ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 2, Spring 1997

HELLINGER INTEGRALS FOR VECTOR FUNCTIONS

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ABSTRACT. One extends the notion of Hellinger integral for vector valued functions with respect to a scalar function, and one gives representation theorems for bounded linear operators $F : Q_0(T, X^*) \to \mathbf{R}$, $F : V_u^0(T, X) \to Y$ and $F : M(T, X) \to Y$ satisfying a certain boundedness condition. Here X and Y are arbitrary Banach spaces. The other spaces are defined in Sections 2, 3 and 4, respectively.

1. Introduction. The basic concept in this paper is the Hellinger integral for point functions or for set functions. The aim of the work is to present the problem of representation of the linear continuous operators using the Hellinger integral. The notion of Hellinger integral appeared when L.V. Kantorovitch considered (in 1940) vector integrals of the Hellinger type in connection with the problem of representation of linear operators. In [**19**] he introduces an integral of Hellinger type for a pair of functions f and g defined on the closed interval $[a, b]$ of the real axis where f has real values and g is a vector function.

The quoted paper contains theorems of representation for linear continuous operators which are defined on an ordered linear space of real functions (M or L) into an ordered linear space or a Banach space. The vector integrals of Hellinger type are also used in the spectral theory of self-adjoint operators (see [**25**]).

Romulus Cristescu [**3**] introduces a vector integral of Hellinger type for functions with values in a Banach space, with respect to a function whose values are operators. He establishes with this integral the general form of the linear continuous operators defined on the space of vector functions which are Bochner integrable. Afterwards, he gives further representations in [**4, 5, 6**].

In 1967, J. Webb, I.I. Alexandrov and H. Salehi considered also different types of vector Hellinger integrals and, using them, they established the general form of some linear operators. In [**34**], J. Webb considered the space of complex quasi-continuous functions on a closed

Received by the editors on April 26, 1994.

interval $[a, b]$ of the real axis, organized as a normed linear space with the "sup" norm, and he proved that any bounded linear functional on this space has a Hellinger type for summable vector set functions. The relationship between Alexandrov's integral and the Hellinger integral is analogous to the one between the Lebesgue integral and the Riemann integral. Alexandrov uses the integral introduced by him to the representation of some linear functionals or for the representation of some linear operators defined on spaces of set summable functions. H. Salehi [**24**] defines a Hellinger integral for measures with matrix values, and he generalizes some of U. Grenander and G. Szego's results [**2**] and some of E.W. Hobson's results [**15**]. The integral introduced by Salehi is important in stochastic process theory, and it allows the generalization of some of P. Masani's results [**21**] and of some of A.N. Kolmogorov's results [**20**].

Some other types of Hellinger vector integrals and some theorems of representation of the linear operators using these integrals were also presented by G. Grigore in [**13**] and [**14**] and by G. Vraciu in [**27, 28, 29**, **30, 31, 32, 33**].

In the first part of this paper, Section 2, we consider the space $Q_0(T, X)$ of all functions defined on the compact interval T with values in the dual space X^*, functions obtained as limits of sequences of step functions on T. As in [**34**] one gives a theorem of representation for linear bounded functionals defined on $Q_0(T, X^*)$.

In Section 3 the space $V_u(T, X)$ is the space of functions $v : T \to X$ with the slope variation bounded with respect to the increasing function u, where X is a Banach space. Here we extend the results from Section 2, proving the theorem of representation for linear bounded operators defined on this space. In Section 3 we consider the space $M(T, X)$ which is the space of functions $f \in L(T, X)$ with the properties:

1. There is a set $\{f_n\}_{n \in \mathbf{N}}$ of simple functions (with values in X) such that $\|f_n(t)\| \leq \lambda$ almost everywhere for all $n \in \mathbf{N}$.

2. $\|f_n(t) - f(t)\| \to 0$ almost everywhere.

We prove that the bounded operators defined on this space can be represented using Hellinger integrals with respect to absolutely continuous functions. This representation is similar to the representation given in [**3**] for continuous operators defined on $L(T, X)$.

2. Hellinger vector integrals. We introduce the Hellinger integral for vector functions with respect to a scalar function, generalizing the basic idea from [**34**]. Using this we shall give a representation for the linear bounded functionals defined on the space $Q_0(T, X^*) = \{x : T \to X^*,\ x \text{ is the limit of step functions on } T\}$.

2.1. *Preliminaries.* Let X be a real Banach space; $T = [a, b]$ a segment of the real axis. We shall denote by $B(T, X)$ the set of the bounded functions, organized as a normed space with respect to the usual operations and the supremum norm $\|x\| = \sup_{t\in T} \|x(t)\|$.

Consider the real functions:

$$R_c(t) = \begin{cases} 0 & \text{if } t \in [a, c) \\ 1 & \text{if } t \in [c, b] \end{cases}$$

for any $c \in (a, b]$ and

$$L_c(t) = \begin{cases} 0 & \text{if } t \in [a, c] \\ 1 & \text{if } t \in (c, b] \end{cases}$$

for any $c \in [a, b)$.

We'll call a step function, any linear combination of the form:

$$s(t) = \sum_{k=1}^{m} \alpha_k R_{c_k}(t) x_k + \sum_{k=m+1}^{n} \alpha_k L_{c_k}(t) x_k$$

with $x_k \in X$.

Let $Q_0(T, X)$ be the closure in $B(T, X)$ of the space of the step functions. $Q_0(T, X)$ is a Banach space.

Definition 2.1. Let $v : T \to X$ and $u : T \to \mathbf{R}$, u strictly increasing. v has bounded slope variation with respect to u if, for every subdivision $\{t_p\}_0^n$ of $T = [a, b]$, there exists a number B such that:

$$\sum_{p=1}^{n-1} \left\| \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right\|_X \le B.$$

The lower bound for all such Bs is called the *slope variation* of v with respect to u over $[a,b]$, and it is denoted by $\operatorname{var}_a^b(dv/du)$.

Definition 2.2. Let $u:[a,b]\to\mathbf{R}$ strictly increasing, $v:[a,b]\to X$ and $w:[a,b]\to\mathcal{B}(X,Y)$ where $\mathcal{B}(X,Y)$ is the space of all the linear bounded operators from X into the Banach space Y. The *Hellinger integral*

$$\int_a^b \frac{(dw)(dv)}{du}\in Y$$

exists if, for any number $\varepsilon>0$, there is a division D_ε such that, for any refinement $\{t_p\}_0^n$ of the division D_ε, we have

$$\bigg\|\int_a^b \frac{(dw)(dv)}{du}-\sum_{p=1}^{n-1}\frac{(\Delta w(t_p))(\Delta v(t_p)}{\Delta u(t_p)}\bigg\|_Y<\varepsilon$$

where

$$\begin{aligned}\Delta u(t_p)&=u(t_p)-u(t_{p-1}),\\ \Delta v(t_p)&=v(t_p)-v(t_{p-1}),\\ \Delta w(t_p)&=w(t_p)-w(t_{p-1}).\end{aligned}$$

The following two properties result directly from the definition of the Hellinger integral.

1. If

$$\int_a^b \frac{(dw)(dv_i)}{du},\quad i=1,2,$$

exists, then there exists

$$\int_a^b \frac{(dw)(dv)}{du}\quad \text{where } v=\alpha v_1+\beta v_2$$

and

$$\int_a^b \frac{(dw)(dv)}{du}=\alpha\int_a^b \frac{(dw)(dv_1)}{du}+\beta\int_a^b \frac{(dw)(dv_2)}{du}.$$

2. If

$$\int_a^b \frac{(dw_i)(dv)}{du},\quad i=1,2,$$

exists, then there exists

$$\int_a^b \frac{(dw)(dv)}{du} \quad \text{where } w = \alpha w_1 + \beta w_2$$

and

$$\int_a^b \frac{(dw)(dv)}{du} = \alpha \int_a^b \frac{(dw_1)(dv)}{du} + \beta \int_a^b \frac{(dw_2)(dv)}{du}.$$

Definition 2.3. Let $u : [a, b] \to \mathbf{R}$ be a strictly increasing function, $v : [a, b] \to X$ and $c \in [a, b)$. $D_u^+(c)$ is the *right derivative* of the function v with respect to the function u at c, if

$$\lim_{t \to c^+} \left\| \frac{v(t) - v(c)}{u(t) - u(c)} - D_u^+ v(c) \right\|_X = 0.$$

$D_u^- v(c)$ can be defined similarly for any $c \in (a, b]$.

We shall prove some propositions that will be used in the next sections.

Lemma 2.1. *If n is an integer greater than 2, and $x_0, x_1, \dots, x_n$ are vectors in X and $\alpha_1, \alpha_2, \dots, \alpha_n$ is a sequence of positive real numbers, then*

$$\sum_{p=1}^{n-1} \left\| \frac{x_{p+1} - x_p}{\alpha_{p+1}} - \frac{x_p - x_{p-1}}{\alpha_p} \right\| \le \frac{1}{\alpha_n} \sum_{q=1}^{n} \alpha_q \left\| \frac{x_n - x_0}{\sum_{q=1}^{n} \alpha_q} - \frac{x_{n-1} - x_0}{\sum_{q=1}^{n-1} \alpha_q} \right\| + \sum_{p=1}^{n-2} \left\| \frac{x_{p+1} - x_0}{\sum_{q=1}^{p+1} \alpha_q} - \frac{x_p - x_0}{\sum_{q=1}^{p} \alpha_q} \right\|.$$

This lemma is a generalization of Lemma 3.1 in [**34**], given in a numeric case.

Lemma 2.2. *If n is an integer greater than 2 and $x_0, x_1, \dots, x_n$ are vectors in X and $\beta_0, \beta_1, \dots, \beta_n$ is an increasing sequence of real* *numbers, then*

$$\sum_{p=1}^{n-1}\left\|\frac{x_{p+1}-x_p}{\beta_{p+1}-\beta_p}-\frac{x_p-x_{p-1}}{\beta_p-\beta_{p-1}}\right\| \geq \sum_{p=1}^{n-1}\left\|\frac{x_{p+1}-x_0}{\beta_{p+1}-\beta_0}-\frac{x_p-x_0}{\beta_p-\beta_0}\right\|.$$

Lemma 2.3. *If* $\operatorname{var}_a^b(dv/du) < \infty$, *then for any* $t \in (a,b]$ *there exists* $D_u^- v(t)$, *and for any* $t \in [a,b)$ *there exists* $D_u^+ v(t)$.

2.2. The Hellinger integral for functions in $Q_0(T,X)$.

Lemma 2.4. *Let* Z *be defined by* $Z = \mathcal{B}(X,Y)$. *Let* w *be a fixed operator* $w_t = R_t w$ *and* $\operatorname{var}_a^b(dv/du) < \infty$. *Then there exists the integral*

$$\int_a^b \frac{(dw_{t^*})(dv)}{du},$$

and this integral is equal to $w(D_u^- v(t^*))$.

Lemma 2.5. *If* $\operatorname{var}_a^b(dv/du) < \infty$ *and* s *is a step function on* $[a,b]$ *with values in* Z, *then there exists the integral*

$$\int_a^b \frac{(ds)(dv)}{du}.$$

Proof. This result can be obtained by the above lemma and by Properties 1 and 2 of the Hellinger integral. □

Theorem 2.1. *If* $\operatorname{var}_a^b(dv/du) < \infty$ *and* $f \in Q_0([a,b],Z)$, *then there exist the integral*

$$\int_a^b \frac{(df)(dv)}{du}$$

and

$$\left\|\int_a^b \frac{(df)(dv)}{du}\right\|_Y \leq \left\{\operatorname{var}_a^b\left(\frac{dv}{du}\right) + \|D_u^- v(b)\|_X\right\}\|f\|. \tag{2.1}$$

Proof. The function f is the limit of a sequence $\{s_k\}$ of step functions. We can assume $\|s_k - f\| < 1/k$, $k \in \mathbf{N}^*$. Let $\{t_p\}_0^n$ be a division of the interval $[a,b]$ and $k \in \mathbf{N}^*$. The integral sum $\sigma(s_k)$ relative to the step function s_k is given by

$$\begin{aligned}\sigma(s_k) &= \sum_{p=1}^{n} \frac{[\Delta s_k(t_p)][\Delta v(t_p)]}{\Delta u(t_p)} \\ &= \sum_{p=1}^{n-1} [s_k(t_p) - s_k(t_{p-1})] \frac{\Delta v(t_p)}{\Delta u(t_p)} \\ &\quad + [s_k(t_n) - s_k(t_{n-1})] \frac{\Delta v(t_n)}{\Delta u(t_n)} \\ &= \sum_{p=1}^{n-1} s_k(t_p) \frac{\Delta v(t_p)}{\Delta u(t_p)} \\ &\quad - \sum_{p=0}^{n-1} s_k(t_p) \frac{\Delta v(t_{p+1})}{\Delta u(t_{p+1})} + s_k(t_n) \frac{\Delta v(t_n)}{\Delta u(t_n)} \\ &= - \sum_{p=1}^{n-1} s_k(t_p) \left(\frac{\Delta v(t_{p+1})}{\Delta u(t_{p+1})} - \frac{\Delta v(t_p)}{\Delta u(t_p)} \right) \\ &\quad + s_k(t_n) \frac{v(t_n) - v(t_{n-1})}{u(t_n) - u(t_{n-1})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\sigma(s_k)\|_Y &\le \|s_k\| \left\{ \sum_{p=1}^{n-1} \left\| \frac{\Delta v(t_{p+1})}{\Delta u(t_{p+1})} - \frac{\Delta v(t_p)}{\Delta u(t_p)} \right\|_X \right. \\ &\qquad \left. + \left\| \frac{v(t_n) - v(t_{n-1})}{u(t_n) - u(t_{n-1})} \right\|_X \right\} \\ &\le \|s_k\| \left(B + \left\| \frac{v(t_n) - v(t_{n-1})}{u(t_n) - u(t_{n-1})} \right\|_X \right).\end{aligned}$$

From the definition of the Hellinger integral and the definition of the left derivative of the function v with respect to u at b, it follows that

$$\left\| \int_a^b \frac{(ds_k)(dv)}{du} \right\|_Y \le \|s_k\| (B + \|D_u^- v(b)\|)$$

and, by taking lim inf with respect to B, we get the inequality (2.1).

If we take $s_k - s_m$ instead of s_k, we obtain

$$\begin{aligned}\left\|\int_a^b \frac{(ds_k)(dv)}{du} - \int_a^b \frac{(ds_m)(dv)}{du}\right\|_Y &= \left\|\int_a^b \frac{(d(s_k - s_m))(dv)}{du}\right\| \\ &\le \|s_k - s_m\|(B + \|D_u^- v(b)\|) \\ &\le \left(\frac{1}{k} + \frac{1}{m}\right)(B + \|D_u^- v(b)\|)\end{aligned}$$

which shows that there exists the limit

$$y_0 = \lim_{k\to\infty} \int_a^b \frac{(ds_k)(dv)}{du}.$$

Let us determine y_0: if $D_1 = \{t'_p\}_0^n$ is a division of $[a,b]$, then the integral sum which corresponds to the function satisfies the relations

$$\begin{aligned}\|\sigma(D_1, f - s_k)\| &= \left\|\sum_{p=1}^{n} \frac{\Delta(f - s_k)(t'_p)\Delta v(t'_p)}{\Delta u(t'_p)}\right\|_Y \\ &\le \|f - s_k\| \left\{\sum_{p=1}^{n-1} \left\|\frac{\Delta v(t'_{p+1})}{\Delta u(t'_{p+1})} - \frac{\Delta v(t'_p)}{\Delta u(t'_p)}\right\|_X \right. \\ &\qquad \left. + \left\|\frac{\Delta v(b) - v(t'_{n-1})}{\Delta u(b) - u(t'_{n-1})}\right\|_X \right\} \\ &\le \frac{2}{k} \operatorname{var}_a^b \left(\frac{dv}{du}\right)\end{aligned}$$

or

$$\|\sigma(D_1, f) - \sigma(D_1, s_k)\|_Y \le \frac{2B}{k}.$$

Since

$$\begin{aligned}\left\|\int_a^b \frac{(df)(dv)}{du} - y_0\right\| &\le \left\|\int_a^b \frac{(df)(dv)}{du} - \sigma(D_1, f)\right\| \\ &\quad + \|\sigma(D_1, f - s_k)\| \\ &\quad + \|\sigma(D_1, s_k) - y_0\| \\ &= \left\|\int_a^b \frac{(df)(dv)}{du} - \sigma(D_1, f)\right\| \\ &\quad + \frac{2B}{k} + O(k).\end{aligned}$$

The result follows by taking the limit with respect to k. □

2.3. *The representation of the linear continuous functionals on the space* $Q_0(T, X^*)$.

Theorem 2.2. *If* $\operatorname{var}_a^b(dv/du) < \infty$, $f \in Q_0(T, Z)$, *then the functional* $F : Q_0(T, X^*) \to \mathbf{R}$ *defined by*

$$F(f) = \int_a^b \frac{(df)(dv)}{du}$$

is linear and its norm is $\operatorname{var}_a^b(dv/du) + \|D_u^- v(b)\|_X$.

Theorem 2.3. *If* F *is a bounded linear functional on* $Q_0([a,b], X^*)$ *and* X *is a reflexive space, then there exists an increasing real function* u *and a vector function* $v : [a,b] \to X$, *with* $\operatorname{var}_a^b(dv/du) < \infty$, *for which*

$$F(f) = \int_a^b \frac{(df)(dv)}{du}, \tag{2.2}$$

for each $f \in Q_0([a,b], X^*)$.

Proof. Let F_g be defined by

$$F_g(x) = F(x_g)$$

where $x_g(t) = x(t)\chi_g(t)$, g is an arbitrary subset of $[a,b]$ and $\chi_g \in Q_0(T, X)$. Let λ and ρ be defined by

$$\lambda(c) = \lim_{t \to c^-} \|F_{(t,c)}\|$$
$$\rho(c) = \lim_{t \to c^+} \|F_{(c,t)}\|.$$

There exists a countable subset δ of $[a,b]$ such that, if t is in $[a,b]$ but not in δ, then $\lambda(t) = \rho(t) = 0$.

We shall consider the function u defined as in [**34**]; $u(t)$ is an increasing function which has jumps at the points where $\lambda(c) \neq 0$ or $\rho(c) \neq 0$. Consider

$$u_t(s) = \begin{cases} u(s) - u(t) & \text{if } t < s \le b \\ 0 & \text{if } a \le s \le t. \end{cases}$$

If $x^* \in X^*$ is a fixed functional, then for $f_{x^*,t}(s) = u_t(s)x^*$ we have $f_{x^*,t} \in Q_0([a,b], X^*)$.

Let

$$g_t : X^* \longrightarrow \mathbf{R}$$

be defined by

$$g_t(x^*) = -F(f_{x^*,t}).$$

The function g_t is in X^{**} so, by reflexivity, $g_t \in X$. We define $\mathbf{v} : T \to X$, $v(t) = g_t$ such that $x^*(v(t)) = g_t(x^*)$. Thus $x^*(v(t)) = -F(f_{x^*,t})$. In [**27**] one proves that the function $\mathbf{v}(t)$ has bounded slope variation with respect to $\mathbf{u}$. We prove now that the functional F can be represented by the integral (2.2).

Consider

$$G(f) = \int_a^b \frac{(df)(dv)}{du}.$$

If $f_{x^*} = R_c(t)x^*$, then by Lemma 2.4, we have

$$\begin{aligned} G(f_{x^*}) &= -\lim_{t\to c_-} \frac{F(u_t(s)x^*) - F(u_c(s)x^*)}{u(t)-u(c)} \\ &= -F\left(\lim_{t\to c_-} \frac{u_t(s)-u_c(s)}{u(t)-u(c)} x^*\right). \end{aligned}$$

Therefore

$$\begin{aligned} G(f_{x^*}) &= F\Big(\lim_{t\to c_-} R_c x^*\Big) + F\left(\lim_{t\to c_-} \frac{u(t)-u(s)}{u(t)-u(c)} \chi_{[t,c]} x^*\right) \\ &= F(R_c x^*) + \lim_{t\to c^-} F_{(t,c)}\left(\frac{u(t)-u(s)}{u(t)-u(c)} x^*\right). \end{aligned}$$

From the definition of $\mathbf{u}(t)$, we have

$$\| \lim_{t\to c^-} \|F_{(t,c)}\| = 0$$

and

$$\lim_{t\to c^-} \frac{u(t)-u(s)}{u(t)-u(c)} = 0.$$

Thus, $G(R_c x^*) = F(R_c x^*)$. In a similar way it can be proved that $G(L_c x^*) = F(L_c x^*)$ so, for any step function s, $G(s) = F(s)$. But the linear span of the step functions is dense in $Q_0([a,b], X^*)$. $\square$

3. The representation of bounded linear operators. We use the integral of Hellinger type introduced in Section 2 for linear operators with respect to vector functions. We shall give a theorem of representation for bounded linear operators defined on the space of vector functions with bounded variation on a closed interval, which takes the value zero at the left end of the interval.

Let X be a Banach space, $T = [a,b]$ a segment of the real axis, $\mathbf{u} : T \to \mathbf{R}$ an increasing function with $\mathbf{u}(a) = 0$ and $\mathbf{v} : T \to X$. We suppose that the function $\mathbf{v}$ has the slope variation bounded with respect to $\mathbf{u}$ (Section 2).

Lemma 3.1. *The set*

$$V_u(T,X) = \left\{ v(t) \mid \operatorname{var}_a^b \left(\frac{dv}{du} \right) < \infty \right\}$$

is a linear space with respect to the usual operations and

$$\|v\|_{V_u} = \|v(a)\| + \operatorname{var}_a^b \left(\frac{dv}{du} \right) + \|D_u^- v(b)\|_X$$

is a norm on this space [**28**].

Lemma 3.2. *The space $V_u(T,X)$ is a Banach space* [**28**].

3.1. *The representation of the bounded linear operators on the space $V_u^0(T,X)$.* Consider $Z = B(X,Y)$ and $Q_0(T,Z)$ the closure of the step functions space. For $w \in Q_0(T,Z)$,

$$\|w\| = \sup_{t \in T} \|w(t)\|.$$

It is clear from Theorems 2.1 and 2.2 that, for any $v \in V_u(T,X)$, there exists

$$F(v) = \int_a^b \frac{(dw)(dv)}{du}$$

and

$$\|F(v)\|_Y \le \|w\|_{\mathcal{B}} \|v\|_{V_u}. \tag{3.3}$$

Lemma 3.3. *$F : V_u^0(T, X) \to Y$ defined above is a bounded linear operator.*

Lemma 3.4. *Let $v \in V_u(T, X)$ and $\varepsilon > 0$. Then there exists a division Δ_1 of the interval $[a, b]$ such that, for any refinement $\Delta_2 = (a = t_0, t_1, \dots, t_n = b)$ of Δ_1 the function $\tilde{v}$ defined by*

$$\tilde{v}(t) = v(t_p) + \frac{u(t) - u(t_p)}{u(t_{p+1}) - u(t_p)}[v(t_{p+1}) - v(t_p)]$$

for any $t \in [t_p, t_{p+1}]$, $p = 0, 1, \dots, n-1$, satisfies $\|v - \tilde{v}\|_{V_u} < \varepsilon$.

Theorem 3.1. *The general form of the bounded linear operators defined on $V_u^0(T, X)$ with values in Y is*

$$F(v) = \int_a^b \frac{(dw)(dv)}{du} \tag{3.4}$$

for some $w \in Q_0(T, Z)$; moreover,

$$\|F\|_{B(V_u^0, Y)} \le \|w\|_{\mathcal{B}}.$$

Proof. As we know from Lemma 3.3, equality (3.4) defines a linear continuous operator on $V_u^0(T, X)$ with values in Y. Conversely, let F be a linear continuous operator that maps $V_u^0(T, X)$ into Y. For fixed $x \in X$ we denote

$$v_{\rho,x}(t) = v_\rho(t) = \begin{cases} u(t)(x) & \text{if } a \le t \le \rho \\ u(\rho)x & \text{if } \rho \le t \le b. \end{cases}$$

Then $v_\rho(t) \in V_u^0(T, X)$ and $v_\rho(a) = u(a)x = 0$.

Let Δ be a division, $\Delta : [a < t_1 < \cdots < t_k < \rho < t_{k+1} < \cdots < t_n < b]$. For all the t_p's to the left of ρ, we have

$$\frac{v_\rho(t_{p+1}) - v_\rho(t_p)}{u(t_{p+1}) - u(t_p)} = \frac{u(t_{p+1})x - u(t_p)x}{u(t_{p+1}) - u(t_p)} = x$$

and for the ones to the right of ρ,

$$\frac{v_\rho(t_{p+1}) - v_\rho(t_p)}{u(t_{p+1}) - u(t_p)} = \frac{u(\rho)x - u(\rho)x}{u(t_{p+1}) - u(t_p)} = 0$$

so that

$$\sum_{p=1}^{n-1} \left\| \frac{v_\rho(t_{p+1}) - v_\rho(t)}{u(t_{p+1}) - u(t_p)} - \frac{v_\rho(t_p) - v_\rho(t_{p-1})}{u(t_p) - u(t_{p-1})} \right\|_X$$
$$= \sum_{p \le k-1} \|x - x\| + \|x - \bar{0}\| + \|\bar{0} - \bar{0}\| = \|x\|.$$

Hence,

$$\operatorname{var}_a^b \left(\frac{dv_\rho}{du} \right) = \|x\|$$

and

$$D_u^- v_\rho(t) = \lim_{t \to b^-} \frac{v_\rho(t) - v_\rho(b)}{u(t) - u(b)} = 0.$$

So, for $\rho < b$, $v_\rho(t) \in V_u^0(T, X)$ and $\|v_\rho\|_{V_u} = \|x\|_X$. For $\rho = b$ all t_p are less than ρ, so

$$\operatorname{var}_a^b \left(\frac{dv_\rho}{du} \right) = 0, \qquad D_u^- v_\rho(b) = x$$

and

$$\|v_\rho\|_{V_u} = \|x\|_X.$$

Define $w(\rho) : [a, b] \to Z$ by $w(\rho)(x) = F(v_{\rho,x})$. But $w(a)(x) = F(v_a) = F(\bar{0}) = \bar{0}$; it follows that $w(a) = 0$.

We shall prove that $w(\rho) \in Q_0(T, Z)$. One can show that there exists $\lim_{\rho \to c^+} w(\rho)$ for any $c \in [a, b)$ and $\lim_{\rho \to c^-} w(\rho)$ for any $c \in (a, b]$. Consider $0 < h_1 < h_2$. We shall compute

$$[w(c + h_2) - w(c + h_1)]x = F(v_{c+h_2} - v_{c+h_1}).$$

We have

$$v_{c+h_2}(t) - v_{c+h_1}(t) = \begin{cases} [u(t) - u(c + h_1)]x & \text{if } t \in [c + h_1, c + h_2] \\ 0 & \text{if } t \le c + h_1 \\ [u(c + h_2) - u(c + h_1)]x & \text{if } t \ge c + h_2. \end{cases}$$

It follows that

$$\|v_{c+h_2}(t) - v_{c+h_1}(t)\|_{V_u} \le [u(c+h_2) - u(c+h_1)]\|x\| \longrightarrow 0$$

for $h_1, h_2 \to 0$. Then $\|w(c+h_2) - w(c+h_1)\| \to 0$, for $h_1, h_2 \to 0$, so that the limit $\lim_{\rho \to c^+} w(\rho)$ exists. In a similar way one can prove that the limit $\lim_{\rho \to c^-} w(\rho)$ exists. Denote

$$G(v) = \int_a^b w \frac{(dw)(dv)}{du}.$$

Since $w \in Q_0(T, Z)$, G is a bounded linear operator defined on $V_u^0(T, X)$ with values in Y (Lemma 3.3). We prove that $G = F$.

Let $\Delta = \{t_p\}$, $p = 1, 2, \dots, n$ be a division of the interval $[a, b]$. Set $x_p = v(t_p) \in X$. The integral sum which corresponds to this division is:

$$\begin{aligned}
\sigma_n(\Delta) &= \sum_{p=1}^{n-1} \frac{(\Delta w(t_p))(\Delta v(t_p))}{\Delta u(t_p)} \\
&= \sum_{p=1}^{n-1} \frac{w(t_{p+1})(x_{p+1}) - w(t_p)(x_{p+1})}{u(t_{p+1}) - u(t_p)} \\
&\quad - \frac{w(t_{p+1})(x_p) + w(t_p)(x_p)}{u(t_{p+1}) - u(t_p)} \\
&= \sum_{p=1}^{n-1} \frac{F(v_{t_{p+1}, x_{p+1}}) - F(v_{t_p, x_{p+1}})}{u(t_{p+1}) - u(t_p)} \\
&\quad - \frac{F(v_{t_{p+1}, x_p}) + F(v_{t_p, x_p})}{u(t_{p+1}) - u(t_p)} \\
&= F\left(\sum_{p=1}^{n-1} \left[\frac{v_{t_{p+1}, x_{p+1}} - v_{t_p, x_{p+1}}}{u(t_{p+1}) - u(t_p)} - \frac{v_{t_{p+1}, x_p} - v_{t_p, x_p}}{u(t_{p+1}) - u(t_p)} \right] \right).
\end{aligned}$$

Keeping track of the definitions of $v_{t_{p+1}, x_{p+1}}$ for each p, it turns out that

$$\sum_{p=1}^{n-1} \left[\frac{v_{t_{p+1}, x_{p+1}} - v_{t_p, x_{p+1}}}{u(t_{p+1}) - u(t_p)} - \frac{v_{t_{p+1}, x_p} - v_{t_p, x_p}}{u(t_{p+1}) - u(t_p)} \right] = \tilde{v}$$

where $\tilde{v}$ was defined in Lemma 3.4. Thus $\sigma_n(\Delta) = F(\tilde{v})$. For each $\varepsilon > 0$ choose Δ such that

$$\|G(v) - \sigma_n(\Delta)\|_Y < \varepsilon$$

and

$$\|v - \tilde{v}\|_{V_u} < \varepsilon.$$

Then

$$\|F(v) - G(v)\| \leq \|F\|\,\|v - \tilde{v}\| + \varepsilon \leq \varepsilon(\|F\| + 1)$$

for any $\varepsilon > 0$. Therefore, $F(v) = G(v)$. □

4. The representation of bounded operators.

4.1. *Preliminaries.* We shall give a theorem of representation of linear continuous operators which satisfy the condition

$$\|U(f)\| \leq \int_0^1 \|f(t)\|\, d\mu(t)$$

on the space $M(T, X)$ of the essentially bounded functions, on a segment T with values in a Banach space X. $M(T, X)$ is the space of the functions $f \in L(T, X)$ which verify the properties:

1. There exists a set $\{f_n\}_{n\in N}$ of simple functions from $L(T, X)$ such that $\|f_n(t)\| \leq \lambda$ almost everywhere for all $n \in \mathbf{N}$.

2. $\|f_n(t) - f(t)\| \to 0$ almost everywhere.

The operators are defined using Hellinger integrals with respect to absolutely continuous functions on the compact interval T.

For $f : [0, 1] \to X$, $G : T \to B(X, Y)$ and Δ a division of T

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

$U(f) = \lim_n S_{\Delta_n}(f)$, where S_Δ is given by

$$S_\Delta(f) = \sum_{i=0}^{m-1} \frac{1}{t_{i+1} - t_i}[G(t_{i+1}) - G(t_i)][f(t_{i+1}) - f(t_i)].$$

Denote

$$U(f) = \int_0^1 \frac{1}{dt}\{(dG(t))(df(t))\}.$$

(See [**3**].)

In the same way, it can be introduced

$$\int_0^1 \frac{1}{du}\{(dG(t))(df(t))\}$$

with u an increasing function. This integral is more general than the one introduced in Section 2.

4.2. *The representation of the bounded operators using Hellinger integrals.* Let X and Y be two Banach spaces, $T = [0,1]$, $L(T,X)$ be the space of Bochner integrable functions. Let $M(T,X)$ be a subspace of $L(T,X)$ and $f \in L(T,X)$.

Theorem 4.1. *Consider the operator defined on $M(T,X)$ with values in Y given by*

$$U(f) = \int_0^1 \frac{(dG(t))(df^*(t))}{dt} \tag{4.5}$$

with

$$f^*(t) = \int_0^t f(t)\, dt \tag{4.6}$$

and G an absolutely continuous function on T into $B(X,Y)$. Then U exists, is linear and satisfies the inequality

$$\|U(f)\| \le \int_0^1 \|f(t)\|\, d\mu(t) \tag{4.7}$$

with μ a positive measure, absolutely continuous with respect to the Lebesgue measure.

Proof. We show that the integral exists for G absolutely continuous and

$$\lim_n S_n = \int_0^1 \frac{(dG)(dF^*)}{dt}.$$

which proves the inequality (4.7) □

Theorem 4.2. *If U is a linear operator defined on $M(T,X)$ with values into Y that satisfy the condition* (4.7), *then U can be represented as in* (4.5) *with G an absolutely continuous function on T with values into $B(X,Y)$.*

Proof. Let U be such an operator. For $G(t) = U(\gamma_t(\tau)x)$ with

$$\gamma_t(\tau) = \begin{cases} 1 & \text{if } \tau \in [0,t) \\ 0 & \text{if } \tau \in [t,1]. \end{cases}$$

We shall prove that G is absolutely continuous. Let $\delta_i = (t_i, t_i')$, $i = 1, \dots, p$ be a system of disjoint intervals. We have

$$\begin{aligned}\sum_{i=1}^{p} \|G(t_i') - G(t_i)\| &= \sum_i \sup_{\|x\|\le 1} \|G(t_{i+1})x - G(t_i)x\| \\ &= \sum_i \sup_{\|x\|\le 1} \|U(\gamma_{t_{i+1}}(\tau)x) - U(\gamma_{t_i}(\tau)x)\| \\ &= \sum_i \sup_{\|x\|\le 1} \|U(\gamma_{t_{i+1}}(\tau) - \gamma_{t_i}(\tau))x\|.\end{aligned}$$

By hypothesis,

$$\begin{aligned}\|U(\gamma_{t_{i+1}}(\tau) - \gamma_{t_i}(\tau))x\| &\le \int_0^1 \|(\gamma_{t_{i+1}}(\tau)) - \gamma_{t_i}(\tau))x\| \, d\mu(t) \\ &= \int_{t_i}^{t_{i+1}} \|x\| \, d\mu(t) \\ &= \|x\|(\mu(t_{i+1}) - \mu(t_i)) \\ &= \|x\|\mu(\delta_i).\end{aligned}$$

Substituting, we have

$$\begin{aligned}\sum_i \|G(t_{i+1}) - G(t_i)\| &\le \sum_i \sup_{\|x\|\le 1} \|x\|(\mu(t_{i+1}) - \mu(t_i)) \\ &\le \sum_i \mu(\delta_i).\end{aligned}$$

If $\sum_i (t_{i+1} - t_i) < \delta$, then $\sum_i \mu(\delta_i) < \varepsilon$, since μ is absolutely continuous with respect to the Lebesgue measure. Then

$$\sum_i \|G(t_{i+1}) - G(t_i)\| < \varepsilon;$$

so G is absolutely continuous.

Consider $f \in M(T, X)$ of the form

$$f(t) = \sum_i \gamma_{[t_i, t_{i+1}]}(t) a_i$$

with $f(t)$ a simple function, t_i the points of a division D, and $a_i \in X$. We have

$$\begin{aligned} f^*(t_{i+1}) - f^*(t_i) &= \int_0^{t_{i+1}} f(t)\,dt - \int_0^{t_i} f(t)\,dt \\ &= \int_{t_i}^{t_{i+1}} f(t)\,dt. \end{aligned}$$

Since $f(t) = a_i$ on $[t_i, t_{i+1}]$,

$$a_i = \frac{f^*(t_{i+1}) - f^*(t_i)}{t_{i+1} - t_i}.$$

Thus

$$\begin{aligned} U(f) &= U\bigg(\sum_i (\gamma_{[t_i, t_{i+1}]} a_i)\bigg) \\ &= \sum_i U[(\gamma_{t_{i+1}} - \gamma_{t_i}) a_i] \\ &= \sum_i [U(\gamma_{t_{i+1}} a_i) - U(\gamma_{t_i} a_i)] \\ &= \sum_i [G(t_{i+1})(a_i) - G(t_i)(a_i)] \\ &= \sum_i \frac{G_i[f^*(t_{i+1}) - f^*(t_i)]}{t_{i+1} - t_i} \end{aligned}$$

with $G_i = G(t_{i+1}) - G(t_i)$, which is

$$\int_0^1 \frac{(dG)(df^*)}{dt}.$$

Let f be an arbitrary element in $M(T,X)$. There exists a set of simple functions $\{f_n\}_{n\in\mathbf{N}}$ such that $f_n \to f$ in the space $M(T,X)$. Then

$$\int_0^1 \|f_n(t) - f(t)\|\, d\mu(t) \longrightarrow 0 \quad \text{for } n \longrightarrow \infty,$$

since μ is absolutely continuous with respect to the Lebesgue measure; therefore $U(f_n) \to U(f)$. Thus $U(f)$ is given by formula (4.5). □

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