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ON MARTINDALE'S THEOREM

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ABSTRACT. Martindale's theorem characterizes prime rings which satisfy a generalized polynomial identity. In the present paper we give further characterizations in terms of the ring only, eliminating thereby the central closure from the formulation of Martindale's theorem.

The definition of a generalized polynomial identity (GPI) and Martindale's characterization of a prime ring satisfying a GPI both make use of the central closure of the ring. Here we give internal characterizations, eliminating thereby the central closure. The possibility of such a characterization may belong to the 'folklore' for specialists in the field; we could not find it in the literature, however, and we think it can be of interest for the nonspecialist because it yields a conceptually simpler approach.

Notice also that the extended centroid, and through it the usual definition of a GPI, is connected with the maximal ring of left quotients of the ring and hence has a slight one-sided flavor, whereas conditions 4 and 5 of our theorem are obviously two-sided.

Theorem. *For a prime ring R, the following conditions are equivalent.*

(1) *R satisfies a* GPI,

(2) *R has a square-cancellable element a such that Ra is a uniform left ideal and aRa is a domain of finite rank over its center*,

(3) *R has a left ideal ${}_RL$ such that* $\mathrm{End}\,{}_RL$ *is a domain of finite rank over its center*,

(4) *R has an element a such that $a^2 \neq 0$ and aRa is a PI-ring,*

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(5) *R satisfies a* GPI *with coefficients from R, i.e., an identity of the form*

$$\sum c_{i_1} x_{i_1} c_{i_2} x_{i_2} \cdots c_{i_{n_i}} x_{i_{n_i}} c_{i_{n_{i+1}}} = 0$$

where the c_j are elements of R or the empty symbol.

(An element $a \in R$ is said to be *square-cancellable* if for all $x \in R$, $a^2x = 0$ implies $ax = 0$ and $xa^2 = 0$ implies $xa = 0$.)

Proof. $1 \Rightarrow 2$. Denote by S the central closure of R. By Martindale's theorem, S has a minimal left ideal L with an idempotent $e \in L$ such that eLe is finite dimensional over its center. If e is defined on $0 \neq V \triangleleft R$, then $0 \neq Ve \subseteq R$, so $(Ve)^2 \neq 0$ as R is prime, hence there is an $a \in Ve \subseteq L \cap R$ such that $(Ve)a \neq 0$, and then $La = L$. Therefore, for $f \in L$ with $fa = a$ we have $Lf = L$, $f^2 = f$, consequently a has an inverse in the division ring fLf. Thus, a is square-cancellable, and if f is defined on a nonzero ideal U of R, then by [**3**, Lemma 1] fUf is an order in fLf, hence aRa is also. By Martindale's theorem, aRa is of finite rank over its center.

$2 \Rightarrow 3$. We choose $L = Ra$; then by [**1**, Theorem 15], R is left nonsingular and hence by [**1**, Theorem 11] aRa and $\mathrm{End}\,_R Ra$ have isomorphic division rings of quotients.

$3 \Rightarrow 4$. Since R is prime, there exists an $a \in L$ with $La \neq 0$. Since a as well as every element of aRa induces an endomorphism of $_RL$ and $\mathrm{End}\, L$ is a domain, it follows that $a^2 \neq 0$ and aRa is isomorphic to a subring of $\mathrm{End}\, L$, which is a PI-ring by the assumption.

$4 \Rightarrow 5$. If aRa satisfies an identity of the form $\sum c_i x_{i_1} \cdots x_{i_{n_i}} = 0$, then $\sum c_i a x_{i_1} a^2 \cdots a^2 x_{i_{n_i}} a = 0$ holds in R.

$5 \Rightarrow 1$ is obvious. $\square$

Remark. A direct proof of $1 \Rightarrow 5$ is implicitly contained in the proof of [**2**, Theorem 3.1].

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