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HUMBERT SURFACES AND TRANSCENDENCE PROPERTIES OF AUTOMORPHIC FUNCTIONS

PAULA BEAZLEY COHEN

Dedicated to Wolfgang M. Schmidt on the occasion of his 60th birthday

1. Introduction and statement of results. Let G be a Zariski connected reductive algebraic group defined over Q such that the abelian part of $G(\mathrm{R})$ is compact. Let $G^o(\mathrm{R})$ be the topological identity component and K be a maximal compact subgroup of $G^o(\mathrm{R})$. Suppose that the quotient $G^o(\mathrm{R})/K$ has an invariant complex structure and hence is isomorphic as a complex manifold to a bounded symmetric domain $\mathcal{D} \subset \mathrm{C}^m$ for an integer $m \geq 1$. A point $z \in \mathcal{D}$ is called a special point if it is the fixed point of a maximal torus $T \subset G$ defined over Q for which $T(\mathrm{R})$ is compact. Suppose that $\mathcal{D}$ is realized in C^m in such a way that the special points are in $\mathcal{D} \cap \overline{\mathrm{Q}}^m$. If Γ is a (neat) arithmetic subgroup of G, there is a Γ-invariant holomorphic map $J = J(\mathcal{D}, \Gamma)$ of $\mathcal{D}$ into a projective space which induces a biregular isomorphism of $\Gamma\backslash\mathcal{D}$ onto a complex quasi-projective variety V [**1**]. Moreover, Faltings showed in [**9**] that the variety V can be defined over $\overline{\mathrm{Q}}$ and that the $\overline{\mathrm{Q}}$-structure of V may be uniquely determined by requiring of $(\mathcal{D}, J, V)$ that all special points $z \in \mathcal{D}$ have $\overline{\mathrm{Q}}$-rational image point $J(z)$ in V. (For modulus varieties of abelian varieties of given PEL-type this was shown in [**22, 23**]. Faltings' approach of course bypasses abelian varieties. For Hilbert modular surfaces, Faltings' proof is written out in [**25**, p. 82]. We call a triple $(\mathcal{D}, J, V)$ as above a normalized model over $\overline{\mathrm{Q}}$ for (G, Γ). It seems reasonable to make the following:

Prediction. *Let $(\mathcal{D}, J, V)$ be a normalized model over $\overline{\mathrm{Q}}$ for (G, Γ) with $\mathcal{D} \subset \mathrm{C}^m$. Then $z \in \mathcal{D} \cap \overline{\mathrm{Q}}^m$ and $J(z) \in V(\overline{\mathrm{Q}})$ if and only if z is a special point.*

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Given the results of [**9**], the "if" part of the above Prediction is a consequence of the definition of a normalized model over $\overline{\mathbf{Q}}$. The "only if" part is supported by the results obtained to date on the transcendence properties of suitably normalized automorphic functions. Recall that, in 1937, Th. Schneider [**17**] showed that the elliptic modular function takes an algebraic value at an algebraic point if and only if the point in question is imaginary quadratic. A generalization of this result to automorphic functions with respect to the norm unit group of a (maximal) order in an indefinite quaternion algebra over Q was obtained in 1972 by Morita [**13**]. In 1992, Shiga [**18, 19**] considered Picard modular functions, the Matsumoto theta map and Siegel modular functions. Picard modular functions were also dealt with by Holzapfel [**10**] in 1993. In [**4, 3, 20**] (henceforth abbreviated as [**3, 20**]) a generalization of Schneider's theorem was obtained which includes all the above results by extending them to the context of modulus varieties for abelian varieties with some additional structure. The setting was that of Shimura's paper [**21**] on analytic families of polarized abelian varieties and automorphic functions. In dimension greater than 1, these results do not in general give information about the transcendence properties of individual automorphic functions, but rather about the nonvanishing of the transcendence degree of the field generated by certain values of all suitably normalized automorphic functions.

We prove in Sections 3 and 4 the Theorem stated below: namely that the Prediction comes true for normalized models $(\mathcal{D}, J, V)$ over $\overline{\mathbf{Q}}$ of (G, Γ) for which $(G, \mathcal{D})$ admits a symplectic embedding. This generalizes the central result given by the Main Theorem and its Corollary in [**4**] and [**20**] and by the Théorème in [**3**]. We owe to Alice Silverberg a suggestion which led to the modular embedding remarks of Section 4 of [**3, 20**] and which was also useful in the present article. Moreover, a key idea having its origin in Shiga's article [**18**] was an inspiration here as it was in [**3, 20**]. Shiga's idea was to apply a special case of Wüstholz's analytic subgroup theorem [**26, 27**] to reduced period matrices of abelian varieties defined over $\overline{\mathbf{Q}}$ in order to generalize Schneider's result to the Siegel modular case. However, the proof in [**18**] presents difficulties and this motivated the Cohen-Shiga-Wolfart collaboration in [**3, 20**]. The proof in Sections 3 and 4 of the more general Theorem of the present paper implies, in particular, an alternative and more direct treatment, for the case of the modulus varieties of [**21**], than the one of [**3, 20**]. Nonetheless, there are a number of other related additional results in [**3, 20**] which we do not mention here and whose proofs are not covered in the present paper or elsewhere. In Sections 5 and 6 of the present article we give one of the simplest examples of the methods of Sections 2, 3 and 4 in dimension greater than 1, namely that of Hilbert modular functions in dimension 2 (Proposition 1). Despite its simplicity, this example brings out the basic ideas and also serves to show that the result of the Theorem is open to refinement (Proposition 2).

We turn now to the statement of results. Let $n \geq 1$ be a positive integer and

$$\mathcal{S}_n = \{z \in \mathrm{M}_n(\mathrm{C}) | z = z^t, \mathrm{Id}_n - z\bar{z} >> 0\} \subset \mathrm{C}^{n^2}$$

the (bounded) Siegel domain of degree n. We shall often work with the unbounded realization of $\mathcal{S}_n$ given by the Hermitian symmetric domain

$$\mathcal{H}_n = \{\tau \in \mathrm{M}_n(\mathrm{C}) | \tau = \tau^t, \mathrm{Im}(\tau) >> 0\}.$$

The domains $\mathcal{S}_n$ and $\mathcal{H}_n$ are related by a biholomorphic transformation $\tau \mapsto z(\tau)$ that preserves $\mathrm{M}_n(\overline{\mathrm{Q}})$. The special points of $\mathcal{S}_n$ and $\mathcal{H}_n$ are in $\mathrm{M}_n(\overline{\mathrm{Q}})$ and correspond to abelian varieties of CM (complex multiplication) type. We redub them CM-points. Let Λ be an R-vector space of dimension $2n$ carrying a symplectic form E. Let

$$\mathrm{Sp}(\Lambda, E) = \{g \in GL(\Lambda) | E(gv, gv') = E(v, v'), v, v' \in \Lambda\}$$

be the symplectic group fixing E. Then $\mathrm{Sp}(\Lambda, E)$ acts on the complex manifold $\mathcal{S}(\Lambda, E)$ of complex structures I on Λ with $E(., I.)$ symmetric positive definite, via

$$(g, I) \longmapsto gIg^{-1}, \qquad g \in \mathrm{Sp}(\Lambda, E),\ I \in \mathcal{S}(\Lambda, E).$$

Let T be the C-linear vector space given by the $+i$ eigenspace of I in $\Lambda \otimes_{\mathrm{R}} \mathrm{C}$. Then there is an ordered C-basis $\{f_1, \ldots, f_n\}$ of T and an R-basis $\{f_1, \ldots, f_{2n}\}$ of Λ such that the matrix of $(f_{n+1}, \ldots, f_{2n})$ with respect to $\{f_1, \ldots, f_n\}$ is given by $\tau \in \mathcal{H}_n$. We write $I = I_\tau = I_z$ where $z = z(\tau)$ (see for example [**16**, Chapter 2, Section 7]). The

Hermitian symmetric space $\mathcal{S}(\Lambda, E)$ may be identified via the Harish-Chandra embedding [**16**, p. 81] with $\mathcal{S}_n$ and, by transport of structure, the group $\mathrm{Sp}(\Lambda, E)$ acts on $\mathcal{S}_n$ and $\mathcal{H}_n$.

Let Λ have a Q-structure, so that $\Lambda = \Lambda_{\mathrm{Q}} \otimes_{\mathrm{Q}} \mathrm{R}$ for a Q-vector space Λ_{Q} of dimension $2n$, satisfying $E(\Lambda_{\mathrm{Q}}, \Lambda_{\mathrm{Q}}) \subset \mathrm{Q}$. We now suppose that there is a strongly equivariant embedding of a $(G, \mathcal{D})$ as in the Prediction into $(\mathrm{Sp}(\Lambda, E), \mathcal{S}_n)$, that is [**16**, p. 92] a pair (ι, F) where ι is an R-homomorphism, defined over Q, of linear algebraic groups

$$\iota : G \to \mathrm{Sp}(\Lambda, E)$$

and a holomorphic embedding

$$F : \mathcal{D} \to \mathcal{S}_n$$

such that for $z \in \mathcal{D}, g \in G^o(\mathrm{R})$ we have

$$F(gz) = \iota(g)F(z), \qquad \iota \circ \theta = \theta_S \circ \iota.$$

Here, respectively, θ (θ_S) are Cartan involutions of G ($\mathrm{Sp}(\Lambda, E)$) at a chosen origin $z_0 \in \mathcal{D}$ ($F(z_0) \in \mathcal{S}_n$). Suppose in addition that F maps the special points of $\mathcal{D}$ to CM-points of $\mathcal{S}_n$. We then say that $(G, \mathcal{D})$ admits a symplectic embedding. We have:

Theorem. *Let $(\mathcal{D}, J, V)$ be a normalized model over $\overline{\mathrm{Q}}$ for (G, Γ) where $(G, \mathcal{D})$ admits a symplectic embedding. Then $z \in \mathcal{D} \cap \overline{\mathrm{Q}}^m$ and $J(z) \in V(\overline{\mathrm{Q}})$ if and only if z is a special point.*

To see that the result of the Theorem in the Siegel modular case, which we will prove in Section 3, gives (in fact a generalization from the context of [**21**] to that of [**22, 23**] of) the result of the Main Theorem and its Corollary in [**3, 20**], consider the construction by Shimura in [**21, 22, 23**] of modulus varieties for families of abelian varieties of a given PEL-type. Namely, let I_z be the complex structure on Λ corresponding to $z \in \mathcal{S}_n$. Let $\mathcal{B}$ be a semi-simple algebra over Q with positive involution realized as a unital subalgebra, with involution induced by E, of

$$\{\alpha \in \mathrm{End}(\Lambda_{\mathrm{Q}}) | \alpha I_{z_0} = I_{z_0} \alpha\}$$

for a fixed $z_0 \in \mathcal{S}_n$. Let

$$G(\mathcal{B}) = \{g \in \mathrm{Sp}(\Lambda, E) | g\alpha = \alpha g, \alpha \in \mathcal{B}\}$$

and

$$\mathcal{D} = \mathcal{D}(\mathcal{B}) = \{z \in \mathcal{S}_n | I_z\alpha = \alpha I_z, \alpha \in \mathcal{B}\}.$$

The Zariski connected identity component G of $G(\mathcal{B})$ is a reductive Hermitian linear algebraic group defined over Q with associated symmetric domain $\mathcal{D} \subset \mathrm{C}^m$. A symplectic embedding of $(G, \mathcal{D})$ is realized here as a natural inclusion of G in $\mathrm{Sp}(\Lambda, E)$ and $\mathcal{D}$ in $\mathcal{S}_n$. We refer to [**22**, p. 319] and [**16**, Chapter 4] for details. The existence of normalized models over $\overline{\mathrm{Q}}$ in the present context was shown in [**22, 23**]. (The above embedding solution has a certain rigidity. Shimura studied solutions in the nonrigid case which are needed in the construction of canonical models, see for example [**6, 7, 11**].)

To state Propositions 1 and 2 we now consider the case where V is a Hilbert modular surface. Throughout the present article, the background reference for Hilbert modular surfaces, from which we shall borrow heavily, is van der Geer's book [**25**]. Hilbert modular surfaces are the moduli varieties for polarized abelian varieties whose endomorphism ring contains a given order in a totally real quadratic number field. More specifically, let F be a totally real quadratic number field and consider, for simplicity of exposition, the order given by the ring Θ of integers of F. Any projective Θ-module of rank 2 is isomorphic to the direct sum $\Theta \oplus \mathcal{A}$ for some fractional ideal $\mathcal{A}$ of Θ. Let $\mathrm{SL}(\Theta \oplus \mathcal{A})$ be the set of matrices in $\mathrm{SL}(2, F)$, of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{where } a, d \in \Theta, b \in \mathcal{A}^{-1}, c \in \mathcal{A},$$

and let $\Gamma = \mathrm{PSL}(\Theta \oplus \mathcal{A}) = \mathrm{SL}(\Theta \oplus \mathcal{A})/\{+1, -1\}$. Via the Galois embeddings of F into R, the group Γ acts on $\mathcal{H}^2$ where $\mathcal{H}$ is the complex upper half plane. Let T be the standard form on $\Theta \oplus \mathcal{A}$ given by

$$T((\theta, a), (\theta', a')) = \mathrm{Tr}_{\mathrm{F/Q}}(\theta a' - a\theta'), \quad (\theta, a), (\theta', a') \in \Theta \oplus \mathcal{A},$$

where $\mathrm{Tr} = \mathrm{Tr}_{\mathrm{F/Q}}$ is the sum of the two Galois conjugates of an element of F. Then [**25**, p. 208] the space $\Gamma\backslash\mathcal{H}^2$ is the moduli space of triples (A, j, r) where A is a polarized abelian complex surface equipped with an injective order homomorphism $j : \Theta \to \mathrm{End(A)}$ and a Θ-module isomorphism $r : \mathrm{H}_1(\mathrm{A}, \mathrm{Z}) \tilde{\to} \Theta \oplus \mathcal{A}$ carrying the Riemann form on A to the standard form T. (The polarization given by the standard form on $\Theta \oplus \mathcal{A}$ is a principal polarization if and only if $\mathcal{A}$ is the Z-dual $\Theta^\vee$ of Θ with respect to Tr.) Let $(\mathcal{H}^2, J_H, V)$ be a normalized model over $\overline{\mathrm{Q}}$ as above so that $J_H : \mathcal{H}^2 \to V(\mathrm{C})$ is a Γ-invariant holomorphic map sending CM-points to $\overline{\mathrm{Q}}$-rational points of V. The transcendence part of the statement of the Theorem can, in the present case, be improved to the following:

Proposition 1. *If $z = (z_1, z_2), z_i \in \mathcal{H}, i = 1, 2$ is not a CM-point and at least one of z_1 or z_2 is in $\mathcal{H} \cap \overline{\mathrm{Q}}$ then $J_H(z) \notin V(\overline{\mathrm{Q}})$.*

We give the proof of Proposition 1 in Section 5 (for $\mathcal{A} = \Theta$ this is a variant of the proof appearing in [**3**]. A different proof for $\mathcal{A} = \Theta$ appears in [**4, 20**]. There is a natural mapping of the space $\mathrm{PSL}(\Theta \oplus \mathcal{A})\backslash\mathcal{H}^2$, which consists of dropping the extra endomorphism structure of the polarized complex abelian surfaces for which it is a modulus variety, into $\mathrm{PSp}(\mathcal{L})\backslash\mathcal{H}_2$ where $\mathcal{H}_2$ is the Siegel upper half space of degree 2, and $\mathrm{Sp}(\mathcal{L})$ is the subgroup of $\mathrm{Sp}(4, \mathrm{Q})$ fixing a symplectic lattice $\mathcal{L}$, with elementary divisors $e_1|e_2$ given by the elementary divisors of the abelian group $\Theta^\vee/\mathcal{A}$. Without loss of generality for the sequel, we always choose the minimal Riemann form of the polarization of a complex abelian surface, so that $e_1 = 1, e_2 = c$ where $c \geq 1$ is a positive integer. Let $(\mathcal{H}_2, J_S, W)$ be a normalized model over $\overline{\mathrm{Q}}$ so that $J_S : \mathcal{H}_2 \mapsto W(\mathrm{C})$ maps CM-points to $\overline{\mathrm{Q}}$-rational points of W. The map J_H (J_S) has projective coordinates Hilbert (Siegel) modular forms of sufficiently high weight defined over $\overline{\mathrm{Q}}$, that is with Fourier expansions with algebraic coefficients at the cusps. For each modulus $J_H(z)$ of a CM-point $z \in \mathcal{H}^2$ the above natural map determines a unique modulus $J_S(\tau)$ of a CM-point $\tau \in \mathcal{H}_2$. As the CM-points are dense in $V(\overline{\mathrm{Q}})$, the natural map induces a morphism defined over $\overline{\mathrm{Q}}$ from V to W. The Humbert surfaces of the title refer to the images of Hilbert modular surfaces in Siegel modular threefolds under these natural maps (see [**25**, Chapter 9] for a detailed treatment). By using the description [**25**, p. 212, Theorem (2.4)] of the representations in $\mathcal{H}_2$ of the irreducible components of the Humbert surfaces together with a modular embedding argument we deduce in Section 6, with notations as above, the following:

Proposition 2. *Suppose that*

$$\tau = \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix} \in \mathcal{H}_2$$

is not a CM-point. Let a, b, c *be integers with* $0 \leq b < 2c$ *and* $\Delta = b^2 - 4ac$ *positive and square-free. Suppose that* $\tau_1 + b\tau_2 + ac\tau_3 = 0$. *Then if either* $\tau_2 + (1/2)(b + \sqrt{\Delta})\tau_3$ *or* $\tau_2 + (1/2)(b - \sqrt{\Delta})\tau_3$ *is an algebraic number then* $J_S(\tau) \notin W(\overline{\mathrm{Q}})$.

2. A transcendence result and an intertwiner proposition. The basic transcendence result needed for the sequel, given in Proposition 3 below, is a special case of [**26**, Theorem 5] itself a consequence of Wüstholz's analytic subgroup theorem (see [**26, 27**]). First we introduce some notation. Let A be an abelian variety defined over $\overline{\mathrm{Q}}$ and T_A be the $\overline{\mathrm{Q}}$-vector space given by the tangent space to A at the origin $\mathcal{O}_A$. Let $T_A(\mathrm{C}) = T_A \otimes_{\overline{\mathrm{Q}}} \mathrm{C}$ with $\exp_A : T_A(\mathrm{C}) \mapsto A(\mathrm{C})$ the exponential map and $\mathcal{L} = \exp_A^{-1}(\mathcal{O}_A)$. Let L denote the realization of $D = \mathrm{End}_{\mathrm{o}}(A) = \mathrm{End}(A) \otimes_{\mathrm{Z}} \mathrm{Q}$ given by the C-linear maps on $T_A(\mathrm{C})$ which preserve $\mathcal{L}_{\mathrm{Q}} = \mathcal{L} \otimes_{\mathrm{Z}} \mathrm{Q}$.

Proposition 3 (G. Wüstholz). *Let* A *be a simple abelian variety of dimension* n *defined over* $\overline{\mathrm{Q}}$. *Any* $\overline{\mathrm{Q}}$*-basis of* T_A *defines an isomorphism* ϕ *of* $T_A(\mathrm{C})$ *with* C^n. *The components of the vectors in* $\phi(\mathcal{L})$ *generate a* $\overline{\mathrm{Q}}$*-vector space of dimension* $2n^2/[L : \mathrm{Q}]$.

More details of the remarks that follow may be found in [**21**, pp. 155–157, 161–162]. We continue to suppose that A is simple so that L is a division algebra over Q with positive involution. Let k be the center of L. Then k is either a totally real field or a totally imaginary quadratic extension of a totally real field. Let L_C be any complex matrix representation of L of degree n. The rational representation of D is equivalent over C to the direct sum of L_C and its complex conjugate representation. In the totally real case, one deduces from this that L_C is equivalent over C to L_0 where $L_0 = \oplus_{\nu=1}^{g} \mu_\nu \chi_\nu$ and $\chi_\nu, \nu = 1, \ldots, g$ for $g = [k : \mathrm{Q}]$ are the mutually inequivalent absolutely irreducible matrix representations of L. The multiplicity $\mu_\nu = \mu$ and the dimension d over $\overline{\mathrm{Q}}$ of χ_ν are independent of ν and one has $\mu^2 = n^2/(g^2 d)$. Let B be in the set $\mathcal{B}$ of intertwiners of L_0 in $\mathrm{GL}_n(\mathrm{C})$. Then the number of nonzero matrix coefficients of B is no greater than $g\mu^2 = n^2/(gd) = n^2/[L:\mathrm{Q}]$. In the totally imaginary case L_C is equivalent over C to L_0 where $L_0 = \oplus_{\nu=1}^{g}(r_\nu \chi_\nu + s_\nu \bar{\chi}_\nu)$ with now $g = (1/2)[k:\mathrm{Q}]$ and, again by comparing with the rational representation, we have for the multiplicities that $r_\nu + s_\nu = q(2n)/[L:\mathrm{Q}]$ for all $\nu = 1, \ldots, g$. Here q^2 is the dimension of L over k. Again, let B be in the intertwiners $\mathcal{B}$ as above. Then the number of nonzero matrix coefficients of B is no greater than $b = \sum_{\nu=1}^{g}(r_\nu^2 + s_\nu^2)$. Now, if $r_\nu s_\nu \neq 0$ for some $\nu = 1, \ldots, g$ then b is strictly less than $\sum_{\nu=1}^{g}(r_\nu + s_\nu)^2 = 2n^2(2q^2 g)/[L:\mathrm{Q}]^2 = 2n^2/[L:\mathrm{Q}]$ as $2q^2 g = [L:\mathrm{Q}]$. If A has complex multiplication then k is totally imaginary and $r_\nu s_\nu = 0$ for all $\nu = 1, \ldots, g$. Conversely, if these latter conditions hold, then [**21**, Proposition 14], implies A has complex multiplication. We have

Proposition 4. *Let $\mathcal{B}$ be the set of intertwiners of L_0 in* $\mathrm{GL}_n(\mathrm{C})$. *Then if A is a simple abelian variety without complex multiplication, any matrix B in $\mathcal{B}$ has strictly less than* $2n^2/[L:\mathrm{Q}]$ *nonzero coefficients.*

In Section 3, we use the results of this section to prove the Theorem of Section 1 for the Siegel modular case. A result in [**16**] on symplectic embeddings then allows us in Section 4 to complete the proof of the Theorem in general.

3. Proof of the theorem in the Siegel modular case. We adopt the notations of Section 2 and let $\mathcal{L}_1 = \mathcal{L}_1(A)$ be the $\overline{\mathrm{Q}}$-vector space of elements of $T_A(\mathrm{C})$ given by the linear combinations over $\overline{\mathrm{Q}}$ in the elements of $\mathcal{L}$. To prove the Theorem in the Siegel modular case, it suffices to do the following:

We suppose that A is an abelian variety defined over $\overline{\mathrm{Q}}$, not of complex multiplication type, for which the $\overline{\mathrm{Q}}$-vector space $\mathcal{L}_1$ in $T_A(\mathrm{C})$ is of dimension $n = \dim(A)$ over $\overline{\mathrm{Q}}$ and we show that this leads to a contradiction.

We begin by supposing that A is simple. Let L be the realization of $D = \mathrm{End}_0(A)$ on $T_A(\mathrm{C})$. As L preserves $\mathcal{L}_1$, any $\overline{\mathrm{Q}}$-basis of $\mathcal{L}_1$ determines a $\overline{\mathrm{Q}}$-matrix representation L_1 of L of degree n. Moreover, as A is defined over $\overline{\mathrm{Q}}$, any $\overline{\mathrm{Q}}$-basis of T_A gives rise to a $\overline{\mathrm{Q}}$-matrix representation L_2 of L of degree n. As we saw in Section 2 any C-matrix representation of L of degree n is equivalent over C to the $\overline{\mathrm{Q}}$-matrix representation L_0 defined in that same section. Therefore, for $i = 1, 2$ the matrix representation L_i is equivalent over C to L_0 and hence is equivalent over $\overline{\mathrm{Q}}$ to L_0 (use for example [**5**, Theorem (29.7), p. 200] applied to the algebra $D_{\overline{\mathrm{Q}}} = D \otimes_{\mathrm{Q}} \overline{\mathrm{Q}}$). Therefore, there is a basis of $T_A(\mathrm{C})$ with elements in $\mathcal{L}_1$ which defines an isomorphism $\phi_1 : T_A(\mathrm{C}) \to \mathrm{C}^n$ inducing the matrix representation L_0 of L. Moreover, there is also a basis of $T_A(\mathrm{C})$ with elements in T_A defining an isomorphism $\phi_2 : T_A(\mathrm{C}) \to \mathrm{C}^n$ which also induces the representation L_0 of L. Therefore, there is a matrix B in $\mathrm{GL}_n(\mathrm{C})$ which intertwines L_0 such that $\phi_2(\mathcal{L}) = B\phi_1(\mathcal{L})$. But $\phi_1(\mathcal{L}) \subset \overline{\mathrm{Q}}^n$ hence the vector components of $\phi_2(\mathcal{L})$ lie in the vector space generated over $\overline{\mathrm{Q}}$ by the non-zero matrix components of B, which are strictly less than $2n^2/[L : \mathrm{Q}]$ in number by Proposition 4 of Section 2, as A was supposed without complex multiplication. But this contradicts Proposition 3 of Section 2.

To deal with the case where A is not simple we use the fact that up to isogeny A decomposes as a direct product of powers of simple abelian varieties A_k defined over $\overline{\mathrm{Q}}$. As A is not of complex multiplication type one of the A_k does not have complex multiplication and clearly the property for A_k that $\mathcal{L}_1(A_k)$ be of dimension $\dim(A_k)$ over $\overline{\mathrm{Q}}$ holds for this abelian variety as we assume it holds for A. Therefore, it is enough to treat the simple case (see also the remarks in Step (i), Lemma 1 of **18**]).

4. Proof of the theorem. With the notations of the Theorem, there is a strongly equivariant embedding (ι, F) of $(G, \mathcal{D})$ into $(\mathrm{Sp}(\Lambda, E), \mathcal{S}_n)$, where Λ has a Q-structure $\Lambda = \Lambda_{\mathrm{Q}} \otimes \mathrm{Q}$ as in Section 1. In order to realize $\mathcal{D}$ and $\mathcal{S}_n$ as bounded symmetric domains one can exploit the Harish-Chandra embedding, and in this way realize F as the restriction of a C-linear map (see [**16**, p. 85]). This map on $\mathcal{D}$ maps the special points to CM-points and is injective, hence an image point is in $\mathcal{S}_n \cap \overline{\mathrm{Q}}^{n^2}$ if and only if its preimage is in $\mathcal{D} \cap \overline{\mathrm{Q}}^m$. For our purposes, we shall need just the "if" part of this last statement as given in the

Lemma below. As Γ is arithmetic, by replacing it by a subgroup of finite index if necessary, one may suppose that $\iota(\Gamma) \subset \iota(G^o(\mathrm{R})) \cap \mathrm{Sp}(\Lambda_{\mathrm{Z}})$ where Λ_{Z} is a Z-module of rank $2n$ for which $\Lambda_{\mathrm{Q}} = \Lambda_{\mathrm{Z}} \otimes_{\mathrm{Z}} \mathrm{Q}$ and $E(\Lambda_{\mathrm{Z}}, \Lambda_{\mathrm{Z}}) \subset \mathrm{Z}$ with $\mathrm{Sp}(\Lambda_{\mathrm{Z}})$ being the subgroup of $\mathrm{Sp}(\Lambda, E)$ fixing Λ_{Z}. Let $(\mathcal{S}_n, J_S, W)$ be a normalized model over $\overline{\mathrm{Q}}$. The map F induces a quotient morphism $\mu : V \to W$ which sends the images of the special points to images of CM-points. As the images of the special points are dense in $V(\overline{\mathrm{Q}})$ the map μ is rational over $\overline{\mathrm{Q}}$. We now have the following Lemma, from which the Theorem follows immediately:

Lemma. *There is a holomorphic embedding F of $\mathcal{D}$ into $\mathcal{S}_n$ compatible with a group injection ι of Γ into $\mathrm{Sp}(\Lambda_{\mathrm{Z}})$ such that $F(\mathcal{D} \cap \overline{\mathrm{Q}}^m) \subset F(\mathcal{D}) \cap \overline{\mathrm{Q}}^{n^2}$ and the induced quotient map $\mu : V \to W$ is a rational morphism defined over $\overline{\mathrm{Q}}$.*

5. Proof of Proposition 1. We now apply the considerations of Sections 2 and 3 to prove Proposition 1. We retain the notations of those sections and of Section 1. Suppose the Proposition is false so that $z = (z_1, z_2)$, $z_i \in \mathcal{H}$, $i = 1, 2$ is not a CM-point, at least one of z_1 or z_2 is in $\mathcal{H} \cap \overline{\mathrm{Q}}$ but $J_H(z) \in V(\overline{\mathrm{Q}})$. Recall that $\Gamma \backslash \mathcal{H}^2$ is the moduli space of triples (A, j, r) as in Section 1. As $J_H(z) \in V(\overline{\mathrm{Q}})$ there is such a triple whose abelian surface A is defined over $\overline{\mathrm{Q}}$ with $A(\mathrm{C})$ isomorphic as a complex torus to $\mathrm{C}^2/(\Theta.z + \mathcal{A}.1)$. Here z is written as the column vector ${}^t(z_1, z_2)$ in C^2 and $1 = {}^t(1, 1)$. The action of $j(F)$ is determined by the action of an element α of F on $w = {}^t(w_1, w_2) \in \mathrm{C}^2$ by $\alpha.w = {}^t(\alpha w_1, \alpha^\sigma w_2)$ where σ is the non-trivial Galois embedding of F into R. Notice that the mutually inequivalent absolutely irreducible representations of F are given by the identity and σ. Let $\mathcal{B}$ be the commutant in $\mathrm{GL}_2(\mathrm{C})$ of the above realization of $j(F)$ as

$$\left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^\sigma \end{bmatrix} \middle| \alpha \in F \right\}.$$

A direct computation shows

$$\mathcal{B} = \left\{ \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \middle| b_1, b_2 \in \mathrm{C}, b_1 b_2 \neq 0 \right\}.$$

Let $T_A(\mathrm{C})$ be the complex tangent space obtained by extension of scalars from T_A as in Section 2. Then there is a basis of T_A with respect to which we obtain also on $T_A(\mathrm{C})$ the above representation of $j(F)$. Therefore, there are generators $\lambda_1 = {}^t(\lambda_{11}, \lambda_{21})$ and $\lambda_2 = {}^t(\lambda_{12}, \lambda_{22})$ over $j(F)$ of the Q-lattice $\mathcal{L}_{\mathrm{Q}}$ and a matrix

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \in \mathcal{B}$$

such that $\lambda_{11} = b_1 z_1$, $\lambda_{21} = b_2 z_2$, $\lambda_{12} = b_1, \lambda_{22} = b_2$. Therefore, if either z_1 or z_2 is algebraic the components of the vectors λ_1 and λ_2 generate a vector space U over $\overline{\mathrm{Q}}$ of dimension at most 3. Suppose that A is simple and that $j(F)$ is in the center of $\mathrm{End}_{\mathrm{o}}(A)$. If $j(F) = \mathrm{End}_{\mathrm{o}}(A)$ then by Proposition 3, Section 2, the dimension of U is no less than 4, so that $\mathrm{End}_{\mathrm{o}}(A)$ strictly contains $j(F)$ and has the maximal dimension 4 over Q. If $j(F)$ is all of the center of $\mathrm{End}_{\mathrm{o}}(A)$ then [**21**, p. 152] $\mathrm{End}_{\mathrm{o}}(A)$ is of dimension 4 over $j(F)$ which is impossible. Therefore the center of $\mathrm{End}_{\mathrm{o}}(A)$ is a totally imaginary quadratic field extension of $j(F)$ equal to all of $\mathrm{End}_{\mathrm{o}}(A)$ and so A has complex multiplication. This contradicts the hypothesis that z is not a CM-point and concludes the proof for the case where the center of $\mathrm{End}_{\mathrm{o}}(A)$ contains $j(F)$ and A is simple.

If A is simple and $j(F)$ is not in the center of $\mathrm{End}_{\mathrm{o}}(A)$ then $\mathrm{End}_{\mathrm{o}}(A)$ is an indefinite quaternion algebra over Q and this case reduces to Morita's result [**13**] mentioned in Section 1. When A is not simple, it is isogenous to a product of two elliptic curves. As $\mathrm{End}_{\mathrm{o}}(A)$ contains $j(F)$ these elliptic curves are isogenous and the Proposition follows from Schneider's theorem on the elliptic modular function [**17**] also mentioned in Section 1. For more details on these two cases see [**25**, Chapter 5].

6. Proof of Proposition 2. More details of the modular embedding remarks of this section may be found in [**25**, Chapter 9, Sections 1 and 2]. We adopt the notation of Section 1 of the present article. Given a positive integer c and a positive square-free integer Δ such that $\Delta = b^2 - 4ac$ for integers a, b with $0 \leq b < 2c$, the Humbert surface H_Δ can be defined as the complex surface in $\mathrm{Sp}(\mathcal{L})\backslash\mathcal{H}_2$, where $\mathcal{L} = \mathcal{L}_c = \mathrm{Z}^3 \times \mathrm{Z}c$, whose irreducible components are given by the image under

$$\pi : \mathcal{H}_2 \to \mathrm{Sp}(\mathcal{L})\backslash\mathcal{H}_2$$

of the

$$\tau = \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix} \in \mathcal{H}_2$$

with $\tau_1 + b\tau_2 + ac\tau_3 = 0$ [**25**, Theorem(2.4)]. As Δ is square-free, by [**25**, Proposition(2.3)], if Θ_Δ is the order of discriminant Δ in $F = \mathbb{Q}(\sqrt{\Delta})$ and A is a complex abelian surface $\mathbb{C}^2/\mathbb{Z}^2 + \tau\mathbb{Z}^2$, $\tau \in \mathcal{H}_2$, then $\mathrm{End}(A)$ contains Θ_Δ precisely when $\pi(\tau)$ is in H_Δ. The discussions of Sections 1 and 5 of the present article go through equally well with Θ replaced by Θ_Δ. In particular, each irreducible component of H_Δ corresponds to a strict ideal class of Θ_Δ containing an ideal $\mathcal{A}$ where $\Theta_\Delta^\vee/\mathcal{A} \simeq \mathbb{Z}/c\mathbb{Z}$. Therefore the natural map

$$\rho : \mathrm{PSL}(\Theta_\Delta \oplus \mathcal{A})\backslash\mathcal{H}^2 \longrightarrow \mathrm{Sp}(\mathcal{L})\backslash\mathcal{H}_2$$

will be determined by a matrix transforming the Θ_Δ-module $\Theta_\Delta \oplus \mathcal{A}$ equipped with the standard form into the symplectic lattice $\mathbb{Z}^3 \times \mathbb{Z}c$ (as on [**25**, pp. 209, 210, 212]). As in the proof of Proposition (2.5) of [**25**, p. 212], a direct calculation shows that the image of

$$\iota : \mathcal{H}^2 \hookrightarrow \mathcal{H}_2$$
$$(z_1, z_2) \longmapsto S^{-1} \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} {}^tS^{-1}$$

where S is the base change matrix

$$S = \begin{bmatrix} 1 & \frac{b+\sqrt{\Delta}}{2} \\ 1 & \frac{b-\sqrt{\Delta}}{2} \end{bmatrix},$$

is the surface defined by $\tau_1 + b\tau_2 + ac\tau_3 = 0$. Therefore, if

$$\tau = \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix}$$

is in the image of ι, then it is the image of (z_1, z_2) where

$$\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} = S \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix} {}^tS.$$

A direct calculation shows that

$$z_1 = \tau_1 + (b + \sqrt{\Delta})\tau_2 + \frac{1}{4}(b + \sqrt{\Delta})^2\tau_3$$
$$z_2 = \tau_1 + (b - \sqrt{\Delta})\tau_2 + \frac{1}{4}(b - \sqrt{\Delta})^2\tau_3.$$

On using the fact that $\tau_1 + b\tau_2 + (1/4)(b^2 - \Delta)\tau_3 = 0$ one finds that

$$z_1 = \sqrt{\Delta}(\tau_2 + \frac{1}{2}(b + \sqrt{\Delta})\tau_3)$$

and

$$z_2 = -\sqrt{\Delta}(\tau_2 + \frac{1}{2}(b - \sqrt{\Delta})\tau_3).$$

Proposition 2 now follows from Proposition 1 with Θ replaced by Θ_Δ, as using the interpretation of the values of J-functions in terms of fields of moduli of abelian varieties, one sees that each $\overline{\mathbb{Q}}$-rational modulus $J_S(\tau), \pi(\tau) \in H_\Delta$ is determined by a $\overline{\mathbb{Q}}$-rational modulus $J_H(z)$.

7. Remarks. 1) An alternative proof of the Lemma of Section 4 in the context of Shimura's modulus varieties for families of abelian varieties of given PEL type [**22, 23**] can be deduced from [**23**, Section 3.3] where an F, as in the statement of the Lemma, compatible with the natural inclusion ι is very explicitly constructed and shown to be a rational map of the complex coordinates of $\mathcal{D}$. As the map sends special points to CM-points and is injective, one can conclude the proof of the Lemma as above.

2) It is well known that Wüstholz's announcement [**26**, Theorem 5] used in Section 2, is a consequence of Wüstholz's analytic subgroup theorem. A discussion was provided in [**4**] and [**20**] but with a specific application to the methods of those articles in mind and certainly it is possible to give a more economic proof of [**26**, Theorem 5].

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URA Géométrie - Analyse - Topologie, Université des Sciences et Technologies de Lille, 59655 Villeneuve d'Ascq cedex, France
E-mail address: pcohen@gat.univ-lille1.fr