2m \Big| \int_\alpha^s \xi(u)\, dW(u) \Big|_K^{2m-2} \int_\alpha^s \xi(u)\, dW(u), \xi(s)\, dW(s) \Big\rangle_K \\
&\quad + \frac{1}{2} \int_\alpha^\beta \mathrm{TRACE} \Big[2m \Big| \int_\alpha^s \xi(u)\, dW(u) \Big|_K^{2m-2} |\xi(s)(\cdot)|_K^2 \\
&\quad + 4m(m-1) \Big| \int_\alpha^s \xi(u)\, dW(u) \Big|_K^{2m-4} \Big\langle \int_\alpha^s \xi(u)\, dW(u), \xi(s)(\cdot) \Big\rangle_K^2 \Big]\, dS.
\end{aligned}
$$

Let $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ be the partition corresponding to ξ. In the following C_1 and C_2 are constants.

$$
\begin{aligned}
E \int_{t_i}^{t_{i+1}} & \Big| \int_\alpha^s \xi(u)\, dW(u) \Big|_K^{4m-4} \Big| \xi(s)^* \Big(\int_\alpha^s \xi(u)\, dW(u) \Big) \Big|_H^2 ds \\
&< C_1 E \int_{t_i}^{t_{i+1}} \Big| \int_\alpha^s \xi(u)\, dW(u) \Big|_K^{4m-2} ds \\
&= C_1 \int_{t_i}^{t_{i+1}} \Big| \xi(t_i)(W(s) - W(t_i)) \\
&\qquad\qquad + \sum_{j=1}^{i-1} \xi(t_j)(W(t_{j+1}) - W(t_j)) \Big|_K^{4m-2} ds \\
&< C_2 E \int_{t_i}^{t_{i+1}} \Big(\|W(s) - W(t_i)\|_o \\
&\qquad\qquad + \sum_{j=1}^{i-1} \|W(t_{j+1}) - W(t_j)\|_o \Big)^{4m-2} ds \\
&< \infty,
\end{aligned}
$$

since, as proved above,

$$
\begin{aligned}
\int_{t_i}^{t_{i+1}} E\|W(s) - W(t_i)\|_o^r\, ds &\le \int_{t_i}^{t_{i+1}} \int_{B_o} \|x\|_o^r \tilde{p}_{s-t_i}(dx)\, ds \\
&= \int_{t_i}^{t_{i+1}} (s - t_i)^{r/2} \int_{B_o} \|x\|_o^r \tilde{p}_1(dx)\, ds \\
&< \infty.
\end{aligned}
$$

From this it follows that

$$E\int_\alpha^\beta \left|\xi(s)^*\left(2m\left|\int_\alpha^s \xi(u)\,dW(u)\right|_K^{2m-2}\int_\alpha^s \xi(u)\,dW(u)\right)\right|_H^2 ds<\infty.$$

Hence, the first term of the right side of (4.2) has zero expectation. Therefore,

$$\begin{aligned}
&E\left|\int_\alpha^\beta \xi(s)\,dW(s)\right|_k^{2m}\\
&\quad=\frac{1}{2}\int_\alpha^\beta E\sum_{j=1}^\infty\left[2m\left|\int_\alpha^s \xi(u)\,dW(u)\right|_K^{2m-2}|\xi(s)(e_j)|_K^2\right.\\
&\qquad\qquad+4m(m-1)\left|\int_\alpha^s \xi(u)\,dW(u)\right|_K^{2m-4}\\
&\qquad\qquad\left.\times\left\langle\int_\alpha^s \xi(u)\,dW(u),\xi(s)(e_j)\right\rangle_K^2\right]ds.
\end{aligned}\tag{4.3}$$

So

$$\begin{aligned}
E\left|\int_\alpha^\beta \xi(s)\,dW(s)\right|_K^{2m}&\le m(2m-1)\\
&\quad\cdot\int_\alpha^\beta E\left|\int_\alpha^s \xi(u)\,dW(u)\right|_K^{2m-2}|\xi(s)|_{HS}^2\,ds.
\end{aligned}$$

An application of Hölder's inequality with $p=m/(m-1)$ and $q=m$ gives us

$$\begin{aligned}
E\left|\int_\alpha^\beta \xi(s)\,dW(s)\right|_k^{2m}&\le m(2m-1)\\
&\quad\cdot\left(\int_\alpha^\beta E\left(\left|\int_\alpha^s \xi(u)\,dW(u)\right|_K^{2m}\right)ds\right)^{(m-1)/m}\\
&\quad\cdot\left(\int_\alpha^\beta E|\xi(s)|_{HS}^{2m}\,ds\right)^{1/m}.
\end{aligned}\tag{4.4}$$

From (4.3) it is clear that the function $t \mapsto E|\int_\alpha^t \xi(u)\,dW(u)|_K^{2m}\,ds$ is monotone increasing. Therefore

$$\int_\alpha^\beta E\left|\int_\alpha^s \xi(u)\,dW(u)\right|_K^{2m} ds \le \int_\alpha^\beta E\left|\int_\alpha^\beta \xi(u)\,dW(u)\right|_K^{2m} ds = (\beta-\alpha)E\left|\int_\alpha^\beta \xi(u)\,dW(u)\right|_K^{2m} ds.$$

So from (4.4) it follows that

$$E\left|\int_\alpha^\beta \xi(u)\,dW(u)\right|_K^{2m} \le m(2m-1) \cdot \left\{(\beta-\alpha)E\left|\int_\alpha^\beta \xi(u)\,dW(u)\right|_K^{2m}\right\}^{(m-1)/m} \cdot \left\{\int_\alpha^\beta E|\xi(s)|_{HS}^{2m}\,ds\right\}^{1/m}.$$

Simplifying the inequality gives

$$E\left|\int_\alpha^\beta \xi(u)\,dW(u)\right|_K^{2m} \le m^m(2m-1)^m(\beta-\alpha)^{m-1}\int_\alpha^\beta E|\xi(u)|_{HS}^{2m}\,ds.$$

This inequality holds for every $\xi \in \mathcal{M}_{\alpha,\beta}^{2m}[\mathcal{L}_2(H;K)]$. Now, using the fact that the process $\{\int_\alpha^t \xi(u)\,dW(u)\}_{\alpha\le t\le\beta}$ is a martingale, we get

$$E\sup_{\alpha\le t\le\beta}\left|\int_\alpha^t \xi(u)\,dW(u)\right|_K^{2m} \le \left(\frac{2m}{2m-1}\right)^{2m} E\left|\int_\alpha^\beta \xi(u)\,dW(u)\right|_K^{2m}.$$

This inequality together with (4.1) gives us the desired estimate. □

Theorem 4.7. *Consider the processes X_1 and X_2 defined by*

$$X_1(t) = \zeta_1 + \int_0^t \xi_1(s)\,dW(s) + \int_0^t \sigma_1(s)\,ds,$$

and

$$X_2(t) = \zeta_2 + \int_0^t \xi_2(s)\,dW(s) + \int_0^t \sigma_2(s)\,ds,$$

where ξ_1 *and* ξ_2 *are in* $\mathcal{M}^2_{0,\infty}[\mathcal{L}_2(H;\mathcal{L}_2(K))]$ *and* σ_1, σ_2 *are in* $\mathcal{M}^1_{0,\infty}[\mathcal{L}(K)]$ *and* ζ_1 *and* ζ_2 *are* $\mathcal{L}(K)$ *valued summable random variables that are* $\mathcal{F}_0$ *measurable. Then*

$$\begin{aligned} X_1(t)X_2(t) = \zeta_1\zeta_2 &+ \int_0^t X_1(s)\xi_2(s)(dW(s)) \\ &+ \int_0^t \xi_1(s)(dW(s))(X_2(s)) \\ &+ \int_0^t \Big(\sigma_1(s)X_2(s) + X_1(S)\sigma_2(s) \\ &\qquad + \sum_{i=1}^{\infty} \xi_1(s)(e_i) \circ \xi_2(s)(e_i)\Big)\, ds. \end{aligned}$$

Proof. We can combine the equations for $X_1(t)$ and $X_2(t)$ in the following form:

$$X(t) = \zeta + \int_0^t \xi(s)\, dW(s) + \int_0^t \sigma(s)\, ds,$$

where $\xi \in \mathcal{M}^2_{0,\infty}[\mathcal{L}_2(H;\mathcal{L}_2(K) \times \mathcal{L}_2(K))]$ is defined by $\xi(s)(h) = (\xi_1(s)(h), \xi_2(s)(h))$ and $\sigma \in \mathcal{M}^1_{0,\infty}[\mathcal{L}(K)\times\mathcal{L}(K)]$ is defined by $\sigma(s)(h) = (\sigma_1(s)(h), \sigma_2(s)(h))$ and $\zeta = (\zeta_1, \zeta_2)$.

Let $G_1 = \mathcal{L}(K) \times \mathcal{L}(K)$, $G_2 = \mathcal{L}(K)$, $N_1 = \mathcal{L}_2(K) \times \mathcal{L}_2(K)$ and $N_2 = \mathcal{L}_2(K)$. Then (N_1, G_1) and (N_2, G_2) are both conditional Banach spaces. Now we apply Ito's formula to the function $\rho : G_1 \to G_2$ defined by $\rho(A,B) = AB$ and to the process $X(t)$. ρ is twice Frechet differentiable; $\rho' : G_1 \to \mathcal{L}(G_1; G_2)$, where $\rho'(A,B)(C,D) = AD + CB$, and $\rho'' : G_1 \to \mathcal{L}(G_1 \times G_1; G_2)$, where $\rho''(A,B)[(C,D),(C',D')] = C'D + CD'$. So

$$\begin{aligned} \rho'(X(t)) &= \rho'(X_1(t), X_2(t))(\xi_1(t), \xi_2(t)) \\ &= \xi_1(t)X_2(t) + X_1(t)\xi_2(t), \end{aligned}$$

and

$$\begin{aligned} \rho''(X(t))(\xi(t) \times \xi(t)) &= \rho''(X_1(t), X_2(t))[(\xi_1(t), \xi_2(t)), (\xi_1(t), \xi_2(t))] \\ &= 2\xi_1(t)\xi_2(t). \end{aligned}$$

So

$$\text{TRACE}\,\rho''(X(t))[\xi(t)\times\xi(t)] = 2\sum_i \xi_1(t)(e_i)\xi_2(t)(e_i).$$

The conclusion of the theorem follows from Ito's formula. □

An important result which will be used in the sequel to prove regularity properties of measures induced by solutions of stochastic differential equations is the Girsanov theorem whose proof is given below (Theorem 4.9).

Lemma 4.8. *Let $f \in \mathcal{M}^2_{0,T}[H]$, and let $E\exp((1+\delta)\int_0^T |f(s)|^2_H\,ds) < \infty$ for some $\delta > 0$. Then the expectation of the random variable*

$$\zeta(t_1,t_2,f) = \exp\left\{\int_{t_1}^{t_2}\langle f(s), dW(s)\rangle - \frac{1}{2}\int_{t_1}^{t_2}|f(s)|^2_H\,ds\right\}$$

is one for every $0 \le t_1 < t_2 \le T$.

Proof. Here we use the fact that the lemma holds when B is finite dimensional (see [**7**]). Now $\zeta(t_1,t_2,f)$ is the limit, almost surely, of a subsequence of the sequence

$$\left\{\exp\left(\int_{t_1}^{t_2}\langle Q_n f(s), dQ_n W(s)\rangle - \frac{1}{2}\int_{t_1}^{t_2}|Q_n f(s)|^2_H\,ds\right)\right\}_{n=1}^{\infty}$$

whose uniform integrability is ensured by the hypothesis of the lemma. The conclusion of the lemma holds because each term of this sequence has expectation one (see [**7**]). □

Theorem 4.9 (Girsanov). *If f and $\zeta(0,T,f)$ are as in Lemma 4.8, then the process $\tilde{W}(t) = W(t) - \int_0^t f(s)\,ds$ is a Brownian motion adapted to $\{\mathcal{F}_t\}_{0\le t\le T}$ with respect to the probability measure $\tilde{\mathcal{P}}(d\omega) = \zeta(0,T,f)\mathcal{P}(d\omega)$.*

Proof. Here again we use the validity of the finite dimensional version of this theorem. For each Q_n (see the paragraph before Theorem 4.2), the process $Q_n\tilde{W}(t) = Q_n W(t) - \int_0^t Q_n f(s)\,ds$ is a

Brownian motion, adapted to $\mathcal{F}_t$, with respect to the measure $\tilde{\mathcal{P}}_n(d\omega) = \zeta(0, T, Q_n f)\mathcal{P}(d\omega)$. For each h, $k \in H$, we have

$$\tilde{E}_n(\langle Q_n(\tilde{W}_t - \tilde{W}_s), h\rangle\langle Q_n(\tilde{W}_t - \tilde{W}_s), k\rangle|\mathcal{F}_s) = \langle Q_n h, Q_n k\rangle t,$$

where $\tilde{E}_n(\cdot|\cdot)$ is the conditional expectation with respect to the measure $\tilde{\mathcal{P}}_n$. Hence the sequence of $\mathcal{F}_s$-measurable random variables

$$\tilde{E}_n(\langle Q_n(\tilde{W}_t - \tilde{W}_s), h\rangle\langle Q_n(\tilde{W}_t - \tilde{W}_s), k\rangle|\mathcal{F}_s)$$

converges to $\langle h, k\rangle t$ almost surely as $n \to \infty$. Now, for each set $A \in \mathcal{F}_s$,

$$\begin{aligned}\tilde{\mathcal{P}}_n(A)\langle Q_n h, Q_n k\rangle t &= \int_A \tilde{E}_n(\langle Q_n(\tilde{W}_t - \tilde{W}_s), h\rangle \\ &\qquad \cdot \langle Q_n(\tilde{W}_t - \tilde{W}_s), k\rangle|\mathcal{F}_s)\zeta(0, T, Q_n f)\, d\mathcal{P} \\ &= \int_A \langle Q_n(\tilde{W}_t - \tilde{W}_s), h\rangle\langle Q_n(\tilde{W}_t - \tilde{W}_s), k\rangle \\ &\qquad \cdot \zeta(0, T, Q_n f)\, d\mathcal{P}.\end{aligned}$$

Taking the limit as $n \to \infty$ while noting the uniform integrability of the sequence of integrands on the right side of the equation, and the fact that $\tilde{\mathcal{P}}_n(A) \to \tilde{\mathcal{P}}(A)$ for each $A \in \mathcal{F}_s$, we obtain

$$\begin{aligned}\langle h, k\rangle t\tilde{\mathcal{P}}(A) &= \lim_n \int_A \tilde{E}_n(\langle Q_n(\tilde{W}_t - \tilde{W}_s), h\rangle \\ &\qquad \cdot \langle Q_n(\tilde{W}_t - \tilde{W}_s), k\rangle|\mathcal{F}_s)\zeta(0, T, Q_n f)\, d\mathcal{P} \\ &= \int_A \tilde{E}(\langle \tilde{W}_t - \tilde{W}_s, h\rangle\langle \tilde{W}_t - \tilde{W}_s, k\rangle|\mathcal{F}_s)\, d\tilde{\mathcal{P}}.\end{aligned}$$

Therefore,

$$\tilde{E}(\langle \tilde{W}_t - \tilde{W}_s, h\rangle\langle \tilde{W}_t - \tilde{W}_s, k\rangle|\mathcal{F}_s) = \langle h, k\rangle t,$$

$\tilde{\mathcal{P}}$ almost surely, and hence $\mathcal{P}$ almost surely. This implies that for each projection $Q : B \to H$ whose restriction to H is orthogonal, the process $Q\tilde{W}_t$ is a standard Brownian motion, with respect to the measure $\tilde{\mathcal{P}}$ in the space QH.

Let f be a bounded continuous function from B to R. Then

$$\begin{aligned}
&\int_B f(Q_n(\tilde{W}_t - \tilde{W}_s))\, d\tilde{\mathcal{P}} \\
&\qquad = \int_{Q_nH} f(x) \frac{1}{\sqrt{2\pi(t-s)}^{\dim Q_nH}} e^{-|x|^2/2(t-s)}\, dx \\
&\qquad = \int_B f(Q_n x) p_{t-s}(dx) \\
&\qquad \longrightarrow \int_B f(x) p_{t-s}(dx),
\end{aligned}$$

since $f(Q_n(x)) \to f(x)$ for each $x \in B_0$, and hence p_{t-s} almost surely. On the other hand,

$$\int_B f(Q_n(\tilde{W}_t - \tilde{W}_s))\, d\tilde{\mathcal{P}} \longrightarrow \int_B f(\tilde{W}_t - \tilde{W}_s)\, d\tilde{\mathcal{P}}.$$

Therefore, $\tilde{\mathcal{P}}((\tilde{W}_t - \tilde{W}_s) \in dx) = p_{t-s}(dx)$.

Now we need to show that if $A \in \mathcal{B}(B)$ and $C \in \mathcal{F}_s$, then

$$\tilde{\mathcal{P}}\{\{(\tilde{W}_t - \tilde{W}_s) \in A\} \cap C\} = \tilde{\mathcal{P}}\{W_t - W_s \in A\}\mathcal{P}(C). \tag{4.5}$$

The fact that for every finite dimensional projection Q, the process $Q\tilde{W}_t$ is a Brownian motion in the space QB implies that (4.5) holds when A is a cylinder set and that the collection of sets A that satisfy (4.5) is an algebra. Furthermore, this collection forms a monotone class (because $\tilde{\mathcal{P}}$ is a probability measure). Therefore, this collection must be the Borel σ-field of B. □

5. Measures induced by B-valued Ito processes. For a fixed $t > 0$, consider the random variable $\int_0^t A(s)\, dW(s)$, where $A(s) = I + K(s)$, $K(s)$ is a deterministic function that takes its values in $\mathcal{L}_2(H)$ with $\int_0^T |K(s)|_{HS}^2\, ds < \infty$ for some $T > 0$. Suppose also that there is some $\varepsilon > 0$ such that $|I + K(s)|_{\mathcal{L}(H)} \geq \varepsilon$ for every $0 < s \leq T$. The last condition imposed on $K(s)$ implies that the restriction $(I + K(s))|_H$ of the bounded linear operator $I + K(s)$ to H is invertible with bounded inverse.

For the moment, we assume that K is a simple function taking values in $\mathcal{L}(B_o; H)$ with jumps at $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$. Then

$$\int_0^t A(s)\, dW(s) = \sum_{j=0}^{n-1} [I + K(t_j)][W(t_{j+1}) - W(t_j)].$$

So for each $h \in B^*$ we have

$$\begin{aligned}\int_\Omega \exp i\Big(h, \int_0^t A(s)\, dW(s)\Big)\, d\mathcal{P} \\ &= \int \exp\bigg(i \sum_{j=0}^{n-1} (h, A(t_j)[W(t_{j+1}) - W(t_j)])\bigg)\, d\mathcal{P} \\ &= \prod_{j=0}^{n-1} \int \exp i(A(t_j)^* h, W(t_{j+1}) - W(t_j))\, d\mathcal{P} \\ &= \prod_{j=0}^{n-1} \exp\Big(- \frac{1}{2}(t_{j+1} - t_j)|A(t_j)^* h|_H^2\Big) \\ &= \exp\Big(- \frac{1}{2}\int_0^t |A(s)^* h|_H^2\, ds\Big).\end{aligned}$$

This equality holds, by a passage to limit, for each K satisfying the conditions stated in the first paragraph of this section. Therefore, for each $h \in B^*$, the function $x \mapsto (h, x)$ is normally distributed, with respect to the measure $\mathcal{P}(\int_0^t A(s)\, dW(s) \in dX)$, with mean zero and variance $|h|_*^2 = \int_0^t |A(s)^* h|_H^2\, ds$. Conditions imposed on K imply that the norms $|\cdot|_*$ and $|\cdot|$ on H are equivalent to each other; hence, the measure $\mathcal{P}(\int_0^t A(s)\, dW(s) \in dx)$ is quasi-invariant and infinitely many times F-differentiable in the directions of H.

Now we use Theorem 4.9 to obtain some results concerning the measures induced by a stochastic differential equation of the type

$$d\xi(t) = A(t) dW(t) + \sigma(t, \xi(t))\, dt, \tag{5.1}$$

where the bounded H-value function σ satisfies the conditions of the existence and uniqueness theorem (see [**12**]), and the coefficient A satisfies the conditions stated above. The solution of (5.1) is a Markov process with transition probability $p(s,x,t,dy) = \mathcal{P}(\xi_{s,x}(t) \in dy)$, where $\xi_{s,x}$ is the solution that satisfies the initial condition $\xi_{s,x}(s) = x$ $\mathcal{P}$ almost surely.

Consider the stochastic processes

$$\xi'_x(t) = x + \int_0^t A(s)\, dW(s)$$

and

$$\tilde{W}(t) = W(t) - \int_0^t A^{-1}(s)\sigma(s, \xi'_x(s))\, ds.$$

By Theorem 4.9, $\tilde{W}(t)$ is a B-valued Brownian motion with respect to the probability measure

$$\tilde{\mathcal{P}}(d\omega) = \exp\left\{ \int_0^T \langle A^{-1}(s)\sigma(s, \xi'_x(s)), dW(s)\rangle - \frac{1}{2}\int_0^T |A^{-1}(s)\sigma(s, \xi'_x(s))|_H^2\, ds \right\} \mathcal{P}(d\omega).$$

Now, using the Girsanov theorem we have

$$\begin{aligned}\mathcal{P}(\xi_x(t) \in dy) &= \tilde{\mathcal{P}}(\xi'_x(t) \in dy)\\ &= E\left(\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}}(\omega) | \xi'_x(t) = y\right)\mathcal{P}(\xi'_x(t) \in dy),\end{aligned}$$

where $E(d\tilde{\mathcal{P}}/d\mathcal{P}|\xi'_x(t) = y) : B \to R$ is a function that belongs to $L^1(\mathcal{P}(\xi'_x \in dy))$. Therefore, the measure $\mathcal{P}_{\xi_x(t)}$, being a measure that has density with respect to a measure that is continuous in the directions of H, is itself continuous in the directions of H (see [**5**]). Furthermore, for each $h \in H$,

$$\begin{aligned}\mathcal{P}(\xi_x(t) \in dy + h) &= \tilde{\mathcal{P}}(\xi'_x(t) \in dy + h)\\ &= E\left(\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}}\Big|\xi'_x(t) = y + h\right)\mathcal{P}(\xi'_x(t) + h \in dy)\\ &\ll \mathcal{P}(\xi'_x \in dy)\\ &= E\left(\frac{d\mathcal{P}}{d\tilde{\mathcal{P}}}(\omega)\Big|\xi'_x(t) = y\right)\mathcal{P}(\xi_x(t) \in dy)\\ &\ll \mathcal{P}(\xi_x(t) \in dy).\end{aligned}$$

Therefore, for each stochastic differential equation of the type (5.1), we obtain a large collection $\{\mathcal{P}_{\xi_x(t)}\}_{x\in B}$ of, in general non-Gaussian, measures that are quasi-invariant and continuous in the directions of H.

Theorem 5.1. *Suppose that the coefficient $\sigma(t,x)$ of (5.1) is n-times H-differentiable in the x-variable with bounded derivatives for each $t>0$. Then the measure $\mathcal{P}_{\xi_x(t)}$ is induced by the solution ξ_x at time $t>0$ is n-times F-differentiable in the directions of H.*

Proof. Let

$$\Gamma(t)=\exp\left\{\int_0^t\langle A^{-1}(s)\sigma(s,\xi_x'(s)),dW(s)\rangle - \frac{1}{2}\int_0^t |A^{-1}(s)\sigma(s,\xi_x'(s))|_H^2\,ds\right\},$$

where $\xi_x'(t)=x+\int_0^t A(s)\,dW(s)$. For each $h\in H$, let $\{\Gamma_h(t)\}_{0<t\le T}$ be the process $\{\Gamma(t)\}_{0<t\le T}$ with $W(t)$ and $\xi_x'(t)$ replaced by the perturbed processes $W_h(t)=W(t)-\int_0^t A(s)^{-1}h\,ds$ and $\xi_{x;h}'(t)=x+\int_0^t A(s)\,dW_h(s)$, respectively. Using Ito's formula with $\rho(x)=e^x$, we see that the process Γ_h satisfies the following stochastic integral equation:

$$\Gamma_h(t)=1+\int_0^t\Gamma_h(s)\langle A(s)^{-1}\sigma(s,\xi_{x;h}'(s)),DW(s)\rangle. \tag{5.2}$$

By Theorem 4.9, the process $\{W_h(s)\}_{0\le s\le t}$ is a B-valued Brownian motion with respect to the probability measure

$$G_h(t)=\exp\left\{\int_0^t\langle A(s)^{-1}h,dW(s)\rangle-\frac{1}{2}\int_0^t|A(s)^{-1}h|_H\,ds\right\}d\mathcal{P}.$$

The process $G_h(t)$ satisfies the following stochastic integral equation:

$$G_h(t)=1+\int_0^t G_h(s)\langle A(s)^{-1}h,dW(s)\rangle. \tag{5.3}$$

By Theorem 4.6 both processes Γ_h and G_h belong to $\mathcal{S}^r_{0,T}[R]$, and the process $\xi'_{x;h}$ belongs to $\mathcal{S}^r_{0,T}[B]$ for each $r \geq 1$. Using an argument similar to that used in [**12**] we can show that, for each $r \geq 1$, the maps $h \mapsto \Gamma_h$ and $h \mapsto G_h$ from H to $\mathcal{S}^r_{0,T}[R]$, and the map $h \mapsto \xi'_{x;h}$ from H to $\mathcal{S}^r_{0,T}[B]$ are Frechet differentiable. It is obvious that $D_h\xi'_{x;h}|_{h=0}(t)(\cdot) = -t(\cdot)$. The derivatives of the maps $h \mapsto \Gamma_h$ and $h \mapsto G_h$ are obtained by formally differentiating equations (5.2) and (5.3) with respect to h and solving the resulting linear stochastic differential equations. This gives us

$$
\begin{aligned}
D_h\Gamma_h\Big|_{h=0}(t)(\cdot) = \bigg\{ -t\int_0^t\int_0^t &\langle A(s)^{-1}[D\sigma(s,\xi'_x(s))(\cdot)], dW(s)\rangle \\
-\int_0^t &\langle A(s)^{-1}\sigma(s,\xi'_x(s)), A(s)^{-1}\rangle\,ds \\
+\,t\int_0^t &\langle A(s)^{-1}[D\sigma(s,\xi'_x(s))(\cdot)], \\
&\qquad A(s)^{-1}\sigma(s,\xi'_x(s))\rangle\,ds\bigg\}\Gamma(t),
\end{aligned}
$$

and

$$
D_hG_h\Big|_{h=0}(t)(\cdot) = \int_0^t \langle A(s)^{-1}(\cdot), dW(s)\rangle.
$$

By the Girsanov theorem, for each $h \in H$ and each function $f \in C_b(B)$ that is differentiable in the directions of H with bounded derivative, the distribution of the random variable $f(\xi'_{x;h}(t))\Gamma_h(t)$ with respect to the measure $G_h(t)\,d\mathcal{P}$ is exactly the same as the distribution of $f(\xi'_x(t))\Gamma(t)$ with respect to $\mathcal{P}$; hence, the function defined by $h \mapsto \int f(\xi'_{x;h}(t))\Gamma_h(t)G_h(t)\,d\mathcal{P}$ is constant. Obviously, the maps $h \mapsto f(\xi'_{x;h}(t))$, $h \mapsto \Gamma_h(t)$, and $h \mapsto G_h(t)$ from H to $L^r(\mathcal{P})$, are all Frechet differentiable. We differentiate with respect to h in the direction of a vector $k \in H$ and then let $h = 0$. The derivative is obviously zero; therefore,

$$\begin{aligned}\int Df(\xi_x'(t))(k)\Gamma(\omega)\mathcal{P}(d\omega) \\ = -\int f(\xi_x'(t))\bigg\{\int_0^t \langle A(s)^{-1}D\sigma(s,\xi_x'(s))(k), dW(s)\rangle \\ + \frac{1}{t}\int_0^t \langle A(s)^{-1}\sigma(s,\xi_x'(s)), A(s)^{-1}k\rangle\,ds \\ - \int_0^t \langle A(s)^{-1}D\sigma(s,\xi_x'(s))(k), A(s)^{-1}\sigma(s,\xi_x'(s))\rangle\,ds \\ - \frac{1}{t}\int_0^t \langle A(s)^{-1}k, dW(s)\rangle\bigg\}\Gamma(t)\,d\mathcal{P} \\ = -\int f(\xi_x'(t))J_k(\omega)\Gamma(t)\,d\mathcal{P}.\end{aligned}$$

Using the equality $\tilde{\mathcal{P}}(\xi_x'(t)\in dx) = \mathcal{P}(\xi_x(t)\in dx)$, where $\tilde{\mathcal{P}} = \Gamma(t)\mathcal{P}$, we obtain

$$\int Df(y)(k)\mathcal{P}_{\xi_x(t)}(dy) = -\int f(x)\tilde{E}(J_k(\omega)\mid \xi_x'(t)=y)\mathcal{P}_{\xi_x(t)}(dy),$$

where $\tilde{E}(\cdot|\cdot)$ denotes conditional expectation with respect to the measure $\Gamma(t)\,d\mathcal{P}$. It follows from Theorem 3.4 that the measure $\mathcal{P}_{\xi_x(t)}$ is Skorokhod-differentiable in the directions of H.

Recall [consequence (B) of Definition 3.2] that there exists a vector-valued measure $[\mathcal{P}_{\xi_x(t)}]'$ such that, for each $h\in B^*$,

$$\Big\langle [\mathcal{P}_{\xi_x(t)}]'(A), h\Big\rangle = d_{S;h}\mathcal{P}_{\xi_x(t)}(A)$$

for each Borel set A. Obviously,

(5.4)

$$\begin{aligned}[\mathcal{P}_{\xi_x(t)}]' = E\Bigg(\bigg\{ &\int_0^t \left(D\sigma(s,\xi_x'(s))^*[A(s)^{-1}]^* - \frac{1}{t}[A(s)^{-1}]^*\right) dW(s) \\ &+ \int_0^t \bigg(\frac{1}{t}[A(s)^{-1}]^* A(s)^{-1}\sigma(s,\xi_x'(s)) \\ &- D\sigma(s,\xi_x'(s))^*[A(s)^{-1}]^* A(s)^{-1}\sigma(s,\xi_x'(s))\bigg)\bigg\} \\ &\times \exp\bigg(\int_0^t \langle A(s)^{-1}\sigma(s,\xi_x'(s)), dW(s)\rangle \\ &- \frac{1}{2}\int_0^t |A(s)^{-1}\sigma(s,\xi_x'(s))|_H^2\, ds\bigg) \bigg| \xi_x'(t) = y\Bigg)\mathcal{P}_{\xi_x'(t)} \\ &= g(y)\mathcal{P}_{\xi_x'(t)}(dy).\end{aligned}$$

Note that $\mathcal{P}_{\xi_x'(t)}$ is quasi-invariant in the directions of H, and

$$[\mathcal{P}_{\xi_x'(t)}]_h = f^h \mathcal{P}_{\xi_x'(t)},$$

where $f^h \to 1$ in $L^p(\mathcal{P}_{\xi_x'(t)})$ for every $p > 1$, as $h \to 0$. So we have

$$[\mathcal{P}_{\xi_x(t)}]_h' = g(y+h) f^h [\mathcal{P}_{\xi_x'(t)}](dy),$$

and $g(y+h)$ is the density of $[P_{\xi_x(t)}]'$ with respect to $\mathcal{P}_{\xi_x'(t)}$ in (5.4) with $\xi_x'(s)$ replaced by $\xi_x'(s) + h$. Now

$$\begin{aligned}\|[\mathcal{P}_{\xi_x(t)}]_h' - [\mathcal{P}_{\xi_x(t)}]'\| &= \|g(\cdot+h)[\mathcal{P}_{\xi_x'(t)}]_h - g(\cdot)\mathcal{P}_{\xi_x'(t)}\| \\ &= \|g(\cdot+h) f^h \mathcal{P}_{\xi_x'(t)} - g(\cdot)\mathcal{P}_{\xi_x'(t)}\| \\ &= \|g(\cdot+h) f^h \mathcal{P}_{\xi_x'(t)} - g(\cdot) f^h \mathcal{P}_{\xi_x'(t)} \\ &\quad + g(\cdot) f^h \mathcal{P}_{\xi_x'(t)} - g(\cdot)\mathcal{P}_{\xi_x'(t)}\| \\ &\le \int \|g(\cdot+h) - g(\cdot)\|_B |f^h| \mathcal{P}_{\xi_x'(t)} \\ &\quad + \int \|g\| |f^h - 1| \mathcal{P}_{\xi_x'(t)}.\end{aligned}$$

Conditions imposed on σ and A ensure that

$$\int \|g(\xi_x'(t)+h) - g(\xi_x')\|_B^r \, d\mathcal{P} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0,$$

for each $r \geq 1$. Therefore, the vector-valued measure $[\mathcal{P}_{\xi_x(t)}]'$ is continuous in the directions of H. This in turn implies that $\mathcal{P}_{\xi_x(t)}$ belongs to $C^1_{F;H}$ [see consequence (D) of Definition 3.2].

Let $k_1, k_2 \in H$, and let $J_{k_1,h}$ denote the random variable J_{k_1} perturbed by replacing W and $\xi'_x(t)$ by W_h and $\xi'_{x;h}(t)$, respectively. By the Girsanov theorem, the distribution of the random variable $f(\xi'_{x;h}(t))J_{k_1,h}(\omega)\Gamma_h(t)$ with respect to the measure $G_h(t)\,d\mathcal{P}$ is independent of h and is the same as the distribution of the random variable $f(\xi'_x(t))J_{k_1}$ with respect to the measure $\mathcal{P}$. Therefore the map

$$h \longmapsto \int Df(\xi'_{x,h}(t))(k_2)J_{k_1,h}\Gamma_h(t)G_h(t)\,d\mathcal{P}$$

is a constant map. Taking the derivative with respect to h in the direction of k_1 and then setting h equal to zero and proceeding as above gives us twice the F-differentiability of $\mathcal{P}_{\xi_x(t)}$ in the directions of H. Continuing in this manner, we obtain n-times F-differentiability of $\mathcal{P}_{\xi_x(t)}$ in the directions of H.

Now we consider a stochastic integral equation of the form

$$\xi_x(t) = x + \int_0^t A(s, \xi_x(s))\,dW(s), \tag{5.5}$$

where the coefficient A satisfies the following conditions:

(A1) $A(t,y) = I + K(t,y)$ where $K : [0,t) \times B \to \mathcal{L}_2(H)$ is a bounded continuous map that is $H - C^2$ in the y variable; $DK(\cdot,\cdot)$ and $D^2K(\cdot,\cdot)$ are bounded continuous maps from $[0,T) \times B$ into $\mathcal{L}_2(H;\mathcal{L}_2(H))$ and $\mathcal{L}_2(H;\mathcal{L}_2(H;\mathcal{L}_2(H)))$, respectively.

The process ξ_x belongs to $\mathcal{S}^p_{0,T}(B)$ for every $p \geq 1$, and the map $x \mapsto \xi_x$ from B to $\mathcal{S}^p_{0,T}(B)$ is twice differentiable in the directions of H. Let $Y = D_x\xi_x$. Then the process $Y \in \mathcal{S}^p_{0,\infty}[L(H)]$ satisfies the following operator-valued stochastic integral equation

$$Y(t) = I + \int_0^t DK(s, \xi_x(s))(Y(s))\,dW(s).$$

An application of Ito's formula to the conditional Banach space $(\mathcal{L}_2(H), \mathcal{L}(H))$, the process Y, and the function $\rho(A) = A^{-1}$ gives us the following linear stochastic integral equation which has a unique solution by the existence and uniqueness theorem [**9**].

$$\bar{Y}(t) = I - \int_0^t \bar{Y}(s) DK(s, \xi_x(s))^* \, dW(s) + \int_0^t \bar{Y}(s) |DK(s, \xi_x(s))|_{HS}^2 \, ds.$$

Theorem 4.7 can now be used to show that $\bar{Y}(\cdot)$ is indeed the inverse process of $Y(\cdot)$.

(A2) Assume that there exists $\varepsilon > 0$ such that for every $h \in H$, $s \in [0, T]$ and $x \in B$, $|A(s, x)h|_H \geq \varepsilon |h|_H$.

Let $\phi^m(s) = A(s, \xi_x(s))^{-1} Y(s)$ if $|A(s, \xi_x(s))^{-1} Y(s)|_{L(H)} \leq m$ and zero otherwise. For each $h \in H$, consider the perturbed process $W^{m,h}(t) = W(t) - \int_0^t \phi^m(s)(h) \, ds$. By Girsanov's theorem, $\{W^{m,h}(s)\}_{0 \leq s < t}$ is a Brownian motion (adapted to $\{\mathcal{F}_s\}_{0 \leq s \leq t}$) with respect to the probability measure $\mathcal{P}_h(d\omega) = G_h(t)\mathcal{P}(d\omega)$ where $G_h(t)$ is the Girsanov density $\exp\{\int_0^t \langle \phi^m(s)h, dW(s)\rangle - \int_0^t |\phi^m(s)|^2 \, ds\}$.

Consider the perturbed stochastic integral equation

$$\begin{aligned} \xi_x^{m,h}(t) &= x + \int_0^t A(s, \xi_x^{m,h}(s)) \, dW^{m,h}(s) \\ &= x + \int_0^t A(s, \xi_x^{m,h}(s)) \, dW(s) \\ &\quad - \int_0^t \{A(s, \xi_x^{m,h}(s)) \phi^m(s)(h)\} \, ds. \end{aligned}$$

The map $h \mapsto \xi_x^{m,h}$ from H to $\mathcal{S}_{0,T}^p[B]$ is Frechet differentiable (as can be proved easily by the method used in [**12**]), and the process $\eta^m = D_h \xi_x^{m,h}|_{h=0}$ satisfies the following stochastic integral equation:

$$\eta^m(t) = \int_0^t DK(s, \xi_x(s))(\eta^m(s)) \, dW(s) - \int_0^t A(s, \xi_x(s)) \phi^m(s) \, ds.$$

By the method of variation of parameters, we see that

$$\eta^m(t) = -Y(t) \int_0^t \bar{Y}(s) A(s, \xi_x^{m,h}(s)) \phi^m(s) \, ds \quad \mathcal{P} \text{ a.s.}$$

Note that almost all sample paths of the processes $\bar{Y}$ and Y are bounded since both belong to the affine space $I + \mathcal{S}^p_{0,T}[\mathcal{L}_2(H)]$ for every $p \geq 1$. By the Lebesgue dominated convergence theorem, for almost all $\omega \in \Omega$, $\eta^m(t)$ converges to $-tY(t)$ as $m \to \infty$.

Next, for each $h \in H$, consider the perturbed stochastic integral equation

$$\begin{aligned} Y^{m,h}(t) &= I + \int_0^t DK(s, \xi_x^{m,h}(s))(Y^{m,h}(s))\, dW^{m,h}(s) \\ &= I + \int_0^t DK(s, \xi_x^{m,h}(s))(Y^{m,h}(s))\, dW(s) \\ &\quad - \int_0^t \{DK(s, \xi_x^{m,h}(s))(Y^{m,h}(s))\phi^m(s)h\}\, ds. \end{aligned}$$

The map $h \mapsto Y^{m,h}$ from H to $\mathcal{S}^p_{0,T}[L(H)]$ is Frechet differentiable for each $p \geq 1$. The process $\zeta^m = D_h Y^{m,h}|_{h=0}$ satisfies the following stochastic integral equation

$$\begin{aligned} \zeta^m(t)(\cdot) = \int_0^t & D^2K(s, \xi_x(s))(\eta^m(s)(\cdot))(Y(s), dW(s)) \\ & - \int_0^t DK(s, \xi_x(s))(Y(s))\phi^m(s)(\cdot)\, ds \\ & + \int_0^t DK(s, \xi_x(s))(\zeta^m(s)(\cdot))(dW(s)). \end{aligned}$$

Since the process

$$\begin{aligned} (5.6) \quad \Big\{ \int_0^u & D^2K(s, \xi_x(s))(\eta^m(s)(\cdot))(Y(s), dW(s)) \\ & - \int_0^u DK(s, \xi_x(s))(Y(s))\phi^m(s)(\cdot)\, ds \Big\}_{0 \leq u \leq T} \end{aligned}$$

belongs to $\mathcal{S}^p_{0,T}[\mathcal{L}_2(H; \mathcal{L}_2(H))]$, it follows from Theorem 4.6 that ζ^m exists as an element in $\mathcal{S}^p_{0,T}[\mathcal{L}^2_2(H)]$. Obviously, the sequence $\{\zeta^m\}$ converges in $\mathcal{S}^p_{0,T}[\mathcal{L}_2(H; \mathcal{L}_2(H))]$ to a process that satisfies equation (5.6) with ζ^m, ϕ^m and η^m replaced by ζ, $A(s, \xi_x(s))^{-1}Y(s)$ and $-tY(t)$, respectively.

Conditions imposed on K and DK give us Frechet differentiability of the map $h \mapsto Y^{m,h}(t)$ from H to $L^p(\mathcal{P})$ for every $p \geq 1$. Therefore, the map $h \mapsto \bar{Y}^{m,h}(t)$ from H to $L^p(\mathcal{P})$ is also differentiable for each $t \geq 0$ and all $p \geq 1$ and

$$\begin{aligned} D_h \bar{Y}^{m,h}(t)\Big|_{h=0}(\cdot) &= -\bar{Y}(t) D_h Y^{m,h}(t)\Big|_{h=0}(\cdot)\bar{Y}(t) \\ &= -\bar{Y}(t)\zeta^m(t)(\cdot)\bar{Y}(t). \end{aligned}$$

Differentiability of the map $h \mapsto G_h(t)$ from H to $L^p(\mathcal{P})$, for each $p \geq 1$, follows easily from the fact that G_h satisfies the stochastic integral equation $G_h(t) = 1 + \int_0^t G_h(s)\langle \phi^m(s)(h), dW(s)\rangle$; to obtain the derivative, simply solve the equation obtained by formally differentiating both sides of this equation with respect to h.

Let f be a bounded uniformly continuous function that is differentiable in the directions of H with bounded derivative. For each $k \in H$, the map

$$\theta : h \longmapsto \int f(\xi_x^{m,h})\langle \bar{Y}^{m,h}(t)(k), e_i\rangle G_h(t)\, d\mathcal{P}$$

is Frechet differentiable. Now, by Theorem 4.9, the distribution of the random variable $f(\xi_x^{m,h}(t))\langle \bar{Y}^{m,h}(t)(k), e_i\rangle$ with respect to the measure $G_h(t)\mathcal{P}$ is exactly the same as the distribution of the random variable $f(\xi_x(t))\langle \bar{Y}(t)(k), e_i\rangle$ with respect to the measure $\mathcal{P}$. Therefore the function θ is a constant function. Differentiating θ with respect to h in the direction of e_i and then setting $h = 0$ gives us

$$\begin{aligned} &\int Df(\xi_x(t))\eta^m(t)(e_i)\langle \bar{Y}(t)(k), e_i\rangle\, d\mathcal{P} \\ &\quad = \int f(\xi_x(s))\Big\{- \langle \bar{Y}(t)(\zeta^m(t)(e_i))\bar{Y}(t)(k), e_i\rangle \\ &\qquad\qquad + \langle \bar{Y}(t)(k), e_i\rangle \int_0^t \langle \phi^m(s)(e_i), dW(s)\rangle\Big\}\, d\mathcal{P}. \end{aligned}$$

Taking the limit as $m \to \infty$, we get
(5.7)

$$\begin{aligned}
&\int Df(\xi_x(t))Y(t)(\langle \bar{Y}(t)(k), e_i\rangle e_i)\, d\mathcal{P} \\
&\quad = -\int f(\xi_x(t)) \bigg\{ \frac{1}{t}\, \langle \bar{Y}(t)(\zeta(t)(e_i))\bar{Y}(t)(k), e_i\rangle \\
&\qquad\qquad - \frac{1}{t}\, \langle \bar{Y}(t)(k), e_i\rangle \int_0^t \langle A(s, \xi_x(s))^{-1}(e_i), dW(s)\rangle \bigg\}\, d\mathcal{P}.
\end{aligned}$$

Now

$$\begin{aligned}
E\bigg|\sum_{i=n}^{m} Df(\xi_x(t))Y(t)(\langle \bar{Y}(t)(k), e_i\rangle e_i)\bigg|^2 \\
&\le C_1 E\bigg|Y(t)\bigg(\sum_{i=n}^{m}\langle \bar{Y}(t)(k), e_i\rangle e_i\bigg)\bigg|^2 \\
&\le C_1 E\bigg\{\Big|Y(t)\Big|^2_{\mathcal{L}(H)} \bigg|\sum_{i=n}^{m}\langle \bar{Y}(t)(k), e_i\rangle e_i\bigg|^2\bigg\} \\
&= C_1 E\bigg\{\Big|Y(t)\Big|^2_{\mathcal{L}(H)} \sum_{i=n}^{m}\langle \bar{Y}(t)(k), e_i\rangle^2\bigg\} \\
&\longrightarrow 0 \quad \text{as} \quad m, n \longrightarrow \infty.
\end{aligned}$$

So the sum of the lefthand side of (5.7) converges to $\int Df(\xi_x(t))(k)\, d\mathcal{P}$. Unfortunately the sum (with respect to i) of the integrand in the righthand side of the equation (5.7) does not always converge in $L^1(d\mathcal{P})$. To ensure convergence, we need to impose further assumptions on K. Below, we give one case in which convergence in $L^1(d\mathcal{P})$ of the right side of (5.7) is ensured. □

Theorem 5.2. *Suppose the coefficient A satisfies the conditions* (A1) *and* (A2) *above. In addition, suppose $K(s, x) = CK'(s, x)$ where $C \in \mathcal{L}(H; B^*)$ and K' satisfies the same conditions mentioned in* (A1) *and* (A2). *Then the measure induced by the solution of the stochastic integral equation* (5.5) *is Skorokhod differentiable in the directions of H.*

Proof. Using the fact that restriction of each bounded linear operator in $\mathcal{L}(B; H)$ to H is Hilbert Schmidt when considered as an element of

$\mathcal{L}(H)$ (see [**9**, Lemma 5.1]), we can prove that C must also be Hilbert Schmidt when considered as an element of $\mathcal{L}(H)$. The argument is as follows. Since the B^*-norm is stronger than the H-norm, C can be thought of as an element $\bar{C}$ of $\mathcal{L}(H)$, and as such, it has an adjoint $\bar{C}^* \in \mathcal{L}(H)$. Next we consider the adjoint $C^* : B^{**} \to H^* = H$, where H^* and H are identified via Riesz representation theorem. The operator $C^*\tau$, where τ is the natural embedding of B into B^{**}, can be considered as a bounded linear map from B to H. Hence $C^*\tau|_H \in \mathcal{L}_2(H)$. But for each $h, k \in H$,

$$\begin{aligned}\langle C^*\tau|_H k, h\rangle &= \langle C^*\tau(k), h\rangle \\ &= \tau(k)(C(h)) \\ &= C(h)(k) \\ &= \langle \bar{C}(h), k\rangle \\ &= \langle h, \bar{C}^* k\rangle.\end{aligned}$$

Therefore the Hilbert Schmidt operator $C^*\tau|_H$ is the adjoint of $\bar{C}$. This implies that $\bar{C}$ is a Hilbert Schmidt operator.

Under the above assumptions, the process Y which solves the equation

$$Y(t) = I + C\int_0^t DK'(s, \xi_x(s))(Y(s))(dW(s))$$

belongs to the affine space $I + \mathcal{S}^p_{0,T}[\mathcal{L}(H; B^*)]$. Its inverse process $\bar{Y}$ which satisfies another linear stochastic integral equation belongs to the same space.

Now we prove the convergence in $L^2(d\mathcal{P})$ of the sum

$$\sum_{i=1}^{\infty}\langle \bar{Y}(t)(k), e_i\rangle \int_0^t \langle A(s, \xi_x(s))^{-1} Y(s)e_i, dW(s)\rangle,$$

where $\bar{Y}(t) = I + Z_1(t)$, $Z_1 \in \mathcal{S}^p_{0,T}[\mathcal{L}(H; B^*)]$ for every $p \geq 1$, and $A(t, \xi_x(t))^{-1}Y(t) = I + Z_2(t)$, $Z_2 \in \mathcal{S}^p_{0,T}[\mathcal{L}_2(H)]$ for all $p \geq 1$.

Therefore,

$$\sum_{i=1}^{\infty}\langle\bar{Y}(t)(k),e_i\rangle\int_0^t\langle A(s,\xi_x(s))^{-1}Y(s)e_i,dW(s)\rangle \tag{5.8}$$
$$=\sum_{i=1}^{\infty}\Big(\langle k,e_i\rangle(e_i,W(t))+\langle Z_1(t)(k),e_i\rangle(e_i,W(t))$$
$$+\langle k,e_i\rangle\int_0^t\langle Z_2(s)e_i,dW(s)\rangle$$
$$+\langle Z_1(t)(k),e_i\rangle\int_0^t\langle Z_2(s)(e_i),dW(s)\rangle\Big).$$

The sum $\sum_{i=1}^{\infty}\langle k,e_i\rangle(e_i,W(t))$ converges in $L^2(d\mathcal{P})$ to the random variable $\langle k,W(t)\rangle$ which is defined $\mathcal{P}$-almost surely.

Next, noting that $Z_1(t)(k)$ takes its values in B^*, we obtain

$$\sum_{i=1}^{n}\langle Z_1(t)(k),e_i\rangle(e_i,W(t))=\Big(Z_1(t)(k),\Big(\sum_{i=1}^{n}(e_i,W(t))\Big)e_i\Big)$$
$$\longrightarrow(Z_1(t)(k),W(t))\quad\text{in}\quad L^1(d\mathcal{P}).$$

Next we have

$$E\Big|\sum_{i=1}^{\infty}\langle k,e_i\rangle\int_0^t\langle Z_2(s)e_i,dW(s)\rangle$$
$$\leq E|k|\sqrt{\sum_{i=1}^{\infty}\Big(\Big\langle\int_0^t Z_2(s)(e_i),dW(s)\Big\rangle\Big)^2}$$
$$=|k|\sqrt{E\sum_{i=1}^{\infty}\int_0^t\langle Z_2(s)(e_i),dW(s)\rangle^2}$$
$$=|k|\sqrt{\sum_{i=1}^{\infty}E\Big(\Big\langle\int_0^t Z_2(s)^*(dW(s)),e_i\Big\rangle\Big)^2}$$
$$=|k|\sqrt{E\Big|\int_0^t\langle Z_2(s)^*,dW(s)\rangle\Big|^2}.$$

Now we consider the last series in (5.8).

$$E\sum_{i=1}^{\infty}\langle Z_1(t)k, e_i\rangle \int_0^t \langle Z_2(s)(e_i), dW(s)\rangle$$
$$\leq E|Z_1(t)k|\left|\int_0^t Z_2(s)^*(dW(s))\right|$$
$$< \infty.$$

By Theorem 3.4, the measure $\mathcal{P}(\xi_x(t) \in dx)$ is Skorokhod differentiable.

Consider the following stochastic differential equation

$$d\xi(t) = A(t, \xi(t))\, dW(t) + \sigma(t, \xi(t))\, dt, \tag{5.9}$$

where the coefficients A and σ satisfy the conditions of the existence and uniqueness theorem. As is well known, there is a B-valued Markov process whose transition probability coincides with $p(t, x; s, dy) = \mathcal{P}(\xi_{x,t}(s) \in dy)$, where the process $\xi_{x,t}$ is the solution that satisfies the initial condition $\xi_{x,t}(t) = x$ almost surely. □

Theorem 5.3. *If the transition probability $p(s, x; t, dy)$ that is associated with the solution of* (5.9) *is twice Skorokhod differentiable in the directions of H, then $(\partial/\partial t)p(s, x; t, dy)$ exists in $\mathcal{D}'$, and the following hold as distributions in $\mathcal{D}'$.*

$$\frac{\partial}{\partial t}p(s, x; t, dy) = -\operatorname{trace}(\sigma(t, y)p(s, x; t, dy))'$$
$$+ \frac{1}{2}\operatorname{TRACE}(A(t, y)^2 p(s, x; t, dy))''.$$

For definitions of trace *and* TRACE*, see consequences* H *and* I *of Definition* 3.2.

Proof. For each $f \in \mathcal{D}$ we have

$$\begin{aligned}
&\frac{1}{h}\int_B f(y)\{p(s,x;t+h,dy) - p(s,x;t,dy)\} \\
&\quad = \frac{1}{h}E\{f(\xi_{s,x}(t+h)) - f(\xi_{s,x}(t))\} \\
&\quad = \frac{1}{h}E\bigg(\int_t^{t+h} \langle A(u,\xi_{s,x}(u))^* D^2 f(\xi_{s,x}(u)), dW(u)\rangle \\
&\qquad\qquad + \int_t^{t+h} \langle Df(\xi_{s,x}(u)), \sigma(u,\sigma_{s,x}(u))\rangle \\
&\qquad\qquad + \frac{1}{2}\operatorname{trace} A(u,\xi_{s,x}(u))^* D^2 f(u,\xi_{s,x}(u)) A(u,\xi_{s,x}(u)))\, du\bigg) \\
&\quad = \frac{1}{h}E\int_t^{t+h} \bigg(\langle Df(\xi_{s,x}(u)), \sigma(u,\xi_{s,x}(u))\rangle \\
&\qquad\qquad \cdot \frac{1}{2}\operatorname{trace} A(u,\xi_{s,x}(u))^* D^2 f(u,\xi_{s,x}(u)) A(u,\xi_{s,x}(u))\bigg)\, du.
\end{aligned}$$

Taking the limit as h approaches zero, and using Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
\left(\frac{\partial}{\partial t}p(s,x;t,dy), f\right) &= E\bigg(\langle Df(\xi_{s,x}(t)), \sigma(t,\xi_{s,x}(t))\rangle \\
&\qquad + \frac{1}{2}\operatorname{trace} A(t,\xi_{s,x}(t))^* D^2 f(t,\xi_{s,x}(t)) A(t,\xi_{s,x}(t))\bigg) \\
&= \int_B \bigg(\langle Df(y), \sigma(t,y)\rangle \\
&\qquad + \frac{1}{2}\operatorname{trace} A(t,y)^* D^2 f(t,y) A(t,y)\bigg) p(s,x;t,dy) \\
&= \bigg(-\operatorname{trace}(\sigma(t,y)p(s,x;t,dy))' \\
&\qquad + \frac{1}{2}\operatorname{TRACE}(A(t,y)^2 p(s,x;t,dy))'', f\bigg).
\end{aligned}$$

To obtain the last equality, we used consequences H and I of Definition 3.2. We should also note that $(\partial/\partial t)p(s,x;t,dy) \in \mathcal{D}'$ because convergence of a net f_α to f in $\mathcal{D}$ means, among other things, that

$Df_\alpha \to Df$ pointwise and boundedly, and trace $D^2 f_\alpha \to$ trace $D^2 f$ pointwise and boundedly. □

REFERENCES

1. V.I. Averbuh, O.G. Smoljanov and S.V. Fomin, *Generalized functions and differential equations in linear spaces*, I. *Differentiable measures*, Trans. Moscow Math. Soc. **24** (1971), 140–184.

2. K. Bichteler and D. Fonken, *A simple version of the Malliavin calculus in dimension N*, Birkhauser, Boston, 1982.

3. V.I. Bogachev, *Some results on differentiable measures*, Math. USSR Sb. **55** (1986), 335–349.

4. ———, *Differentiability of measures in the sense of Skorokhod*, Theory Probab. Appl. **33** (1988), 330–334.

5. Yu.L. Daletskii and S.V. Fomin, *Measures and differential equations in infinite dimensional spaces*, Nauka, Moscow, 1983.

6. N. Dunford and J.T. Schwartz, *Linear operators*, I. *General theory*, Interscience, New York, 1958.

7. A. Friedman, *Stochastic differential equations and applications*, Vol. 1, Academic Press, New York, 1975.

8. L. Gross, *Abstract Wiener spaces*, Math. Statist. Probab. **2** (1965), 31–42.

9. H-H. Kuo, *Gaussian measures in Banach spaces*, Lecture Notes in Math. **463** (1975).

10. ———, *Differentiable measures*, Chinese J. Math. **2** (1974), 188–198.

11. ———, *Stochastic integrals in abstract Wiener space*, Pacific J. Math. **41** (1972), 469–483.

12. ———, *Stochastic integrals in abstract Wiener space* II: *Regularity properties*, Nagoya Math. J. **50** (1973), 89–116.

13. ———, *Distribution theory on Banach spaces*, Lecture Notes in Math. **526** (1975), 143–156.

14. ———, *Potential theory associated with Uhlenbeck-Ornstein process*, J. Funct. Anal. **21** (1976), 63–75.

Department of Mathematics and Physics, Fort Valley State College, 1005 State College Dr., Fort Valley, GA 31030