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ON LARGE VALUES OF BINARY FORMS

B. BRINDZA

1. Introduction. Let $F(X, Y)$ be an irreducible binary form with rational integral coefficients of degree $n \geq 3$. Moreover, let m be a positive integer having s distinct prime factors. Improving several earlier results, Bombieri and Schmidt [**2**] proved that the equation

$$(1) \qquad |F(x,y)| = m \quad \text{in coprime integers } x \text{ and } y$$

has at most cn^{s+1} solutions where c is an effectively computable constant (the solutions (x, y) and $(-x, -y)$ are considered as the same). Most likely the bound cn^{s+1} is not the best possible; however, it does not seem to be very easy to improve it in this generality. Langmann [**7**] showed that n^s can be achieved for almost every m. For further results and different approaches related to the number of solutions of (1) we refer to [**3, 4, 8, 9, 10**]. The purpose of this note is to derive a bound linear in n, provided that m is large enough. More exactly, we have

Theorem. *There exists an effectively computable constant C depending only on the discriminant of F, such that $m > C$ and $n > 5$ imply that the number of solutions of* (1) *does not exceed*

$$n\left(\left(n^{4/(n-2)}\right)^s + 6\right).$$

Remarks. The term $n^{4/(n-2)}$ can be improved a bit and it is certainly bounded, however its "shape" shows that if s is fixed then the dependence of the bound above is improving in s as n is getting larger.

Schmidt [**8**] conjectures the existence of a bound like $c(F)(\log m)^{c'}$, $m > 1$, where $c(F)$ may depend on F and c' is an absolute constant.

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It is known that

$$s = \log\log m + O(\sqrt{\log\log m})$$

for almost every m; therefore our Theorem implies Schmidt's conjecture, but just for almost every m.

2. Preliminaries. The *Mahler height* of a binary form

$$G(X,Y) = b(X-\beta_1 Y)\cdots(X-\beta_n Y)$$

is defined by

$$M(G) = |b|\prod_{i=1}^{n}\max\{1,|\beta_i|\}.$$

The following profound lemma is a generalized effective version of the Birch-Merriman theorem.

Lemma 1. *Let $G_1(X,Y)\in \mathbf{Z}[X,Y]$ be an irreducible binary form of degree $n\ge 3$ with discriminant D. Then $n \le (2/\log 3)\log|D(G_1)|$ and there exists a binary form $G_2(X,Y)\in\mathbf{Z}[X,Y]$ equivalent*[1] *to G_1 for which $M(G_2) < C_1$ where C_1 is an effectively computable constant depending only on D.*

Proof. It is a simple consequence of Theorem 1 in [**6**] and Corollary 1 in [**5**]. □

Following the notation of Bombieri and Schmidt [**2**] the height of an integer point (x,y) is defined by

$$H(x,y) = \max\{|x|,|y|\}.$$

Lemma 2 (Bombieri and Schmidt [**2**]). *Let (x,y) be a solution to* (1) *with $y\ne 0$. Then*

$$\min_{\substack{\alpha \\ F(\alpha,1)=0}} \min\left\{1,\left|\alpha-\frac{x}{y}\right|\right\} \le \frac{(2\sqrt{n}M(F))^n m}{(H(x,y))^n}.$$

Lemma 3 (Thue-Siegel principle). *Let* (x, y) *and* (x', y') *be two solutions to* (1) *with* $yy' \neq 0$*,*

$$\min_{1\le j\le n}\left|\alpha_j-\frac{x}{y}\right|=\left|\alpha_1-\frac{x}{y}\right|;\qquad \min_{1\le j\le n}\left|\alpha_j-\frac{x'}{y'}\right|=\left|\alpha_1-\frac{x'}{y'}\right|.$$

For $0 < t < \sqrt{2/n}$ *and* $\sqrt{2-nt^2} < \tau < t$ *put* $\lambda = 2/(t-\tau)$, $\delta = (nt^2+\tau^2-2)/(n-1)$, $A = (t^2n/(2-nt^2))((1/n)\log M(F)+1/2)$*. If we have* $|\alpha_1| \le 1$*,*

$$\left|\alpha_1-\frac{x}{y}\right| < \min\{1, t-\tau, (e^{A+\log 3}H(x,y))^{-\lambda}\},$$

$$\left|\alpha_1-\frac{x'}{y'}\right| < \min\left\{1, \frac{1}{2}(t-\tau)^2, (e^{A+\log 3}H(x',y'))^{-\lambda}\right\}$$

then we also have

$$\log 3 + A + \log H(x',y') < \frac{1}{\delta}(\log 3 + A + \log H(x,y)). \tag{2}$$

Proof. It is a special case of an up-to-date version of Thue-Siegel principle done by Bombieri and Mueller [**1**], (cf. [**2**]). □

3. Proof of the Theorem. The number of solutions of (1) is the same for equivalent forms and equivalent forms have the same discriminant, that is, we may assume by using Lemma 1 that $M(F)$ and n are bounded by a constant depending only on the discriminant of F. The binary form F can be factorized as

$$F(X,Y) = a(X-\alpha_1 Y)\cdots(X-\alpha_n Y)$$

where $a; \alpha_1, \dots \alpha_n$ are the leading coefficient and the zeros of the associated polynomial $F(X,1)$, respectively. It will be convenient in much of the sequel to write (1) in the form

$$|(ax-a\alpha_1 y)\cdots(ax-a\alpha_n y)| = |a|^{n-1}m$$

where $a\alpha_1, \dots, a\alpha_n$ are pairwise distinct algebraic integers. Let D and $\mathbf{K}$ be denote the discriminant and the splitting field of $F(X,1)$, respectively. A prime ideal $\mathfrak{p}$ in $\mathbf{K}$ is called *essential* if either $\mathfrak{p}$ does not divide Da or

$$\mathrm{ord}_{\mathfrak{p}} m > n \max_{i \neq j} \mathrm{ord}_{\mathfrak{p}} a^2(\alpha_i - \alpha_j) > 0. \tag{3}$$

If (x, y) is a solution to (1) and $\mathfrak{p}|m$ is an essential prime ideal, then there exists a unique $1 \leq k \leq n$ for which

$$\max_{1 \leq i \leq n} \mathrm{ord}_{\mathfrak{p}}(ax - a\alpha_i y) = \mathrm{ord}_{\mathfrak{p}}(ax - a\alpha_k y);$$

furthermore,

$$\begin{aligned} \sum_{\substack{1 \leq i \leq n \\ i \neq k}} \mathrm{ord}_{\mathfrak{p}}(ax - a\alpha_i y) &\leq \sum_{\substack{1 \leq i \leq n \\ i \neq k}} \mathrm{ord}_{\mathfrak{p}} a^2(\alpha_k - \alpha_i) \\ &\leq \mathrm{ord}_{\mathfrak{p}} a^{n(n-1)} D. \end{aligned} \tag{4}$$

The index k is called the *location* of $\mathfrak{p}$ with respect to the solution (x, y) and denoted by $\mathrm{loc}\,(\mathfrak{p}; x, y)$. By taking any $\mathbf{Q}$-isomorphism ϕ of $\mathbf{K}$, the prime ideal $\phi(\mathfrak{p})$ is also essential and its location depends only on $\mathrm{loc}(\mathfrak{p}; x, y)$ and ϕ. We now write m in the form $m = m_0 m_E$, where m_0 and m_E are relatively prime positive integers for which m_E is divisible by the essential prime ideal divisors of m, only, and m_0 is not divisible by any of them. The inequality $|a| \leq M(F)$ and Lemma 1 imply that m_0 is bounded by an effective constant depending only on D. Let $p_1^{k_1} > \cdots > p_t^{k_t}$ denote the distinct prime factors of m_E. Then $1 \leq t \,(\leq s)$ provided that m is large enough.

For a *rational* $0 < \omega < 1$ put

$$\mathcal{N}(\omega) = 1 + n^{[t\omega]+1},$$

and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be a fixed set of prime ideals (in $\mathbf{K}$) with $\mathfrak{p}_l | p_l$, $l = 1, \ldots, t$. If (1) has at least $\mathcal{N}(\omega)$ solutions, then we get at least two among them, (x, y) and (x', y'), say, for which

$$\mathrm{loc}\,(\mathfrak{p}_l; x, y) = \mathrm{loc}(\mathfrak{p}_l; x', y'), \qquad l = 1, \ldots, [t\omega] + 1.$$

The greatest common divisor of the principal ideals generated by $ax - a\alpha_i y$ and $ax' - a\alpha_i y'$, respectively, divides $a(xy' - x'y), 1 \leq i \leq n$

which is a nonzero rational integer; hence, (3) and (4) yield that $p_1^{k_1}\cdots p_{[t\omega]+1}^{k_{[t\omega]+1}}$ divides $a^{n(n-1)}Da(xy'-x'y)$, therefore

$$m_E^{([t\omega]+1)/t} < p_1^{k_1}\cdots p_{[t\omega]+1}^{k_{[t\omega]+1}} \leq 2|a^{n^2-n+1}D|(\max\{|x|,|y|,|x'|,|y'|\})^2.$$

Let d denote the denominator of ω. From the inequality

$$m^{1/t} \geq m^{1/s} \geq (\log m)^{c_1},$$

where c_1 is an effective absolute constant, we obtain

$$\begin{aligned} m_E^{([t\omega]+1)/t} &\geq m_E^{(1/t)(t\omega+1/d)} \\ &\geq m_E^{\omega} m_0(2|a|^{n^2-n+1}|D|) \\ &\geq m^{\omega}(2|a|^{n^2-n+1}|D|), \end{aligned}$$

provided that m is large enough compared with ω and D. In other words (1) has at most $n^{[t\omega]+1}$ solutions satisfying

$$H(x,y) \leq m^{\omega/2}$$

(under the condition imposed on m). By taking

$$\omega = 2\left(\frac{1}{n-2}+\frac{1}{n-1}\right)$$

it remains to estimate the number of "larger" solutions $(x_1,y_1),\ldots,(x_k,y_k)$ with

$$\begin{gathered} \min_{1\leq j\leq n}\left|\alpha_j - \frac{x_i}{y_i}\right| = \left|\alpha_1 - \frac{x_i}{y_i}\right|; \qquad i=1,\ldots,k, \\ m^{1/(n-2)+1/(n-1)} < H(x_1,y_1) \leq \cdots \leq H(x_k,y_k). \end{gathered}$$

Following the "gap principle" argument of Bombieri and Schmidt [**2**] based upon Lemma 2 one can obtain

$$\begin{aligned} H(x_k,y_k) &\geq \left((2\sqrt{n}M(F))^{-n/(n-2)} m^{-1/(n-2)} H(x_1,y_1)\right)^{(n-1)^{k-1}} \\ &\geq (2\sqrt{n}M(F))^{-n(n-1)^{k-1}/(n-2)} m^{(n-1)^{k-2}}, \qquad k\geq 2. \end{aligned} \tag{5}$$

If $|\alpha_1| \le 1$ and we choose $t = \sqrt{n + a^2}, \tau = bt$ (cf. [**2**]) with $b = 0.21$ and $a = 0.01$, say, then we get $0 < t < \sqrt{2/n}, \sqrt{2 - nt^2} < \tau < t$, $0 < \delta < 25n^2/2$, $\lambda = 2/((1-b)t) < n - (3/2)$, $n \ge 6$,

$$\frac{(2\sqrt{n}M(F))^n m}{(H(x_i, y_i))^n} < \min\{1, t - \tau, e^{A + \log 3}(H(x_i, y_i))^{-\lambda}\}, \qquad i \ge 2;$$

therefore, for $k > 2$ the solutions (x_2, y_2) and (x_k, y_k) (if any) satisfy all the conditions of Lemma 3 supposing m is large enough. The comparison of inequalities (2) and (5) leads to $k \le 6$.

In case of $|\alpha_1| > 1$ the whole argument can be repeated, noting that

$$\begin{aligned} \left|\frac{1}{\alpha_1} - \frac{y_i}{x_i}\right| &\le \left|\frac{y_i}{x_i}\right|\left|\alpha_1 - \frac{x_i}{y_i}\right| \\ &\le \frac{|x_i| + m^{\frac{1}{n}}}{|\alpha_1 x_i|}\left|\alpha_1 - \frac{x_i}{y_i}\right| \\ &< 2\left|\alpha_1 - \frac{x_i}{y_i}\right|, \end{aligned}$$

$i = 1, \ldots, k$. Since α_1 is an arbitrary zero of $F(X, 1)$ the Theorem is proved. □

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ENDNOTES

1. Two binary forms, say G_1 and G_2, are said to be equivalent if there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ such that $G_2(X, Y) = G_1(aX + bY, cX + dY)$.

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Kuwait University, Department of Mathematics, P.O. Box 5969, 13060 Safat, Kuwait