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QUATERNIONIC BUNDLES ON ALGEBRAIC SPHERES

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ABSTRACT. It is shown that for $n \geq 4$ there are nonfree rank 1 algebraic quaternionic vector bundles on the n-sphere which are topologically trivial. For $n \geq 5$ it is shown that there are uncountably many such bundles.

1. Introduction. An old question asks whether there is a bijection between algebraic and topological vector bundles on spheres. More precisely, let $\mathbf{F}$ be one of $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$, and let $VB_k^{\mathbf{F}}(S^n)$ be the set of isomorphism classes of topological $\mathbf{F}$-vector bundles of rank k on the n-sphere S^n. Let $A_n = \mathbf{R}[x_0, \dots, x_n]/(\sum x_i^2 - 1)$ be the coordinate ring of S^n, and let $P_k(\mathbf{F} \otimes_{\mathbf{R}} A_n)$ be the set of isomorphism classes of finitely generated projective $\mathbf{F} \otimes_{\mathbf{R}} A_n$-modules of rank k. The question then is whether $P_k(\mathbf{F} \otimes_{\mathbf{R}} A_n) \to VB_k^{\mathbf{F}}(S^n)$ is a bijection.

The following results are known about this question.

(1) [**16**]. The stable version of the conjecture is true, i.e., $K_0(\mathbf{F} \otimes_{\mathbf{R}} A_n) \to K_{\mathbf{F}}^0(S^n)_{\text{top}}$ is an isomorphism for all n and for $\mathbf{F} = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$.

(2) [**17**]. The conjecture is true if A_n is replaced by the localization $(A_n)_S$ where $S = \{1 + f_1^2 + \dots + f_s^2 \mid f_i \in A_n, s \geq 0\}$.

(3) [**15**]. For $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, it is true for $k \leq 1$ and all n.

(4) [**1**] (see also [**14**]). If $\mathbf{F} = \mathbf{R}$, it is true for $n \leq 2$.

(4) (Murthy, see [**15**]). If $\mathbf{F} = \mathbf{C}$, it is true for $n \leq 3$.

(5) [**15**]. If $\mathbf{F} = \mathbf{H}$ it is true for $n \leq 1$ (and also for $k = 0$ and all n).

Case (6) was observed by the referee of [**15**] who remarked that $\mathbf{H} \otimes_{\mathbf{R}} A_1$ is a principal ideal domain [**12**, Theorem 5.3] (see also Corollary 5.2).

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One of the main purposes of this paper is to show that the conjecture is actually false for $\mathbf{F} = \mathbf{H}$ if $n \geq 4$.

Theorem A. *If $n \geq 4$, there is a stably free, nonfree $\mathbf{H} \otimes_{\mathbf{R}} A_n$-module Q of rank 1 which is topologically trivial.*

In Section 2 I will give a quick proof based on well-known results of Ojanguren, Parimala, Sridharan, and Wood. A more computational approach (see Sections 7, 8) gives more explicit examples and shows that if $n \geq 5$, there are an infinite number of such modules. The cases of S^2 and S^3 are still open, but I will show that the analogue of Theorem A holds for a number of other quadric hypersurfaces, even in dimensions 2 and 3.

A theorem of Suslin [**13**] shows that the example of Ojanguren and Sridharan is peculiar to the case of rank 1 projective modules and that projective modules of rank greater than 1 over a polynomial ring $\mathbf{H}[x_1, \dots, x_n]$ are free. This gives some hope that the conjecture may yet be true in the quaternionic case for the case of modules of rank at least 2.

Section 3 contains some further results for more general fields, not necessarily real. Section 4 gives an algebraic version of case (2) above for rings of Krull dimension at most 3. Section 5 contains a generalization of [**12**, Theorem 5.3] which was used in case (6) above.

2. Real quadric hypersurfaces. I will prove here a more general version of Theorem A which applies to real hypersurfaces $X \subset \mathbf{R}^{n+1}$ defined by an equation $q(x) = 1$ where q is a nondegenerate quadratic form.

Theorem 2.1. *Let q be a nondegenerate quadratic form in $n+1$ variables over $\mathbf{R}$, and let $A = \mathbf{R}[x_0, \dots, x_n]/(q(x) - 1)$. If $n \geq 4$ or if $n \geq 2$ and q is isotropic, there is a stably free, nonfree $\mathbf{H} \otimes_{\mathbf{R}} A$-module Q of rank 1 which induces a topologically trivial bundle on $X = \{x \in \mathbf{R}^{n+1} \mid q(x) = 1\}$.*

In Section 5 I will show that if $n = 1$, all projective $\mathbf{H} \otimes_{\mathbf{R}} A$-modules are free. Thus the only open cases are those of S^2, S^3, and the imaginary hypersurfaces $x^2+y^2+z^2+1=0$ and $w^2+x^2+y^2+z^2+1=0$.

Ojanguren and Sridharan [**10**, Proposition 1] have constructed a nonfree stably free module over the polynomial ring $D[x,y]$ in two variables over a noncommutative division ring D. If $a, b \in D$ with $c = ab-ba \in D^*$, the module is the kernel of the map $D[x,y]^2 \to D[x,y]$ sending (λ, μ) to $\lambda(x+a)+\mu(y+b)$. I will denote this module by $P(x+a, y+b)$. Further discussion can be found in [**7**, Section 3] and [**3**, pp. 18–19] (see also Section 3).

For the proof of Theorem 2.1 we will need a stronger version of this due to Parimala and Sridharan [**11**] which shows that there are in fact an infinite number of isomorphism classes of such modules over $\mathbf{H}[x,y]$.

Theorem 2.2 [**11**]. *The modules* $P(x+ti, y+j)$ *over* $\mathbf{H}[x,y]$ *with* $t \in \mathbf{R}$*,* $t > 0$*, are all distinct.*

The proof of this in [**11**] involves some rather complicated calculations. I would like to thank Raja Sridharan for showing me the following very simple proof.

Proof. If we invert the central element x^2+t^2, the element $x+ti$ becomes a unit so $P(x+ti, y+j)$ becomes free and therefore extended from $\mathbf{H}[x]_{x^2+t^2}$. If $P(x+si, y+j) \approx P(x+ti, y+j)$ with $s^2 \neq t^2$, Quillen's patching theorem [**7**, Chapter V, Theorem 1.6] shows that $P(x+ti, y+j)$ also becomes extended when $(x^2+t^2)-(x^2+s^2) = t^2-s^2$ is inverted. Since this element is a unit, this would imply that $P(x+ti, y+j)$ is extended from $\mathbf{H}[x]$ and is therefore free since $\mathbf{H}[x]$ is a principal ideal domain. This would contradict the theorem of Ojanguren and Sridharan. □

It follows immediately that the same result holds for the modules $P(x+ti, y+j)$ defined, as above, by the unimodular row $(x+ti, y+j)$ over a polynomial ring $\mathbf{H}[x, y, z_1, \ldots, z_m]$ in more variables. We just factor out the ideal $(z_1, \ldots, z_m)$ to get the modules of Theorem 2.1.

Another proof of these results will be given in Section 7 where we will also see that $P(x+ti, y+j) \approx P(x-ti, y+j)$. This explains the restriction to $t > 0$ in Theorem 2.2.

We will also need to use [**12**, Proposition 6.1] which I will restate in the following form. As usual, $U(R)$ denotes the group of units of R.

Theorem 2.3 [**12**]. *Let G be a finite group acting on an $\mathbf{R}$-algebra A, and let $B = A^G$. Suppose that $U(\mathbf{H} \otimes_{\mathbf{R}} A) = \mathbf{H}^*$. Then there are only a finite number of isomorphism classes of rank 1 projective $\mathbf{H} \otimes_{\mathbf{R}} B$-modules which become free over $\mathbf{H} \otimes_{\mathbf{R}} A$.*

This is proved in [**12**] by observing that the usual Galois descent argument embeds the set of such isomorphism classes in $H^1(G, U(\mathbf{H} \otimes_{\mathbf{R}} A)) = H^1(G, \mathbf{H}^*)$. The extension is not required to be Galois for this. Now $H^1(G, \mathbf{H}^*)$ is easily seen to be finite using the well-known classification of finite subgroups of $\mathbf{H}^*$.

Remark. More generally, if a finite group G acts on a Lie group L with a finite number of connected components, then $H^1(G, L)$ is finite. Borel pointed out to me that this can easily be deduced from results of Hochschild and Mostow. In particular, I would like to thank Borel for the reference [**5**] which provides the necessary justification for the following simple proof: If G acts trivially on L, then $H^1(G, L)$ is just $\mathrm{Hom}\,(G, L)$ modulo conjugation. If K is a maximal compact subgroup of L, $\mathrm{Hom}\,(G, K)/\mathrm{conj} \to \mathrm{Hom}\,(G, L)/\mathrm{conj}$ will be onto by [**5**, Chapter 15, Theorem 3.1] so it is enough to consider the case of K. By [**9**, Section 5.3] each $f : G \to K$ will have a neighborhood U in $\mathrm{Hom}\,(G, K)$ so small that if g lies in U then $ag(G)a^{-1} \subset f(G)$ for some a lying in a very small neighborhood W of 1. If U and W are small enough, this implies that g is conjugate to f since $ag(\sigma)a^{-1} = f(\tau)$ will be so near $f(\sigma)$ that $f(\tau) = f(\sigma)$. So the conjugacy classes are open in $\mathrm{Hom}\,(G, K)$ and therefore a finite number cover $\mathrm{Hom}\,(G, K)$ by compactness. The general case follows by replacing L by the semi-direct product $L \rtimes G$. We have a map $\theta : H^1(G, L) \to \mathrm{Hom}\,(G, L \rtimes G)/\mathrm{conj}$ sending a cocycle f to the section $s(\sigma) = (f(\sigma), \sigma)$. One easily checks that the center $Z(G)$ acts transitively on the fibers of θ sending $f(\sigma)$ to $zf(\sigma)$ for $z \in Z(G)$. Since $\mathrm{Hom}\,(G, L \rtimes G)/\mathrm{conj}$ is finite, it follows that $H^1(G, L)$ is also finite.

Lemma 2.4. *Let q and A be as in Theorem* 2.1*. If $n \geq 4$ or if $n \geq 2$ and q is isotropic, then $U(\mathbf{H} \otimes_{\mathbf{R}} A) = \mathbf{H}^*$.*

For the case $A = A_n$ this follows from a result of R. Wood [**19**] which shows that all polynomial maps $S^n \to S^m$ are constant if $n \geq 2^s > m$ for some s. We need only note that if $u = f_0 + f_1 i + f_2 j + f_3 k$ is a unit of $\mathbf{H} \otimes_{\mathbf{R}} A_n$, then u can be normalized so that $x \mapsto (f_0(x), \dots, f_3(x))$ defines a polynomial mapping $S^n \to S^3$. It would be interesting to know if the converse of Wood's result is true. In other words, if all polynomial maps $S^n \to S^m$ are constant, is there an s with $n \geq 2^s > m$? For example, is there a nonconstant polynomial map $S^{48} \to S^{47}$?

Proof of Lemma 2.4 (Wood [**19**]). Let $u = f_0 + f_1 i + f_2 j + f_3 k$ be a nontrivial unit of $\mathbf{H} \otimes_{\mathbf{R}} A$. Then $r = u\bar{u} = \sum f_i^2$ lies in A^* which is equal to $\mathbf{R}^*$ by [**15**, Lemma 9.1]. So in the polynomial ring $\mathbf{R}[x_0, \dots, x_n]$ we have $\sum f_i^2 - r = (q-1)G$. We can assume that q does not divide the leading forms of the f_i, otherwise we can reduce the degree of f_i by dividing by $q - 1$. Let $\max \deg f_i = d$, and let g_i be the homogeneous part of f_i of degree d. Then at least one of the g_i is nonzero. We have $\sum g_i^2 = qH$ where H is the leading form of G. Since $n \geq 2$, q is irreducible and therefore $\mathbf{R}[x_0, \dots, x_n]/(q)$ is a domain. Let K be its quotient field. In K we have $\sum g_i^2 = 0$ so the level $s(K)$ is at most 3 [**6**, Chapter 11, Section 2]. Now if q is isotropic, K is rational and so real so $s(K) = \infty$. If q is nonisotropic, we can assume that $q = \sum x_i^2$. By [**6**, Chapter 11, Theorem 2.8], we have $s(K) = 2^k$ where $2^k \leq n < 2^{k+1}$ so $s(K) \geq 4$ if $n \geq 4$. □

To prove Theorem 2.1, we let $G = \mathbf{Z}/2\mathbf{Z}$ act on A by $x_0 \mapsto -x_0$ and $x_i \mapsto x_i$ for $i \neq 0$. Then $B = A^G = \mathbf{R}[x_1, \dots, x_n]$. Following the argument of [**12**, Proposition 6.1], consider the modules $P(x_1+ti, x_2+j)$ defined by the unimodular rows $(x_1 + ti, x_2 + j)$ over $\mathbf{H} \otimes_{\mathbf{R}} B$ with $t \in \mathbf{R}^*$. Since these are all nonisomorphic, Theorem 2.3 shows that one of them must remain nonfree when tensored with A, which gives us the required module $Q = A \otimes_B P(x_1 + ti, x_2 + j)$. Since $P(x_1 + ti, x_2 + j)$ becomes free when we localize by inverting $x_1^2 + t^2$ so does Q. Since $x_1^2 + t^2$ has no zeros on X, Q is topologically trivial by the following lemma.

Let X be a topological space which is either (1) compact or (2) paracompct, finite dimensional, and with a finite number of connected components. Let $C(X)$ be the ring of continuous real or complex functions on X. By [**18**] the category of finitely generated projective $C(X)$-modules is equivalent to the category of (real or complex, resp.) vector bundles on X. If $f : A \to C(X)$ is a ring homomorphism and P is a finitely generated projective A-module, then $C(X) \otimes_A P$ corresponds to a vector bundle on X which we call the bundle induced by P.

Lemma 2.5. *If there is an element $s \in A$ such that P_s is free and $f(s)$ has no zeros on X, then P induces the trivial bundle on X.*

Proof. Since $f(s)$ is a unit of $C(X)$, f factors through the localization $A \to A_s$. Since $A_s \otimes_A P = P_s$ is free, so is $C(X) \otimes_A P$ and the vector bundle is therefore trivial. □

In Sections 7 and 8 I will show that, with the possible exception of the hypersurface defined by $x_0^2 + x_1^2 + \cdots + x_4^2 + 1 = 0$, $Q = A \otimes_B P(x_1 + ti, x_2 + j)$ will be nonfree for any $t > 0$ if $n \geq 4$ (and if x_1 and x_2 are chosen properly when $n = 4$).

Remark. If one could extend Lemma 2.4 to the modules $Q = A \otimes_B P(x_1 + ti, x_2 + j)$ by showing that $\mathrm{Aut}\,(Q) = \mathbf{H}^*$ (or even that $\mathrm{Aut}\,(Q)$ is a Lie group with a finite number of components) the above argument would show that the map taking $P(x_1 + ti, x_2 + j)$ to Q is finite to one. In particular, this would show that there are $2^{\aleph_0}$ distinct isomorphism classes of stably free nonfree Q.

3. Other quadric hypersurfaces. The results of this section are based on a generalization of the theorem of Ojanguren and Sridharan [**10**, Proposition 1]. We begin by recalling the original construction [**10**; **7**, Section 2.2; **3**, pp. 18–19]. Let Λ be a ring, and let $a, b \in \Lambda$ be such that $c = ab - ba$ lies in Λ^*, the group of units of Λ. Let x, y be central elements of Λ. Then $(x + a, y + b)$ is a unimodular row since $(y + b)(x + a) - (x + a)(y + b) = -c$. Let $\varphi : \Lambda^2 \to \Lambda$ by $(\lambda, \mu) \mapsto \lambda(x + a) + \mu(y + b)$. Then φ is onto, so its kernel P, here denoted by $P = P(x+a, y+b)$, is projective and $\Lambda \oplus P \approx \Lambda^2$. We want to show that P is not free under suitable hypotheses. If we assume that $\Lambda^2 \approx \Lambda^n$ implies $n = 2$, then P is free if and only if $P \approx \Lambda$.

Lemma 3.1. *Let D be a domain such that $D^2 \approx D^n$ implies $n = 2$. Let $a, b \in D$ with $c = ab - ba$ in D^*. Let $\Lambda = D[x, y_1, \dots, y_n]$ be the polynomial ring over D. Let $f(x, y_1, \dots, y_n)$ be a polynomial over the center of D such that $f(-a, y_1, \dots, y_n)$ is a nonconstant polynomial. Then $P(x + a, f(x, y_1, \dots, y_n) + b)$ is not free.*

Proof. Following a remark of Bass [**3**, p. 19] we observe that the projection $pr_2 : \Lambda^2 \to \Lambda$ gives an isomorphism $P \approx I$ where I is the left ideal of all $\lambda \in \Lambda$ with $\lambda(f + b) \in \Lambda(x + a)$. The kernel is zero since if $(\lambda, 0)$ lies in P then $\lambda(x + a) = 0$ so $\lambda = 0$.

Define a section σ of φ by $\sigma(\lambda) = (-\lambda c^{-1}(f + b), \lambda c^{-1}(x + a))$. Then $P = \operatorname{im}(1 - \sigma\varphi)$. An easy calculation shows that the images of (1,0) and (0,1) under $pr_2 \circ (1 - \sigma\varphi)$ are $-(x+a)c^{-1}(x+a)$ and $1 - (f+b)c^{-1}(x+a)$ so I is the left ideal generated by these elements.

If P is free then $I = \Lambda g$ is principal. Write

(1) $(x + a)c^{-1}(x + a) = hg$.

(2) $1 - (f + b)c^{-1}(x + a) = h_1 g$.

It is clear from (1) that g involves only x and is of degree ≤ 2. We can assume that g is monic since its leading term divides the unit c^{-1}.

Case 1. $\deg g = 0$. Then $g = 1$. Therefore $1 \in \Lambda g$ so there is an element $(\xi, 1)$ in P. It follows that $\xi(x + a) + (f + b) = 0$ showing that $f + b \in \Lambda(x + a)$. Now $f(-a, y_1, \dots, y_n) + b \equiv f + b \equiv 0 \bmod \Lambda(x + a)$ which is clearly impossible since the left side is nonzero and does not involve x.

Case 2. $\deg g = 2$. Then h is a unit since it divides c^{-1}. Therefore, $g = h^{-1}(x + a)c^{-1}(x + a) \in \Lambda(x + a)$. By (2) we get $1 \in \Lambda(x + a)$, a contradiction.

Case 3. $\deg g = 1$. Here $g = x + d$ for some $d \in D$. Since $g \in I$, there is some $(\xi, x + d)$ in P and so $(x + d)(f + b) \equiv 0 \bmod \Lambda(x + a)$. Let

$f_0 = f(-a, y_1, \dots, y_n)$. We have $f \equiv f(-a, y_1, \dots, y_n) \bmod \Lambda(x+a)$ so $(x+d)(f_0+b) \equiv 0 \bmod \Lambda(x+a)$ and so $(x+d)(f_0+b)-(f_0+b)(x+a) \equiv 0 \bmod \Lambda(x+a)$. Since f_0 and a clearly commute, the lefthand side is $(d-a)f_0 + db - ba$. This contains no x and so is zero. Since D is a domain and f_0 is nonconstant, we have $d = a$ and $ab = db = ba$ which contradicts the choice of a and b. □

Taking $f = y$ and $n = 1$ gives the original example of Ojanguren and Sridharan [**10**]. By making use of an observation of Murthy, we can extend this example to the case of "sufficiently split" quadrics.

Theorem 3.2. *Let D be a noncommutative division ring. Let q be a quadratic form over the center of D which is the sum of a hyperbolic form and a form representing 1. Then there is a nonfree stably free rank 1 projective module Q over $\Lambda = D[x_0, \dots, x_n]/(q-1)$.*

Proof. Write $q = uv + w^2 + q'(w, y_3, \dots, y_n)$ where all terms of q' contain some y_i. Choose $a, b \in D$ with $c = ab - ba$ in D^*, and let $Q = P(u+a, w+b)$. It will suffice to show that $Q/(y_3, \dots, y_n)Q$ is nonfree, so we can assume that $\Lambda = D[u, v, w]/(uv + w^2 - 1)$. We now use Murthy's observation that Λ embeds in the polynomial ring $\Gamma = D[x, y]$ by $u \mapsto x$, $v \mapsto -y(2+xy)$, $w \mapsto 1 + xy$. We only need the existence of this map here but the fact that it is injective is easily verified since the basis elements $u^a v^b$ and $u^a v^b w$ of Λ map into monic polynomials with distinct leading terms. It will suffice to show that $D[x, y] \otimes_\Lambda Q$ is not free. This module is $P(x + a, 1 + xy + b)$ over $D[x, y]$, and the result follows from Lemma 3.1. □

4. The Lissner-Moore argument. Let $S = \{1 + f_1^2 + f_2^2 + \cdots + f_s^2 \mid f_i \in A_n, s \geq 0\}$. In [**17**, Theorem 11.1] it was shown, using topological methods, that there is a one to one correspondence between isomorphism classes of finitely generated projective $\mathbf{H} \otimes_{\mathbf{R}} A_n$-modules and isomorphism classes of quaternionic vector bundles on S^n. I will show algebraically that this is so for $n \leq 3$ using ideas of Lissner and Moore [**8**]. See also [**3**, Section 5.6].

Theorem 4.1 (cf. [**8**]). *Let A be an algebra over $\mathbf{R}$, let $S =$*

$\{1 + f_1^2 + f_2^2 + \cdots + f_s^2 \mid f_i \in A, s \geq 0\}$, and let $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$. Let M be a Λ_S-module, and let $\xi \in M$. Then ξ is unimodular over Λ_S if and only if it is unimodular over A_S.

Proof. If ξ is unimodular over Λ_S, let $f : M \to \Lambda_S$ with $f(\xi) = 1$. Let $\Re : \Lambda \to A$ send λ to its real part. Then $\Re \circ f : M \to A_S$ sends ξ to 1 so ξ is unimodular over A_S. If ξ is unimodular over A_S, let $g : M \to A_S$ with $g(\xi) = 1$, and define, following [**8**], $h : M \to \Lambda_S$ by $h(x) = g(x) - ig(ix) - jg(jx) - kg(kx)$. Then h is easily seen to be a Λ_S-homomorphism and $h(\xi) = 1 + ia + jb + kc$ where $a = -g(ix)$, etc. Let $s = 1 + a^2 + b^2 + c^2$, and let $f(x) = h(x)(1 - ia - jb - kc)s^{-1}$. Then f is a Λ_S-homomorphism and $f(\xi) = 1$ as required. □

Recall that a ring R is called real if $r_1^2 + r_2^2 + \cdots + r_s^2 = 0$ with $r_i \in R$, implies $r_i = 0$ for all i. The ring A_n clearly has this property since its quotient field is a pure transcendental extension of $\mathbf{R}$.

Corollary 4.2. *Let A be a real affine domain over $\mathbf{R}$, let $S = \{1 + f_1^2 + f_2^2 + \cdots + f_s^2 \mid f_i \in A, s \geq 0\}$, and let $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$. If* $\operatorname{Krull\,dim} A \leq 3$, *then all finitely generated projective Λ_S-modules are free.*

Proof. We again follow [**8**]. The quotient field K of A is a real field so $H \otimes_{\mathbf{R}} K$ is a division algebra. Let P be a finitely generated projective Λ_S-module. Then $P \otimes_{A_S} K$ is a vector space over $\mathbf{H} \otimes_{\mathbf{R}} K$ showing that the rank of P over A is divisible by 4. If $P \neq 0$, $\operatorname{rk} P \geq 4$ so P has a unimodular element over A_S by Serre's theorem [**2, 4**]. Therefore, P has a unimodular element over Λ_S by Theorem 4.1, so $P \approx \Lambda_S \oplus Q$ and Q is free by induction on the rank. □

5. Principal ideal rings. The following is a somewhat more general version of [**12**, Theorem 5.3].

Theorem 5.1. *Let A be a finitely generated commutative $\mathbf{R}$-algebra. If $\mathbf{C} \otimes_{\mathbf{R}} A$ is a principal ideal domain, then $\mathbf{H} \otimes_{\mathbf{R}} A$ is a left and right principal ideal ring. Moreover, all ideals of $\mathbf{H} \otimes_{\mathbf{R}} A$ are projective.*

Note that $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$ need not be a domain, e.g., if $A = \mathbf{R}[x,y]/(x^2+y^2+1)$ then $u = 1 + ix + jy$ has $u\bar{u} = 0$ and Λu has rank 2 over A. In particular, Λu is not free although it is projective.

Proof. Let $K = Q(A)$ be the quotient field of A. Then $\mathbf{H} \otimes_{\mathbf{R}} K$ is either a division algebra or $\mathcal{M}_2(K)$. Let I be a left ideal of $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$. We can find another left ideal J such that $KI \oplus KJ = K\Lambda$. It will suffice to show that $I \oplus J$ is principal and projective so we can assume that $KI = K\Lambda$. Therefore, $\mathfrak{a} = \operatorname{Ann}_A(\Lambda/I) = I \cap A \neq 0$ and hence Λ/I has finite length as an A-module, A being at most one-dimensional. We use induction on the length $l(\Lambda/I)$. If $I < J < \Lambda$ and $J = \Lambda x$, then $K\Lambda = KJ = K\Lambda x$ so x is a unit in $K\Lambda$ and $Ix^{-1} < Jx^{-1} = \Lambda$ so $I \approx I' = Ix^{-1}$ where $l(\Lambda/I') < l(\Lambda/I)$. This shows that it is enough to consider the case where I is a maximal left ideal. In this case $\mathfrak{a} = \operatorname{Ann}_A(\Lambda/I)$ is prime since if $ab \in \mathfrak{a}$ then $ab\Lambda/I = 0$ but Λ/I is simple so either $b\Lambda/I = 0$ or $b\Lambda/I = \Lambda/I$ and $a\Lambda/I = 0$. Since $\mathfrak{a} \neq 0$, we see that $\mathfrak{a} = \mathfrak{m}$ is a maximal ideal of A and hence $A/\mathfrak{m} = \mathbf{R}$ or $\mathbf{C}$. It follows that $\Lambda/\mathfrak{m}\Lambda = \mathbf{H}$ or $\mathcal{M}_2(\mathbf{C}) = \mathbf{H} \otimes_{\mathbf{R}} \mathbf{C}$. Since Λ/I is a simple module over this, $\Lambda/I = \mathbf{H}$ or $\mathbf{C}^2$. Therefore, $\dim_{\mathbf{R}} \Lambda/I = 4$ in either case so that Λ/I is isomorphic to $\mathbf{H}$ as an $\mathbf{H}$-module. We will identify Λ/I with $\mathbf{H}$. The action of an element a of A on Λ/I commutes with that of $\mathbf{H}$ and so is given by right multiplication by an element $\varphi(a)$ of $\mathbf{H}$, $a \circ q = q\varphi(a)$. Clearly $\varphi : A \to \mathbf{H}$ is a homomorphism of $\mathbf{R}$-algebras. Its image is commutative and so lies in a maximal commutative subfield of $\mathbf{H}$. Therefore, we can choose a standard basis $1, i, j, k$ for $\mathbf{H}$ such that $\varphi(A)$ lies in $\mathbf{C} = \mathbf{R} + \mathbf{R}i$. This implies that $\mathbf{C} \subset \mathbf{H} = \Lambda/I$ is stable under $\mathbf{C} \otimes_{\mathbf{R}} A$ for this choice of $\mathbf{C}$ in $\mathbf{H}$. Let $J = I \cap (\mathbf{C} \otimes_{\mathbf{R}} A)$. Then $(\mathbf{C} \otimes_{\mathbf{R}} A)/J = \mathbf{C}$ since it is the image of $\mathbf{C} \otimes_{\mathbf{R}} A$ in $\mathbf{H} = \Lambda/I$. Now $\Lambda J \subset I$ and $\Lambda/\Lambda J = (\mathbf{C}A \otimes j\mathbf{C}A)/(J \oplus jJ) = \mathbf{C}A/J \oplus j\mathbf{C}A/J$ has dimension four over $\mathbf{R}$ showing that $\Lambda J = I$. By hypothesis, J is principal and therefore so is I.

Let $I = \Lambda x$. Since $KI = K\Lambda$, the map $\Lambda \to \Lambda x$ by $\lambda \mapsto \lambda x$ is an isomorphism showing that I is free.

Corollary 5.2. *Let A be a finitely generated $\mathbf{R}$-algebra which is real. If $\mathbf{C} \otimes_{\mathbf{R}} A$ is a principal ideal domain, then $\mathbf{H} \otimes_{\mathbf{R}} A$ is also a left and right principal ideal domain and so all finitely generated projective $\mathbf{H} \otimes_{\mathbf{R}} A$-modules are free.*

Proof. As in the proof of Corollary 4.2, $\mathbf{H} \otimes_{\mathbf{R}} Q(A)$ is a division algebra so $\mathbf{H} \otimes_{\mathbf{R}} A$ is a domain. □

In particular, Theorem 5.1 shows that the rings $\mathbf{H}[x,y]/(x^2 \pm y^2 \pm 1)$ are all left and right principal ideal rings since $\mathbf{C}[x,y]/(x^2 \pm y^2 \pm 1) \approx \mathbf{C}[u,v]/(uv-1) \approx \mathbf{C}[u,u^{-1}]$. Except for $\mathbf{H}[x,y]/(x^2+y^2+1)$, these rings are even left and right principal ideal domains by Corollary 5.2.

6. A criterion for freeness. Let $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$ where A is a commutative $\mathbf{R}$-algebra. Let $x, y \in A$, $t \in \mathbf{R}^*$ and consider the stably free module $P = P(x+ti, y+j)$ over Λ. We will always assume that y^2+1 is regular in A so that $y+j$ is regular and, therefore, as in Section 3, the projection $pr_1 : P \to \Lambda$ maps P isomorphically onto the left ideal I generated by $1 + (x+ti)(-k/2t)(y+j) = -(k/2t)(y+j)(x-ti)$ and $(y+j)(-k/2t)(y+j) = -(k/2t)(y^2+1)$. Thus, $I = \Lambda(y^2+1) + \Lambda(y+j)(x-ti)$. The object of this section is to give a simple criterion for I to be principal.

Note that Λ is free as a left $\mathbf{C}A$-module with base $1, j$. Any other base (χ, ω) is given by

$$\begin{pmatrix} \chi \\ \omega \end{pmatrix} = E \begin{pmatrix} 1 \\ j \end{pmatrix}$$

where $E \in GL_2(\mathbf{C}A)$. Choose

$$E = \begin{pmatrix} 1 & -(i/2t)y(x+ti) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}.$$

Then $\omega = y+j$ and $\chi = 1-(i/2t)y(x+ti)(y+j) = 1+y^2-(i/2t)y(y+j)(x-ti)$. Note that χ lies in I.

Lemma 6.1. *I is a free $\mathbf{C}A$-module with base χ, ω' where $\omega' = \omega(1+y^2)$.*

Proof. Since χ and ω are linearly independent over $\mathbf{C}A$, it is clear that χ and ω' are because $1+y^2$ is regular. Let $L = \mathbf{C}A\chi \oplus \mathbf{C}A\omega' \subset I$. Since $\Lambda = \mathbf{C}A\chi \oplus \mathbf{C}A\omega$, $\Lambda(1+y^2) = \mathbf{C}A(1+y^2)\chi + \mathbf{C}A\omega'$ lies in L. Modulo $\Lambda(1+y^2)$ we have $y\chi \equiv (i/2t)(y+j)(x-ti)$ so $\mathbf{C}A(y+j)(x-ti) \subset L$. Since $(y-j)(y+j)(x-ti) \in \Lambda(1+y^2)$, we have $j(y+j)(x-ti) \in L$, but $\Lambda = \mathbf{C}A + \mathbf{C}Aj$ so $\Lambda(y+j)(x-ti) \subset L$. □

Lemma 6.2. *Assume that $1+y^2$ is regular. Let $f \in \Lambda$. Then $I = \Lambda f$ if and only if $f \in I$ and $\bar{f} f = u(1+y^2)$ for some $u \in A^*$.*

Proof. I is equal to Λf if and only if f, jf is a base for I over $\mathbf{C}A$. This will be the case if and only if there is some $X \in GL_2(\mathbf{C}A)$ with

$$\begin{pmatrix} f \\ jf \end{pmatrix} = X \begin{pmatrix} \chi \\ \omega' \end{pmatrix}. \tag{6.1}$$

Let $f = a + bj$ with $a, b \in \mathbf{C}A$. Then $jf = \bar{a}j - \bar{b}$ so

$$\begin{pmatrix} f \\ jf \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix}.$$

Therefore we need

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix} = X \begin{pmatrix} 1 & 0 \\ 0 & y^2+1 \end{pmatrix} E \begin{pmatrix} 1 \\ j \end{pmatrix}.$$

Since 1 and j are linearly independent over $\mathbf{C}A$, this is equivalent to

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = X \begin{pmatrix} 1 & 0 \\ 0 & y^2+1 \end{pmatrix} E. \tag{6.2}$$

Suppose that $I = \Lambda f$. Then $\det X = u \in \mathbf{C}A^*$, $\det E = 1$ and

$$\det \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = a\bar{a} + b\bar{b} = f\bar{f}$$

so $f\bar{f} = u(y^2+1)$. Since $f\bar{f}$ and y^2+1 lie in A and y^2+1 is regular, it follows that u lies in A and so does u^{-1}.

Conversely, if $f \in I$ and $f\bar{f} = u(y^2+1)$, define X by (6.1) above. We have $X \in \mathcal{M}_2(\mathbf{C}A)$ since (χ, ω') is a $\mathbf{C}A$-base for I. As above, (6.2) holds so $f\bar{f} = (y^2+1)\det X$. Therefore, $\det X = u \in A^*$ so $X \in GL_2(\mathbf{C}A)$. □

Corollary 6.3. *Assume that $1 + y^2$ is regular. Let $f \in \Lambda$. Then $I = \Lambda f$ if and only if $f\bar{f} = u(y^2+1)$ for some $u \in A^*$ and $(y+j)(x-ti) \equiv 0 \bmod \Lambda f$.*

Proof. If $I = \Lambda f$, these conditions clearly hold. For the converse we need only show that $f \in I$. Let $(y+j)(x-ti) = gf$. Then $(y+j)(x-ti)\bar{f} = gf\bar{f} = (y^2+1)ug = (y+j)(y-j)ug$ so $(x-ti)\bar{f} = (y-j)ug$. Conjugating gives $f(x+ti) = u\bar{g}(y+j)$ showing that $(f, -u\bar{g}) \in P$. Therefore, $f = pr_1(f, -u\bar{g}) \in I$. □

An application of this criterion will be given in Section 8. We conclude this section with the following lemma which will be used in Section 7.

Lemma 6.4. *Assume that $1+y^2$ is regular. Then $\Lambda/I \approx \mathbf{C}A/(y^2+1)$ as a $\mathbf{C}A$-module under an isomorphism sending $1 \in \Lambda/I$ to $y(x+ti)$ and j to $x-ti$.*

Proof. By Lemma 6.1 we have $\Lambda/I = (\mathbf{C}A\chi + \mathbf{C}A\omega)/(\mathbf{C}A\chi + \mathbf{C}A\omega') = \mathbf{C}A\omega/\mathbf{C}A\omega' = \mathbf{C}A/(y^2+1)$. Let E be the matrix chosen at the beginning of this section. Then

$$\begin{aligned}\begin{pmatrix}1\\ j\end{pmatrix} &= E^{-1}\begin{pmatrix}\chi\\ \omega\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ -y & 1\end{pmatrix}\begin{pmatrix}1 & (i/2t)y(x+ti)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\chi\\ \omega\end{pmatrix}.\end{aligned}$$

Multiplying this out, we get $1 = \chi + (i/2t)y(x+it)\omega$ and $j = \omega - y\chi - (i/2t)y^2(x+it)\omega$ so after identifying $\mathbf{C}A\omega/\mathbf{C}A\omega'$ with $\mathbf{C}A/(y^2+1)$, 1 maps to $(i/2t)y(x+it)$ and j to $1-(i/2t)y^2(x+it) \equiv (i/2t)(x-ti) \bmod y^2+1$. We multiply this isomorphism by the unit $-2ti$ to get the required one. □

It follows immediately that $I \neq \Lambda(y^2+1)$ unless y^2+1 is a unit in $\mathbf{C}A$ since otherwise $\Lambda/I = \Lambda/\Lambda(y^2+1)$ would be free of rank 2 over $\mathbf{C}A/(y^2+1)$. This is also obvious from Lemma 6.2.

7. A large set of examples. The proof of Section 2 leaves us uncertain as to which of the modules $P(x_1+ti, x_2+j)$ will be the required example. I will show here that for $n \geq 5$, any one will do and, moreover, they are all nonisomorphic for $t > 0$.

Theorem 7.1. *Let A be a real algebra which is free as a module* *over a poynomial subring $\mathbf{R}[x, y]$, on a basis which includes 1. Let $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$, and suppose that for an infinite set of real numbers s we have $U(\Lambda/\Lambda(y-s)) = \mathbf{H}^*$. Then the stably free Λ-modules $P(x+ti, y+j)$ with $t \neq 0$ are all nonfree and $P(x+ri, y+j) \approx P(x+ti, y+j)$ if and only if $r = \pm t$.*

This gives a new proof of Theorem 2.2 since $\mathbf{R}[x, y, z_1, \ldots, z_m]$ clearly satisfies the hypotheses of Theorem 7.1

Corollary 7.2. *Let $q = \sum a_i x_i^2$ be a nondegenerate quadratic form in $n+1$ variables over $\mathbf{R}$, and let $A = \mathbf{R}[x_0, \ldots, x_n]/q(x)-1)$. Write $q' = q - a_0 x_0^2 = \sum_{i \geq 1} a_i x_i^2$. If $n \geq 5$, or if $n \geq 3$ and q' is isotropic, then the stably free $\mathbf{H} \otimes_{\mathbf{R}} A$-modules $P(x_1+ti, x_0+j)$ with $t \in \mathbf{R}$, $t > 0$ are all nonfree and nonisomorphic.*

We let $x = x_1$ and $y = x_0$. The freeness hypothesis is clear and the condition on units for all real s except $\pm 1/\sqrt{a_0}$ follows from Lemma 2.4. Of course, if q is isotropic we can make q' isotropic by a suitable choice of x_0.

Corollary 7.3. *If $n \geq 5$, there are $2^{\aleph_0}$ elements of $P_1(\mathbf{H} \otimes_{\mathbf{R}} A_n)$ which map to the trivial element of $VB_k^{\mathbf{H}}(S^n)$.*

The proof of Theorem 7.1 makes use of the following two lemmas. The notation is as in Section 6. In particular, we write $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$, and we have $P(x+ti, y+j) \approx I = \Lambda(y^2+1) + \Lambda(y+j)(x-ti)$.

Lemma 7.4. *Let A be a real algebra which is free as a module over a polynomial subring $\mathbf{R}[y]$, on a basis which includes 1. Let $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$, and suppose that for an infinite set of real numbers s we have $U(\Lambda/\Lambda(y-s)) = \mathbf{H}^*$. If $\lambda, \mu \in \Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$ and $\lambda\mu = (y^2+1)^m$ for some m, then $\lambda, \mu \in \mathbf{H}[y]$.*

Proof. Let $s \in \mathbf{R}$ be one of the specified real numbers. Let λ map to λ_s in $\Lambda/\Lambda(y-s)$. Then $\lambda_s \mu_s = (s^2+1)^m$ is a unit so λ_s lies in $\mathbf{H}^*$. Therefore, $\lambda = \alpha + (y-s)f$ for some $f \in \Lambda$. Let ω_i be a basis for A

over $\mathbf{R}[y]$ with $\omega_0 = 1$. Then $\{\omega_i\}$ is also a basis for Λ over $\mathbf{H}[y]$, and we can write $\lambda = \alpha + (y - s)\sum f_i(y)\omega_i$ with $f_i(y) \in \mathbf{H}[y]$. If $s' \neq s$ is another real number with $U(\Lambda/\Lambda(y - s')) = \mathbf{H}^*$, then $\lambda_{s'}$ lies in $\mathbf{H}^*$ so that $\alpha + (s' - s)\sum f_i(s')\omega_i' \in \mathbf{H}$ where ω_i' is the image of ω_i in $\Lambda/\Lambda(y - s')$. Since $A/A(y - s') = \mathbf{R}[y]/(y - s') \otimes_{\mathbf{R}[y]} A$, the ω_i' form a basis for $A/A(y-s')$ over $\mathbf{R}$ and also for $\Lambda/\Lambda(y-s')$ over $\mathbf{H}$. Therefore, we see that $f_i(s') = 0$ for $i \neq 0$. Since this holds for infinitely many values of s', it follows that $f_i(y) = 0$ for $i \neq 0$ and $\lambda = \alpha + (y - s)f_0(y)$ lies in $\mathbf{H}[y]$. □

Lemma 7.5. *Let A be a real algebra which is free as a module over a polynomial subring $\mathbf{R}[x, y]$ on a basis which includes* 1. *Let $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$ and $I = \Lambda(y^2 + 1) + \Lambda(y + j)(x - ti)$ with $t \neq 0$. Then $I \cap \mathbf{H}[y] = (y^2 + 1)\mathbf{H}[y]$.*

Proof. For any ring homomorphism $\mathbf{R}[x, y] \to R$, R will be a direct summand, and therefore a subring, of $R\otimes_{\mathbf{R}[x,y]} A$. Clearly, $(y^2+1)\mathbf{H}[y]$ is contained in $I \cap \mathbf{H}[y]$. Let $f(y)$ belong to $I \cap \mathbf{H}[y]$. After dividing f by $y^2 + 1$ and multiplying by a constant, we can assume that $f = 1$ or $y + \alpha$ with $\alpha \in \mathbf{H}$. But $f = 1$ is impossible since then $I = \Lambda$ contradicting Lemma 6.4 because $\mathbf{C}A/(y^2 + 1) \supset \mathbf{C}[x, y]/(y^2 + 1) \neq 0$. If $f = y + \alpha = y + \beta + \gamma j$, with $\beta, \gamma \in \mathbf{C}$, the image of f under the isomorphism of Lemma 6.4 is $0 = (y+\beta)y(x+ti)+\gamma(x-ti)$. All terms lie in the subring $\mathbf{C}[x, y]/(y^2 + 1)$. Now $(y + \beta)y \equiv \beta y - 1 \bmod y^2 + 1$ so $(\beta y - 1)(x + ti) + \gamma(x - ti) = 0$. Examining the coefficient of xy shows that $\beta = 0$. We now get $\gamma(x - ti) = (x + ti)$. The coefficient of x shows that $\gamma = 1$, and we are left with an obvious contradiction if $t \neq 0$. □

Proof of Theorem 7.1. Suppose that $P(x+ti, y+j)$ and $P(x+ri, y+j)$ are isomorphic. Then so are $I = \Lambda(y^2 + 1) + \Lambda(y + j)(x - ti)$ and $J = \Lambda(y^2 + 1) + \Lambda(y + j)(x - ri)$. Let $f : J \approx I$. Since $I_{y^2+1} = \Lambda_{y^2+1} = J_{y^2+1}$, we see that f has the form $f(\xi) = \xi\alpha$ where $\alpha \in \Lambda_{y^2+1}$. Since $f(y^2 + 1) = (y^2 + 1)\alpha$ lies in I and so in Λ, we have $\alpha = \lambda(y^2 + 1)^{-1}$ for some $\lambda \in I \subset \Lambda$. Similarly, $\alpha^{-1} = \mu(y^2 + 1)^{-1}$ for some $\mu \in J \subset \Lambda$. Multiplying these expressions we get $\lambda\mu = (y^2 + 1)^2$. By Lemmas 7.4 and 7.5, we see that λ lies in $(y^2 + 1)\mathbf{H}[y]$ so that $\alpha \in \mathbf{H}[y]$. Similarly, $\alpha^{-1} \in \mathbf{H}[y]$ so $\alpha \in \mathbf{H}^*$.

A similar argument applies if $P(x+ti, y+j)$ is free. In this case we have $f : \Lambda \approx I$ and $f(\xi) = \xi\alpha$ where $\alpha \in I \subset \Lambda$ and $\alpha^{-1} = \mu(y^2+1)^{-1}$ for some $\mu \in \Lambda$. We have $\alpha\mu = (y^2+1)$ showing that α lies in $(y^2+1)\mathbf{H}[y]$ while $\mu \in \mathbf{H}[y]$. Write $\alpha = \lambda(y^2+1)$ with $\lambda \in \mathbf{H}[y]$. Then $\lambda\mu = 1$ so $\lambda \in \mathbf{H}^*$. Since $\alpha = \lambda(y^2+1)$ we see that $I = \Lambda(y^2+1)$ contradicting the last remark in Section 6.

To finish the proof we must show that $I = J\alpha$ with $\alpha \in \mathbf{H}^*$ is impossible unless $r = \pm t$. If $I = J\alpha$, then $(y+j)(x-ri)\alpha$ maps to 0 in $\Lambda/I \approx \mathbf{C}A/(y^2+1)$. Write $\alpha = a + bj$ with $a, b \in \mathbf{C}$. Then $(y+j)(x-ri)\alpha = y(x-ri)a + y(x-ri)bj + (x+ri)\bar{a}j - (x+ri)\bar{b}$ maps to $[y(x-ri)a - (x+ri)\bar{b}]y(x+ti) + [y(x-ri)b + (x+ri)\bar{a}](x-ti) = 0$ or, since $y^2 \equiv -1$ in $\mathbf{C}A/(y^2+1)$, $[(x-ri)a + (x+ri)\bar{b}y](x+ti) = [y(x-ri)b + (x+ri)\bar{a}](x-ti)$. All terms lie in the subring $\mathbf{C}[x,y]/(y^2+1)$. Comparing coefficients of y gives $(x+ri)\bar{b}(x+ti) = (x-ri)b(x-ti)$ and therefore we also get $(x-ri)a(x+ti) = (x+ri)\bar{a}(x-ti)$. Comparing coefficients of x^2 gives $a = \bar{a}$ and $b = \bar{b}$, and the equations reduce to $2ix(r+t)b = 0$ and $2ix(t-r)a = 0$ showing that $\alpha = 0$ unless $r = \pm t$.

Finally observe that $Ij = \Lambda(y^2+1)j + \Lambda(y+j)(x-ti)j = \Lambda(y^2+1) + \Lambda(y+j)(x+ti)$ showing that $P(x+ti, y+j) \approx P(x-ti, y+j)$. □

8. Another proof of Theorem A. The criterion of Corollary 6.3 can be used to give still another proof of Theorem A which shows that all of the modules $P(x+ti, y+j)$ for $t \neq 0$ are nonfree even in the case of S^4 which is not covered by the method of Section 7. I do not know if all the modules $P(x+ti, y+j)$ with $t > 0$ are distinct in this case. The method applies to all smooth real quadric hypersurfaces having a real point.

Theorem 8.1. *Let* $A = \mathbf{R}[x_0, \dots, x_n]/(q(x)-1)$ *where* $q = \sum a_i x_i^2$ *is a nondegenerate quadratic form such that* $q(1,0,\dots,0) = 1$. *Let* $\Lambda = \mathbf{H} \otimes_{\mathbf{R}} A$, *and let* $t \in \mathbf{R}^*$. *If* $n \geq 4$, *then the* Λ*-module* $P(x_1+ti, x_2+j)$ *is stably free, nonfree, and topologically trivial.*

Proof. Since $w = 1 + x_2^2$ is invertible in the ring of continuous functions on the hypersurface, the topological triviality follows from the fact that P_w is free by Lemma 2.5.

Murthy has given an algebraic description of the stereographic projection $S^n - (1, 0, \dots, 0) \to \mathbf{R}^n$. The same procedure applies to our rings A. Note that $q = x_0^2 + \sum a_i x_i^2$. Write $u = 1 - x_0$ and $y_i = x_i/u$ for $i = 1, \dots, n$. Then $1 + \sum a_i y_i^2 = 2/u$ so $(A_n)_u = \mathbf{R}[y_1, \dots, y_n]_s$ where $s = 1 + \sum a_i y_i^2$. Note that $u = 2/s$ so $x_i = 2y_i/s$ for $i = 1, \dots, n$. Therefore, it will suffice to prove the following lemma.

Lemma 8.2. *If $n \geq 4$, $P(2y_1 s^{-1} + ti, 2y_2 s^{-1} + j)$ is not free over $\mathbf{H}[y_1, \dots, y_n]_s$ where $s = 1 + \sum a_i y_i^2$.*

Proof. By Corollary 6.3, we must show that there is no $f \in \mathbf{H}[y_1, \dots, y_n]_s$ with $f\bar{f} = \beta(4y_2^2 s^{-2} + 1)$ and $(2y_2 s^{-1} + j)(2y_1 s^{-1} - ti) \equiv 0 \bmod \Lambda f$. Here β is a unit of $\mathbf{R}[y_1, \dots, y_n]_s$ so $\beta = \alpha s^m$ where $\alpha \in \mathbf{R}^*$. Clearly $\alpha > 0$ since $f\bar{f}$ and $4y_2^2 s^{-2} + 1$ are positive. Replacing f by $f\alpha^{-1/2}$, we can assume that $f\bar{f} = s^m(4y_2^2 s^{-2} + 1)$. Clearing denominators and replacing f by fs^k for some k gives us the equations

$$f\bar{f} = s^N(y^2 + s^2) \quad \text{where } y = 2y_2 \tag{8.1}$$

and

$$\varphi f = s^M(y + sj)(x - sti) \quad \text{where } x = 2y_1 \tag{8.2}$$

in the ring $\Gamma = \mathbf{H}[y_1, \dots, y_n]$. □

Lemma 8.3. *$\Gamma/s\Gamma$ is a domain if $n \geq 4$.*

Proof. $\Gamma/s\Gamma = \mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}[y_1, \dots, y_n]/(s)$. It is sufficient to show that the quotient field K of $\mathbf{R}[y_1, \dots, y_n]/(s)$ does not split $\mathbf{H}$. As is well known, this is equivalent to showing that the level of K is at least 4, the level $s(K)$ being the least m such that -1 is the sum of m squares in K [**6**, p. 304(3)]. Now K is the function field of the quadric $1 + a_1 y_1^2 + \cdots + a_n y_n^2 = 0$. Let z be an indeterminate. Then $s(K) = s(K(z))$ [**6**, p. 304(5)]. Let $z_0 = z$, $z_i = zy_i$. Then $K(z)$ is the function field of the affine quadric cone $z_0^2 + a_1 z_1^2 + \cdots + a_n z_n^2 = 0$. If some a_n is negative, this field is rational over $\mathbf{R}$ and so has level ∞. If all a_n are positive, the level of this field is 2^k where $2^k \leq n < 2^{k+1}$ by [**6**, Chapter 11, Theorem 2.8]. Therefore, $s(K) \geq 4$ if $n \geq 4$.

Now suppose that $N > 0$ in (8.1). Then s divides f or $\bar{f}$ by Lemma 8.3. Since $\bar{s} = s$, $s|f$ in any case. Replacing f by $s^{-1}f$ and φ by $s\varphi$ now reduces N. Therefore, we can assume that $N = 0$. This implies that s does not divide f otherwise (8.1) would imply that $y^2 \equiv 0 \bmod s$.

If $M > 0$, this and Lemma 8.3 show that s divides φ so we can replace φ by $s^{-1}\varphi$ reducing M. Therefore we can also assume that $M = 0$. Letting $g = -k\varphi$, where $k = ij$ is the quaternion unit, we now see that the equations

$$f\bar{f} = (y^2 + s^2) \tag{8.3}$$

$$gf = (s + jy)(st + ix) \tag{8.4}$$

have a solution in Γ. Replacing (8.4) by the difference (8.4) $-$ t(8.3) gives

$$hf = (s + jy)(ix + jty) \tag{8.5}$$

where $h = g - t\bar{f}$.

Write $s = 1 + S$ where $S = \sum a_i y_i^2$ is homogeneous of degree 2. By (8.3), f has degree at most 2 so we can write $f = f_0 + f_1 + f_2$ where f_i is homogeneous of degree i. Now $f\bar{f} = 1 + (y^2 + 2S) + S^2$ so $f_0\bar{f}_0 = 1$, $f_2\bar{f}_2 = S^2$. Replace f by $f_0^{-1}f$ so that $f_0 = 1$. Since $f_2 \neq 0$, $\deg f = 2$, and so $\deg h = 1$ by (8.5). Let $h = h_0 + h_1$. Then $h_0 f_0 = 0$ by (8.5) so $h_0 = 0$ and $h = h_1$ is homogeneous of degree 1. Therefore, (8.5) gives $h + hf_1 + hf_2 = (1 + jy + S)(ix + jty)$ so $h = ix + jty$ and $hf_1 = jy(ix + jty)$. Also $hf_2 = S(ix + jty)$ showing that $f_2 = S$.

Now $(ix + jty)f_1 = jy(ix + jty)$ so $f_1 = \alpha y$ for some $\alpha \in \mathbf{H}$ and $(ix + jty)\alpha = j(ix + jty)$. Comparing coefficients of y shows that $\alpha = j$, and comparing coefficients of x gives the contradiction $ij = ji$. □

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REFERENCES

1. J. Barge and M. Ojanguren, *Fibrés algébriques sur une surface réelle*, Comment. Math. Helv. **62** (1987), 616–629.

2. H. Bass, *Algebraic K-theory*, W.A. Benjamin, Inc., New York, 1968.

3. ———, *Some problems in "classical" algebraic K-theory*, Algebraic K-Theory II, Lecture Notes in Math. **342**, Springer, Berlin-New York, 1973, 3–73.

4. D. Eisenbud and E.G. Evans, *Generating modules efficiently*: *Theorems from algebraic K-theory*, J. Algebra **27** (1973), 278–305.

5. G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, 1965.

6. T-Y. Lam, *The algebraic theory of quadratic forms*, W.A. Benjamin, Inc., Reading, MA, 1973.

7. ———, *Serre's conjecture*, Lecture Notes in Math. **635**, Springer, Berlin-New York, 1978.

8. D.Lissner and N. Moore, *Projective modules over certain rings of quotients of affine rings*, J. Algebra **15** (1970), 72–80.

9. D. Montgomery and L. Zippen, *Topological transformation groups*, Interscience, New York, 1955.

10. M. Ojanguren and R. Sridharan, *Cancellation of Azumaya algebras*, J. Algebra **18** (1971), 501–505.

11. S. Parimala and R. Sridharan, *Projective modules over polynomial rings over division rings*, J. Math. Kyoto Univ. **15** (1975), 129–148.

12. S. Parimala and R. Sridharan, *Projective modules over quaternion algebras*, J. Pure Appl. Algebra **9** (1977), 181–193.

13. A.A. Suslin, *Structure of projective modules over rings of polynomials in the case of a noncommutative ring of coefficients*, Trudy Mat. Inst. Steklov **148** (1978), 233–252, 279. English translation in Proc. Steklov Inst. Math. (1980), 245–267.

14. R.G. Swan, *Algebraic vector bundles on the 2-spheres*, Rocky Mountain J. Math. **23** (1993), 1443–1469.

15. ———, *Vector bundles, projective modules, and the K-theory of spheres*, in *Algebraic topology and algebraic K-theory* (W. Browder, ed.), Princeton University Press, Princeton, 1987, 432–522.

16. ———, *K-theory of quadric hypersurfaces*, Ann. Math. **122** (1985), 113–153.

17. ———, *Topological examples of projective modules*, Trans. Amer. Math. Soc. **230** (1977), 201–234.

18. ———, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962), 264–277.

19. R. Wood, *Polynomial maps from spheres to spheres*, Invent. Math. **5** (1968), 163–168.

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