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ON CLS-MODULES

A. TERCAN

In this note we consider CLS-modules. Let R be a ring with identity, and let M be a right R-module which is the direct sum of its submodules M_1 and M_2. At this case, we show that if M_1 and M_2 are CLS-modules such that M_1 is M_2-injective, then M is a CLS-module. In particular, if M_1 is a CS-module and M_2 is a CLS-module such that M_1 is M_2-injective, then M is a CLS-module.

Throughout this paper all rings will have identities and all modules will be unital. Let R be any ring and M a right R-module. A submodule N of M is called a *complement* (in M) if N has no proper essential extension in M, and the module M is called a CS-*module* provided every complement in M is a direct summand of M (see, for example, [**2, 3, 6, 7**]).

Recall that a direct sum of CS-modules need not be a CS-module (see, for example, [**10**, Example 10]). In [**6**, Theorem 1] Kamal and Muller proved that a module M_R is CS if and only if $M = Z_2(M) \oplus N$ where $Z_2(M)$ and N are CS-modules and $Z_2(M)$ is N-injective. Recently in [**5**, Theorem 8] Harmanci and Smith showed that if $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a finite direct sum of relatively injective modules M_i, $1 \leq i \leq n$, then M is a CS-module if and only if M_i is a CS-module for each $1 \leq i \leq n$. Kamal and Muller's theorem [**6**, Theorem 1] allows us to consider nonsingular CS-modules. In this paper we define CLS-modules as a generalization of CS-modules, and we think of when the finite direct sums of CLS-modules is a CLS-module.

Let R be a ring and M a right R-module. We will use $Z(M)$ and $Z_2(M)$ to indicate, respectively, the singular submodule of M and the Goldie torsion (second singular) submodule of M.

Definition 1. A submodule N of M is a *closed submodule* of M provided M/N is nonsingular. Note that the concept 'closed submodule' has been used by some other authors. For example,

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according to [**4**], closed submodule is in the sense of complement as in this note. On the other hand, in [**7**, p. 19], complement and closed submodules are the same.

Example 2. Let K be a field and V a vector space over K such that $\dim_K V \geq 2$. Let

$$R = \begin{bmatrix} K & V \\ 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} k & v \\ 0 & k \end{bmatrix} : k \in K, v \in V \right\};$$

then R is a commutative ring such that it contains a complement ideal which is not a closed ideal.

Proof. Let $E = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. Then E is essential in R_R. Let $F_v = \begin{bmatrix} 0 & Kv \\ 0 & 0 \end{bmatrix}$, $v \in V$. Suppose that $G \leq R$ such that F_v is essential in G. Thus, F_v is essential in $G \cap E$ and hence $F_v = G \cap E$. Let $\begin{bmatrix} k & w \\ 0 & k \end{bmatrix} \in G$ for some $w \in V$, $0 \neq k \in K$. Let $x \in V$ such that $x \notin Kv$. Thus,

$$\begin{bmatrix} k & w \\ 0 & k \end{bmatrix} \begin{bmatrix} 0 & (1/k)x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in G \cap E.$$

Therefore, $x \in Kv$, a contradiction. Thus, $k = 0$. Hence, $G \leq E$ so $F_v = G$. It follows that F_v is a complement in R_R for all $v \in V$. But $E^2 = 0$ so $E^2 \leq F_v$. However, E is not contained in F_v. Thus F_v is not a closed ideal of R. □

The next lemma is taken from [**8**, Lemma 2.3], and its proof is given for completeness.

Lemma 3. *Let M_R be a module.*

(i) *Every closed submodule is a complement.*

(ii) *If M is nonsingular, then every complement is closed.*

Proof. (i) Suppose K is a closed submodule of M. Let N be a submodule of M such that K is essential in N. Then $N/K \leq Z(M/K)$

so that $N/K = Z(M/K)$ so that $N/K = 0$, and hence $K = N$. Thus, K is a complement in M.

(ii) Suppose K is a complement submodule of M which is not a closed submodule. Then M/K is not nonsingular. There exists $m \in M$, $m \notin K$, such that $mE \leq K$ for some essential right ideal E of R. Let $r \in R$, $k \in K$ and consider $mr + k$. Let

$$F = \{r \in R : rs \in E\}.$$

Then F is essential in R_R and $(mr + k)F \leq K$. If $mr + k \neq 0$, then $(mr + k)F \neq 0$ and hence $K \cap (mr + k)R \neq 0$. Thus, K is essential in $mR + K$ so that K is not a complement in M.

Definition 4. A module M_R is called a CLS-*module* provided every closed submodule of M is a direct summand of M.

Clearly, over a commutative integral domain, any torsion module is a CLS-module. Moreover,

Corollary 5. (i) *Every* CS-*module is a* CLS-*module.*

(ii) *Every nonsingular* CLS-*module is a* CS-*module.*

Proof. By Lemma 3. □

The following example illustrates that CLS-modules actually differ from CS-modules.

Example 6. Let p be any prime integer, and let M be the $\mathbf{Z}$-module $(\mathbf{Z}/\mathbf{Z}p) \oplus (\mathbf{Z}/\mathbf{Z}p^3)$. Then M is a CLS-module but is not a CS-module.

Proof. Since $M_{\mathbf{Z}}$ is singular, then M is a CLS-module. Now let $K = \mathbf{Z}(1 + \mathbf{Z}p, p + \mathbf{Z}p^3)$. Then K is a complement in M of order p^2 which is not a direct summand of M. Thus $M_{\mathbf{Z}}$ is not a CS-module, see [**11**]. □

The following result shows that CLS-modules behave like CS-modules in terms of direct summands.

Lemma 7. *Any direct summand of a* CLS*-module is a* CLS*-module.*

Proof. Suppose $M = K \oplus K'$ for some submodules K and K' of M. Let L be a closed submodule of K. Since

$$\frac{M}{L \oplus K'} = \frac{K \oplus K'}{L \oplus K'} \cong \frac{K}{L}$$

then $L \oplus K'$ is a closed submodule of M so that $L \oplus K'$ is a direct summand of M which gives that L is a direct summand of M. Then L is a direct summand of K. It follows that K is a CLS-module. □

Note that a direct sum of CLS-modules need not be a CLS-module in general, as the following example illustrates.

Let M be the $\mathbf{Z}$-module $\mathbf{Z} \oplus \mathbf{Z}_2$ where $\mathbf{Z}_2 = \{a/b : a, b \in \mathbf{Z}, b \text{ is odd}\}$. Now, obviously, $M_{\mathbf{Z}}$ is torsion-free and $\mathbf{Z}, \mathbf{Z}_2$ are CLS-modules. But M is not a CLS-module (see [**7**, p. 19]).

Proposition 8. *A right R-module M is a* CLS*-module if and only if there exists a submodule M' of M such that $M = Z_2(M) \oplus M'$ and M' is a* CS*-module. In this case M' is $Z_2(M)$-injective.*

Proof. Suppose that M is a CLS-module. Thus, $Z_2(M)$ is a direct summand of M so that $M = Z_2(M) \oplus M'$ for some submodule M' of M. Note that M' is nonsingular and, by Lemma 7, a CLS-module and hence a CS-module by Corollary 5. Conversely, suppose $M = Z_2(M) \oplus M'$ for some CS-module M'. Let K be a closed submodule of M. Then $Z(M) \leq K$ and hence $Z_2(M) \leq K$. Thus $K = Z_2(M) \oplus (K \cap M')$. Now $M/K \cong M'/(K \cap M')$ so that $K \cap M'$ is a closed submodule of M'. Hence, by Corollary 5, $M' = (K \cap M') \oplus K'$ for some submodule K'. Then $M = K \oplus K'$. It follows that M is a CLS-module. The second part is obvious. □

Theorem 9. *Suppose that a right R-module M is a direct sum of $M_1 \oplus M_2$ of* CLS*-modules M_1 and M_2 such that M_1 is M_2-injective. Then M is a* CLS*-module.*

Proof. Let N be a closed submodule of M. Then M/N is nonsingular.

Now $M_1/(N \cap M_1) \cong (M_1 + N)/N$ implies $N \cap M_1$ is a closed submodule of M_1. Thus $N \cap M_1$ is a direct summand of M_1 and hence of M. It follows that $N \cap M_1$ is a direct summand of N so $N = (N \cap M_1) \oplus K$ for some submodule K of N. Let $\pi_i : M \to M_i$, $i = 1, 2$, denote the canonical projections. Consider the diagram

$$\begin{array}{ccccl} 0 & \longrightarrow & K & \xrightarrow{\alpha} & M_2 \quad \text{exact} \\ & & {\scriptstyle\beta}\big\downarrow & & \\ & & M_1 & & \end{array}$$

where $\alpha = \pi_2|_K$ and $\beta = \pi_1|_K$. Note that α is a monomorphism and M_1 is M_2-injective. Thus, there exists a homomorphism $\varphi : M_2 \to M_1$ such that $\varphi\alpha = \beta$. Let

$$L = \{x + \varphi(x) : x \in M_2\}.$$

Then it can easily be checked that L is a submodule of M and $L \cong M_2$. Moreover, $M = M_1 \oplus L$. If $k \in K$, then $k = m_1 + m_2$ for some $m_i \in M_i$, $i = 1, 2$. Then

$$m_1 = \beta(k) = \varphi\alpha(k) = \varphi(m_2),$$

and this implies that $k = \varphi(m_2) + m_2 \in L$. Thus, $K \subseteq L$. Since

$$\frac{M}{N} = \frac{M_1}{N \cap M_1} \oplus \frac{L}{K},$$

then L/K is nonsingular, i.e., K is a closed submodule of L. But $L \cong M_2$, so that K is a direct summand of L. Thus, N is a direct summand of M. It follows that M is a CLS-module. □

Let n be a positive integer and $M_1, M_2, \dots, M_n$ are right R-modules. Then these modules are called *relatively injective* if M_i is M_j-injective for all $1 \leq i \neq j \leq n$, see [**5**]. Then we have the similar result of [**5**, Theorem 8] for the finite direct sums of CLS-modules which are as follows.

Theorem 10. *Let R be a ring and M a right R-module such that $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a finite direct sum of relatively injective* *modules M_i, $1 \leq i \leq n$. Then M is a* CLS*-module if and only if M_i is a* CLS*-module for each $1 \leq i \leq n$.*

Proof. The necessity is clear by Lemma 7. The converse follows by induction on n and using Theorem 9. □

Corollary 11. *Suppose that a nonsingular right R-module M is a direct sum $M_1 \oplus M_2$ of* CS*-modules M_1, M_2 such that M_1 is M_2-injective. Then M is a* CS*-module.*

Proof. By Corollary 5 and Theorem 9. □

The next result has also been proved by Harmanci and Smith [**5**, Theorem 4].

Corollary 12. *Suppose that a right R-module M is a direct sum $M_1 \oplus M_2$ of* CS*-modules M_1, M_2 such that M_1 is M_2-injective and M_2 is nonsingular. Then M is a* CS*-module.*

Proof. It is clear that $Z_2(M) = Z_2(M_1)$ is a direct summand of M_1. Thus, $M_1 = Z_2(M) \oplus M_1'$ for some nonsingular submodule M_1' of M_1. Now

$$M = Z_2(M) \oplus M_1' \oplus M_2.$$

Note that M_1' is M_2-injective, M_1' is a CS-module and $M_1' \oplus M_2$ is nonsingular. By Corollary 11, $M_1' \oplus M_2$ is a CS-module. But by [**6**, Theorem 1] $Z_2(M)$ is M_1'-injective and hence $Z_2(M)$ is $(M_1' \oplus M_2)$-injective. Again, by [**6**, Theorem 1], M is a CS-module. □

Corollary 13. *Suppose $M = M_1 \oplus M_2$ where M_1 and M_2 are* CS*-modules such that M_1 is M_2-injective. Then M is a* CS*-module if and only if $Z_2(M)$ is a CS module.*

Proof. The necessity is clear by [**6**, Theorem 1]. Conversely, suppose that $Z_2(M) = Z_2(M_1) \oplus Z_2(M_2)$ is a CS-module. There exist submodules M_1' of M_1 and M_2' of M_2 such that $M_1 = Z_2(M_1) \oplus M_1'$ and $M_2 = Z_2(M_2) \oplus M_2'$. Then $M = [Z_2(M_1) \oplus Z_2(M_2)] \oplus (M_1' \oplus M_2')$.

By [**6**, Theorem 1] and the fact that M_1 is M_2-injective, we know that $Z_2(M_1) \oplus Z_2(M_2)$ is $(M_1' \oplus M_2')$-injective. Also $M_1' \oplus M_2'$, being nonsingular, is a CS-module by Corollary 11. Hence, by [**6**, Theorem 1], M is a CS-module. □

Recall that any CS-module M also satisfies the following properties.

(i) Every semisimple submodule of M is essential in a direct summand of M, and

(ii) Every submodule of M has a complement which is a direct summand of M.

A module which satisfies property (i) (property (ii)) is called *weak* CS-*module* (*module with* (C_{11})), see [**9, 11**].

Finally we state some examples which illustrate that there is no relationship between CLS-modules and weak CS-modules, modules with (C_{11}) and CLS-modules.

Example 14. Let R be as in Example 2. Then R_R is an indecomposable module. Since R_R has no proper closed submodules, then R_R is a CLS-module which is neither weak CS-module nor module with (C_{11}).

Example 15. Let M be the $\mathbf{Z}$-module $\mathbf{Z} \oplus \mathbf{Z}_2$ where $\mathbf{Z}_2 = \{a/b : a, b \in \mathbf{Z}, b \text{ is odd}\}$. Then $M_{\mathbf{Z}}$ is not a CLS-module. But it is a weak CS-module by [**9**, Corollary 1.17].

Example 16. Let $M_{\mathbf{Z}} = \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots$. Then M satisfies (C_{11}) but is not a CLS-module.

Proof. By [**11**, Corollary 5.1], $M_{\mathbf{Z}}$ satisfies (C_{11}). Now suppose that $\varphi : M \to \mathbf{Q}$ is an epimorphism. Let $K = \ker \varphi$. Thus, $M/K \cong \mathbf{Q}$ which is nonsingular. Hence K is a closed submodule of M. If K were a direct summand of M, then we would have $M = K \oplus L$ for some submodule L of M. Thus, $L \cong \mathbf{Q}$, which is a contradiction. It follows that $M_{\mathbf{Z}}$ is not a CLS-module. □

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Hacettepe University, Department of Mathematics, Beytepe Campus, 06532 Ankara, Turkey