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NONRESONANCE CONDITIONS ON THE POTENTIAL WITH RESPECT TO THE FUČIK SPECTRUM FOR THE PERIODIC BOUNDARY VALUE PROBLEM

PATRICK HABETS, PIERPAOLO OMARI AND FABIO ZANOLIN

ABSTRACT. The existence of periodic solutions to a class of second order nonlinear ordinary differential equations is established under some rather mild restrictions on the behavior of the primitive of the nonlinearity with respect to the Fučik spectrum of the periodic problem.

1. Introduction. In this paper we study the solvability of the periodic boundary value problem

$$u'' + g(u) = h(t), \tag{1.1}$$

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \tag{1.2}$$

where $g : \mathbf{R} \to \mathbf{R}$ is continuous and $h : [0, 2\pi] \to \mathbf{R}$ is Lebesgue integrable. The conditions we consider relate the asymptotic behavior of $g(s)$ and of its primitive $G(s) = \int_{[0,s]} g(\xi)\, d\xi$, with the Dancer-Fučik spectrum of the positively homogeneous problem

$$u'' + \mu u^+ - \nu u^- = 0, \tag{1.3}$$

subject to the boundary conditions (1.2). We recall that the Dancer-Fučik spectrum $\mathcal{S}$ (cf. [**4, 12**]) is made by all pairs $(\mu, \nu) \in \mathbf{R}^2$ such that (1.3)–(1.2) has nontrivial solutions. Precisely, it can be expressed as

$$\mathcal{S} = \bigcup_{m \in \mathbf{N}} \mathcal{C}_m,$$

where

$$\mathcal{C}_0 = \{(\mu, \nu) : \mu\nu = 0\}$$

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and, for $m \geq 1$,

$$\mathcal{C}_m = \left\{(\mu,\nu) : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2}{m}\right\}.$$

This research, initiated in [**4, 12**], found new interesting motivations by some recent studies in the theory of suspended bridges involving the consideration of asymmetric nonlinearities (cf. [**19**]). In the literature, various conditions have been introduced in order to guarantee nonresonance, that is, the existence of solutions to (1.1)–(1.2) for any given h. Usually, such conditions require that $g(s)/s$ does not interfere asymptotically with the critical branches $\mathcal{C}_m$ in the sense that

$$q_\pm < \liminf_{s\to\pm\infty} \frac{g(s)}{s} \leq \limsup_{s\to\pm\infty} \frac{g(s)}{s} < Q_\pm, \tag{1.4}$$

where, for some $m \in \mathbf{N}$,

$$(q_-, q_+) \in \mathcal{C}_m \quad \text{and} \quad (Q_-, Q_+) \in \mathcal{C}_{m+1}. \tag{1.5}$$

Note that, in the symmetric case $q_- = q_+ = m^2$ and $Q_- = Q_+ = (m+1)^2$, (1.4) and (1.5) are the classical nonresonance conditions with respect to the spectrum $\{m^2 : m \in \mathbf{N}\}$ of the differential operator $-d^2/dt^2$, with periodic boundary conditions on $[0, 2\pi]$, considered by Mawhin in [**20**] for the solvability of (1.1)–(1.2).

In subsequent papers, assumptions (1.4)–(1.5) have been generalized in various directions, see [**25, 18, 16, 11, 7, 13, 6, 8**]. In particular, in [**13**] (adapting a technique introduced in [**9**]), the strict inequalities in (1.4) have been replaced by the weaker ones

$$q_\pm \leq \liminf_{s\to\pm\infty} \frac{g(s)}{s} \leq \limsup_{s\to\pm\infty} \frac{g(s)}{s} \leq Q_\pm, \tag{1.6}$$

and nonresonance has been achieved by requiring, for instance, that

$$q_+ < \liminf_{s\to+\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \limsup_{s\to+\infty} \frac{2G(s)}{s^2} < Q_+, \tag{1.7}$$

or analogous conditions at $-\infty$. According to (1.6) and (1.7), one can consider, in particular, nonlinear functions $g(s)$ such that the ratio $g(s)/s$ oscillates between two consecutive eigenvalues m^2 and $(m+1)^2$.

Actually, when $m = 0$, even more general conditions have been introduced with respect to $\mathcal{C}_0$ and for more general equations as well. Indeed, in [**14**] it is proved that the assumptions

$$\limsup_{s\to\pm\infty} \frac{g(s)}{s} \le Q_\pm \quad \text{and} \quad \limsup_{s\to+\infty} \frac{2G(s)}{s^2} < Q_+, \tag{1.8}$$

(or a similar condition at $-\infty$) imply the solvability of (1.1)–(1.2) for any bounded h, whenever

$$\inf_{s\in\mathbf{R}} g(s) = -\infty \quad \text{and} \quad \sup_{s\in\mathbf{R}} g(s) = +\infty. \tag{1.9}$$

This result holds true also for the Liénard equation

$$u'' + f(u)u' + g(u) = h(t), \tag{1.10}$$

with the boundary conditions (1.2), where $f : \mathbf{R} \to \mathbf{R}$ is an arbitrary continuous function. A similar problem was previously considered in [**22**] and [**17**].

More recently, in the symmetric case $q_- = q_+ = m^2$ and $Q_- = Q_+ = (m+1)^2$, with $m \ge 1$, the above-recalled result in [**13**] has been extended in [**23**], by assuming (1.5)–(1.6), but replacing condition (1.7) with

$$m^2 < \limsup_{s\to+\infty} \frac{2G(s)}{s^2} \tag{1.11a}$$

and

$$\liminf_{s\to+\infty} \frac{2G(s)}{s^2} < (m+1)^2, \tag{1.11b}$$

or similar conditions at $-\infty$, which allow us to consider some larger classes of nonlinearities g than those in [**13**] (see the example given in [**23**]).

The aim of this paper is twofold: on the one hand, we obtain an existence result with respect to two consecutive branches $\mathcal{C}_m$ and $\mathcal{C}_{m+1}$, which generalizes the above quoted result in [**13**] in the direction of [**23**]; on the other, we improve the main theorem in [**14**] for what concerns the nonresonance condition with respect to $\mathcal{C}_1$. Some results in [**6, 10**] are completed as well. Precisely, the following results hold.

Theorem 1.1. *Suppose that* $(Q_-, Q_+) \in \mathcal{C}_1$. *Assume*

$$\limsup_{s\to\pm\infty} \frac{g(s)}{s} \le Q_\pm$$

and

$$\liminf_{s\to-\infty} \frac{2G(s)}{s^2} < Q, \quad \text{or} \quad \liminf_{s\to+\infty} \frac{2G(s)}{s^2} < Q_+.$$

Then problem (1.1)–(1.2) *has at least one solution for any given* $h \in L^\infty(0, 2\pi)$, *provided that* (1.9) *holds.*

Theorem 1.2. *Suppose that* $(q_-, q_+) \in \mathcal{C}_m$ *and* $(Q_-, Q_+) \in \mathcal{C}_{m+1}$, *for some* $m \ge 1$. *Assume that*

$$q_\pm \le \liminf_{s\to\pm\infty} \frac{g(s)}{s} \le \limsup_{s\to\pm\infty} \frac{g(s)}{s} \le Q_\pm.$$

Suppose also that at least one of the following conditions holds

$$q_- < \limsup_{s\to-\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \liminf_{s\to-\infty} \frac{2G(s)}{s^2} < Q_-,$$
$$q_- < \limsup_{s\to-\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \liminf_{s\to+\infty} \frac{2G(s)}{s^2} < Q_+,$$
$$q_+ < \limsup_{s\to+\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \liminf_{s\to-\infty} \frac{2G(s)}{s^2} < Q_-,$$
$$q_+ < \limsup_{s\to+\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \liminf_{s\to+\infty} \frac{2G(s)}{s^2} < Q_+.$$

Then problem (1.1)–(1.2) *has at least one solution, for any given* $h \in L^1(0, 2\pi)$.

Theorem 1.1 is a consequence of a more general result which is stated and proved in Section 2 for the Liénard equation (1.10), under a weaker condition than (1.9). In the same section we produce as well a variant of this result for the variational equation (1.1),where an Ahmad-Lazer-Paul condition at the eigenvalue 0 is considered. Section 3 is devoted to the proof of Theorem 1.2. Actually, in Sections 2 and 3 we produce a class of assumptions on the nonlinearity $g(s)$, which include the above-mentioned hypotheses on $G(s)/s^2$ and are expressed by means of asymptotic conditions on ratios of the form

$$\frac{1}{\Phi(s)}\int_{[0,s]} p(\xi)\, d\Phi(\xi),$$

where Φ is a suitable convex function satisfying the Δ_2-condition near infinity (see [**1**]) and $g(s)/s - p(s) \to 0$ as $s \to +\infty$ (similar conditions are considered at $-\infty$). We refer to the appendix for a more detailed discussion of this topic.

We point out that the proofs given here simplify in a significant way the argument in [**23**], as well as that in [**13**] and [**14**], where (1.7) and (1.8) were exploited only through their equivalence to suitable density conditions. In particular, we observe that the method of proof in [**23**] makes use in an essential way, of the symmetry of the problem as well as the properties of the usual (linear) spectrum and, therefore, cannot be adapted to the present setting, which is definitely nonsymmetric.

We also stress that in both Theorem 1.1 and 1.2, the conditions on $2G(s)/s^2$ cannot be replaced by similar ones on $g(s)/s$. Indeed, in view of the nonexistence result in [**5**, Theorem 5.2] we know that one can find pairs of points (q_-, q_+) and (Q_-, Q_+) such that the segment joining them intersects the critical set $\mathcal{S}$, and mappings g satisfying

$$q_\pm \le \liminf_{s\to\pm\infty} \frac{g(s)}{s} < \limsup_{s\to\pm\infty} \frac{g(s)}{s} < Q_\pm$$

or

$$q_\pm < \liminf_{s\to\pm\infty} \frac{g(s)}{s} < \limsup_{s\to\pm\infty} \frac{g(s)}{s} \le Q_\pm,$$

for which problem (1.1)–(1.2) has no solution for some smooth function h.

Finally, we notice that similar results can be obtained for equation (1.1) with Dirichlet or Neumann boundary conditions.

2. Nonsymmetric nonlinearities with coefficients between $\mathcal{C}_0$ and $\mathcal{C}_1$. Let us consider the periodic problem for the Liénard equation

$$u'' + f(u)u' + g(u) = h(t), \tag{2.1}$$

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \tag{2.2}$$

with $f, g : \mathbf{R} \to \mathbf{R}$ continuous functions and $h \in L^\infty(0, 2\pi)$. In what follows, for any real number s, the notation $s^+ = \max\{s, 0\}$ and $s^- = \max\{-s, 0\}$ is used.

Theorem 2.1. *Assume that* $(a, b) \in \mathcal{C}_1 = \{(\mu, \nu) : 1/\sqrt{\mu} + 1/\sqrt{\nu} = 2\}$ *and suppose that* g *can be written in the form*

$$g(s) = p(s)s^+ - q(s)s^- + r(s),$$

where $p, q, r : \mathbf{R} \to \mathbf{R}$ *are continuous functions such that*

(i) $0 \le p(s) \le a$, $0 \le q(s) \le b$ *for all* s;

(ii) $\liminf_{s \to -\infty} (1/s) \int_{[0,s]} q(\xi)\, d\xi < b$ *or*

$$\liminf_{s \to +\infty} \frac{1}{s} \int_{[0,s]} p(\xi)\, d\xi < a; \tag{2.3}$$

(iii) $\lim_{|s| \to +\infty} r(s)/s = 0$;

(iv) *there exist constants* A *and* B *such that*

$$g(A) \le h(t) \le g(B), \quad \text{for a.e. } t \in [0, 2\pi]. \tag{2.4}$$

Then problem (2.1)–(2.2) *has at least one solution.*

Proof. We start by observing that A is an upper solution and B is a lower solution. Therefore, if $A \ge B$ the conclusion follows from known results (see, e.g., [**2**]) and assumptions (i), (ii) and (iii) are not needed. Accordingly, we can assume in (iv) $A < B$. Moreover, it is not restrictive to suppose that $A < 0 < B$. Indeed, if it is not the case, one simply makes the change of variable $v := u - (A + B)/2$.

We will use the following homotopy

$$u'' + \lambda f(u)u' + p_\lambda(u)u^+ - q_\lambda(u)u^- + r_\lambda(t, u) = 0 \tag{2.5$_\lambda$}$$

with $\lambda \in [0,1]$ and

$$\begin{aligned} p_\lambda(s) &:= \lambda p(s) + (1-\lambda)(a/2), \\ q_\lambda(s) &:= \lambda q(s) + (1-\lambda)(b/2), \\ r_\lambda(t,s) &:= \lambda r(s) - \lambda h(t). \end{aligned}$$

To apply degree theory, we need to define in $H^1(0,2\pi)$ an open bounded set Ω, with $0 \in \Omega$ such that no solution of (2.5_λ)–(2.2), with $\lambda \in [0,1[$, belongs to the boundary of Ω. Basically, our proof consists in building such a set. We will write the proof only in the case (2.3) is satisfied.

Claim 1. *If u is a solution of (2.5_λ)–(2.2), for some $\lambda \in [0,1[$, then*

$$\min u \neq B \quad \textit{and} \quad \max u \neq A. \tag{2.6}$$

Proof of Claim 1. Suppose, by contradiction, there is a solution u of (2.5_λ)–(2.2), for some $\lambda \in [0,1[$, such that $\min u = u(t_0) = B$ (> 0). Hence, as $(1-\lambda)(a/2)B + \lambda(g(B) - h(t)) \geq (1-\lambda)(a/2)B > 0$, for almost every t, and $u'(t_0) = 0$, there exist numbers $\varepsilon, \delta > 0$ (depending on u and λ) such that $(1-\lambda)(a/2)u^+(t) + \lambda(g(u(t)) - h(t)) \geq \varepsilon$, $|f(u(t))u'(t)| \leq \varepsilon/2$ and $u(t) > 0$ for almost every $|t - t_0| \leq \delta$. Then we have almost everywhere on this interval,

$$\begin{aligned} -u''(t) = \lambda f(u(t))u'(t) + \lambda g(u(t)) + (1-\lambda)(a/2)u^+(t) \\ - (1-\lambda)(b/2)u^-(t) - \lambda h(t) \geq \varepsilon/2 > 0. \end{aligned}$$

Hence, taking $t_1 \in [t_0 - \delta, t_0[$, we get

$$u(t_1) - u(t_0) = \int_{[t_1,t_0]} u''(\xi)(\xi - t_1)\, d\xi < 0,$$

which is impossible. Similarly, one proves the second part of (2.6). $\square$

According to Claim 1, in the sequel u will denote a solution of (2.5_λ)–(2.2), for some $\lambda \in [0,1[$, such that, for at least one t,

$$B < u(t) < A. \tag{2.7}$$

By condition (2.3) we can find $\eta > 0$ and a sequence $(R_n)_n$ such that $R_n \to +\infty$ and

$$\eta R_n \leq \int_{[0,R_n]} (a - p(\xi))\, d\xi \tag{2.8}$$

Claim 2. *For any $n_0 \geq 0$, there exists $n \geq n_0$ such that, for all $\lambda \in [0,1[$, (2.5_λ)–(2.2) has no solution u satisfying* (2.7) *such that*

$$\max u = R_n. \tag{2.9}$$

Proof of Claim 2. Suppose to the contrary that there exists n_0 such that, for all $n \geq n_0$, we can find a solution u_n of (2.5_λ)–(2.2), for some $\lambda_n \in [0,1[$, that satisfies (2.7) and (2.9). In this case the function $v_n = u_n/||u_n||$, where $||\cdot||$ denotes the H^1-norm, solves the equation

$$v_n'' + \lambda_n f(u_n) v_n' + p_{\lambda n}(u_n) v_n^+ - q_{\lambda n}(u_n) v_n^- + \frac{r_{\lambda_n}(t, u_n)}{||u_n||} = 0 \tag{2.10}$$

and satisfies the boundary conditions (2.2). Notice that, by (iii), for any $\varepsilon > 0$,

$$\left| \frac{r_{\lambda_n}(\cdot, u_n)}{||u_n||} \right|_\infty \leq (\varepsilon |u_n|_\infty + K_\varepsilon + |h|_\infty) ||u_n|| \leq \kappa \varepsilon,$$

for some fixed constant $\kappa > 0$, provided that n is sufficiently large. Hence, it follows that

$$\frac{r_{\lambda_n}(t, u_n(t))}{||u_n||} \to 0, \quad \text{uniformly a.e. on } [0, 2\pi].$$

Moreover, possibly passing to a subsequence, we still denote by $(v_n)_n$, we have

$$v_n \to v, \quad \text{weakly in } H^1(0, 2\pi),$$

and

$$v_n \to v, \quad \text{uniformly on } [0, 2\pi].$$

We can also suppose that $\lambda_n \to \lambda \in [0,1]$. We start by observing that, by (2.7), $v(t) = 0$, for at least one t, and $v \not\equiv 0$. Indeed, if $v \equiv 0$, multiplying equation (2.10) by v_n and integrating, we get

$$\int_{[0,2\pi]} |v_n'|^2 = \int_{[0,2\pi]} p_{\lambda n}(u_n)|v_n^+|^2 + \int_{[0,2\pi]} q_{\lambda n}(u_n)|v_n^-|^2 + \int_{[0,2\pi]} \frac{r_{\lambda_n}(t, u_n(t))}{||u_n||} \to 0,$$

as $n \to +\infty$. This is impossible as $v_n \to v$ uniformly and $||v_n|| = 1$.

Further, since the sequences $(p_{\lambda n}(u_n))_n$ and $(q_{\lambda n}(u_n))_n$ are bounded in $L^\infty(0, 2\pi)$ by (i), we can also assume that

$$p_{\lambda n}(u_n) \to p_0, \qquad q_{\lambda n}(u_n) \to q_0,$$

in the L^∞-weak* topology, and

$$0 \le p_0(t) \le a, \qquad 0 \le q_0(t) \le b,$$

for almost every $t \in [0, 2\pi]$.

To conclude the proof of Claim 2, we need the following result.

Claim 3. *Extending p_0, q_0, v by 2π-periodicity on* $\mathbf{R}$*, there exist an interval $[\alpha, \alpha + 2\pi]$ and a constant $\sigma > 0$ such that, for almost every $t \in [\alpha, \alpha + \pi/\sqrt{a}]$,*

$$p_0(t) = a, \qquad v(t) = \frac{\sigma}{\sqrt{a}} \sin \sqrt{a}(t - \alpha),$$

and, for almost every $t \in [\alpha + \pi/\sqrt{a}, \alpha + 2\pi]$,

$$q_0(t) = b, \qquad v(t) = -\frac{\sigma}{\sqrt{b}} \sin \sqrt{b}\left(t - \alpha - \frac{\pi}{\sqrt{a}}\right).$$

Proof of Claim 3. We recall that there exists at least one $t \in [0, 2\pi[$ such that $v(t) = 0$. Then we suppose for a moment that $v(t) \ge 0$ on

$[0, 2\pi]$. Integrating equation (2.10) on $[0, 2\pi]$ and going to the limit, we get

$$\int_{[0,2\pi]} p_0 v^+ = 0,$$

which implies that $p_0 v^+ \equiv 0$ almost everywhere on $[0, 2\pi]$. Multiplying equation (2.10) by v_n, integrating and going to the limit, we derive

$$\int_{[0,2\pi]} |v'|^2 = 0.$$

Hence, v must be constant and therefore $v \equiv 0$, which is impossible. Similarly, one can conclude that $v(t) \le 0$ on $[0, 2\pi]$ is impossible. Accordingly, v changes sign on $[0, 2\pi]$ and therefore

$$M_n = \max u_n \to +\infty, \qquad m_n = \min u_n \to -\infty.$$

Let $[\alpha_n, \beta_n]$ and $[\gamma_n, \delta_n]$ be intervals such that

$$\begin{gathered} u_n > 0 \quad \text{on }]\alpha_n, \beta_n[\,, \qquad u_n < 0 \quad \text{on }]\gamma_n, \delta_n[\,, \\ u_n(\alpha_n) = u_n(\beta_n) = 0 = u_n(\gamma_n) = u_n(\delta_n), \\ \max_{[\alpha_n,\beta_n]} u_n = M_n, \qquad \min_{[\gamma_n,\delta_n]} u_n = m_n. \end{gathered}$$

Passing to subsequences, we can suppose that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\gamma_n \to \gamma$, $\delta_n \to \delta$ and

$$\begin{gathered} v \ge 0 \quad \text{on } [\alpha, \beta], \qquad v \le 0 \quad \text{on } [\gamma, \delta], \\ v(\alpha) = v(\beta) = 0 = v(\gamma) = v(\delta), \end{gathered}$$

and $[\alpha, \beta]$, $[\gamma, \delta]$ have disjoint interiors. We will prove that $\beta - \alpha \ge \pi/\sqrt{a}$ and $\delta - \gamma \ge \pi/\sqrt{b}$. Let us consider the first inequality. Multiplying equation (2.10) by $u_n \cdot ||u_n||$ and integrating on $[\alpha_n, \beta_n]$ we get, by (iii), for any $\varepsilon > 0$,

$$\begin{aligned} \int_{[\alpha_n,\beta_n]} |u_n'|^2 &= \int_{[\alpha_n,\beta_n]} p_{\lambda n}(u_n) |u_n^+|^2 \\ &\quad + \int_{[\alpha_n,\beta_n]} r_{\lambda n}(t, u_n(t)) u_n \\ &\le \int_{[\alpha_n,\beta_n]} p_{\lambda n}(u_n) |u_n|^2 + \varepsilon \int_{[\alpha_n,\beta_n]} |u_n|^2 \\ &\quad + (K_\varepsilon + |h|_\infty) \int_{[\alpha_n,\beta_n]} u_n. \end{aligned}$$

Dividing by $||u_n||^2$ and passing to the limit as $n \to +\infty$, we have

$$\int_{[\alpha,\beta]} |v'|^2 \le \int_{[\alpha,\beta]} (p_0 + \varepsilon)|v|^2$$

and then, going to the limit as $\varepsilon \to 0$,

$$\int_{[\alpha,\beta]} |v'|^2 \le \int_{[\alpha,\beta]} p_0 |v|^2 \le a \int_{[\alpha,\beta]} |v|^2. \tag{2.11}$$

Using the Poincaré inequality, we get

$$\int_{[\alpha,\beta]} |v'|^2 \le a\left(\frac{\beta-\alpha}{\pi}\right)^2 \int_{[\alpha,\beta]} |v'|^2,$$

which implies $\beta - \alpha \ge \pi/\sqrt{a}$. Similarly, one proves the inequality $\delta - \gamma \ge \pi/\sqrt{b}$. Now, as $(\beta-\alpha)+(\delta-\gamma) \le 2\pi$ and $\pi/\sqrt{a} + \pi/\sqrt{b} = 2\pi$, we conclude that

$$\beta - \alpha = \frac{\pi}{\sqrt{a}} \quad \text{and} \quad \delta - \gamma = \frac{\pi}{\sqrt{b}}.$$

Finally, Poincaré inequality yields

$$\int_{[\alpha,\beta]} |v'|^2 \le a \int_{[\alpha,\beta]} |v|^2 \le \int_{[\alpha,\beta]} |v'|^2$$

and

$$v(t) = \frac{\sigma}{\sqrt{a}} \sin \sqrt{a}(t-\alpha), \quad \text{on } [\alpha,\beta].$$

Coming back to (2.11), we also get

$$p_0(t) = a, \quad \text{a.e. on } [\alpha,\beta].$$

The first part of Claim 3 is then proved. Similarly, one gets the second part. □

According to Claim 3, we have that $\lambda_n \to \lambda = 1$, $p(u_n) \to p_0$ and $q(u_n) \to q_0$, in the L^∞-weak* topology. Since $v_n \to v$ uniformly in

$[0, 2\pi]$, Claim 3 also implies that, for all large n, v_n takes positive and negative values. Hence we can pick t'_n and $t''_n \in [t'_n, t'_n + 2\pi]$ such that

$$u_n(t'_n) = 0 \quad \text{and} \quad u_n(t''_n) = \max u_n = R_n,$$

with $t'_n \to \alpha$ and $t''_n \to \alpha + \pi/(2\sqrt{a})$. (Of course, also in this context we suppose u_n, as well as the other involved functions, extended by 2π-periodicity.) We now observe that

$$\int_{[\alpha,\alpha+\pi/\sqrt{a}]} (p_0 - p(u_n))^2 \to 0, \quad \text{as } n \to +\infty.$$

Indeed, since $0 \le p(u_n) \le a \equiv p_0$, on $[\alpha, \alpha + \pi/\sqrt{a}]$ and $p(u_n) \to p_0$ in the L^∞-weak* topology, we have, as $n \to +\infty$,

$$\int_{[\alpha,\alpha+\pi/\sqrt{a}]} (p_0 - p(u_n)) \to 0,$$

and hence

$$\int_{[\alpha,\alpha+\pi/\sqrt{a}]} (p_0 - p(u_n))^2 \le a \int_{[\alpha,\alpha+\pi/\sqrt{a}]} |p_0 - p(u_n)| \to 0.$$

Then it follows, for each $\varepsilon > 0$ and all large n, we have, by (2.8) and (2.2),

$$\begin{aligned}\eta R_n &\le \int_{[0,Rn]} (a - p(\xi))\, d\xi \\ &= \int_{[t'_n,t''_n]} (a - p(u_n(t)))u'_n(t)\, dt \\ &\le |u'_n|_2 \bigg[\bigg(\int_{[t'_n,\alpha]} (a - p(u_n(t)))^2 \bigg)^{1/2} \\ &\qquad + \bigg(\int_{[\alpha,t''_n]} (p_0 - p(u_n(t)))^2 \bigg)^{1/2} \bigg] \\ &\le \varepsilon |u'_n|_2.\end{aligned} \tag{2.12}$$

Multiplying equation (2.10) by $u_n \cdot ||u_n||$ and integrating on $[0, 2\pi]$, we get

$$\begin{aligned}|u'_n|_2^2 &= \int_{[0,2\pi]} p_{\lambda n}(u_n)|u_n^+|^2 + \int_{[0,2\pi]} q_{\lambda n}(u_n)|u_n^-|^2 \\ &\quad + \int_{[0,2\pi]} r_{\lambda n}(t, u_n(t))u_n \\ &\le c_1(|u_n|_\infty^2 + 1),\end{aligned} \tag{2.13}$$

for some constant $c_1 > 0$. Moreover, using Claim 3 and the uniform convergence of $(v_n)_n$ to v, we deduce that, for $K > 0$ small enough and n large,

$$\frac{1}{2}\sqrt{\frac{b}{a}} < \frac{1/\sqrt{a} - K}{1/\sqrt{b} + K} \le -\frac{\max v_n}{\min v_n}$$
$$= -\frac{\max u_n}{\min u_n} \le \frac{1/\sqrt{a} + K}{1/\sqrt{b} - K} < 2\sqrt{\frac{b}{a}}.$$

Hence, for some constant $c_2 > 0$, $|u_n|_\infty \le c_2 R_n$. Accordingly, we obtain from (2.12) and (2.13)

$$\eta R_n \le \varepsilon \sqrt{c_1}(c_2 R_n + 1),$$

which yields a contradiction if ε is chosen sufficiently small and n sufficiently large. Thus, Claim 2 is proven. □

Claim 4. *For each n large enough, every solution u of* (2.5_λ)–(2.2)*, for any $\lambda \in [0, 1]$, with* $\max u \le R_n$*, is such that*

$$\min u > -2\sqrt{\frac{a}{b}} R_n =: S_n.$$

Proof of Claim 4. If it were not the case, we should have sequences $(\lambda_n)_n$ and $(u_n)_n$, where u_n solves (2.5_λ)–(2.2), with $\lambda = \lambda_n$, such that

$$\max u_n \le R_n \quad \text{and} \quad \min u_n \le -2\sqrt{\frac{a}{b}} R_n.$$

As before, going to subsequences, one proves that $u_n/||u_n|| \to v$, uniformly on $[0, 2\pi]$, where v is such that

$$\max v = \frac{\sigma}{\sqrt{a}} \quad \text{and} \quad \min v = -\frac{\sigma}{\sqrt{b}},$$

for some $\sigma > 0$. Hence, we easily get a contradiction. □

Consider now the set

$$\Omega = \{u \in H^1(0, 2\pi) : A < \max u < R, S < \min u < B, |u'|_2 < C\},$$

where we choose $R = R_n$ and $S = S_n$, with n so large that $R > B$, $S < A$ and Claims 2 and 4 hold; moreover, we take $C = \sqrt{c_3} + 1$, where

$$c_3 = (a + b + 1)(\max\{|S|, R\})^2 + K_1 \max\{|S|, R\}$$

and K_1 is such that $|r_\lambda(t, s)| \le |s| + K_1$, for almost every $t \in [0, 2\pi]$, $s \in \mathbf{R}$, and $\lambda \in [0, 1]$.

Assume that u is a solution to (2.5_λ)–(2.2) for some $\lambda \in [0, 1[$, with $u \in \overline{\Omega}$, that is, $A \le u(t) \le B$, for at least one t; $S \le u(t) \le R$ for every t; $|u'|_2 \le C$. Then, we have

$$\begin{aligned} |u'|_2^2 = & \int_{[0,2\pi]} p_\lambda(u)|u^+|^2 + \int_{[0,2\pi]} q_\lambda(u)|u^-|^2 \\ & + \int_{[0,2\pi]} r_\lambda(t, u)u \\ & \le (a + b)|u|_\infty^2 + (|u|_\infty^2 + K_1|u|_\infty) \\ & \le (a + b + 1)(\max\{|S|, R\})^2 + K_1 \max\{|S|, R\} \\ & := c_3 \end{aligned}$$

and therefore $|u'|_2 < C$.Moreover, by Claim 1, we get $A < u(t) < B$, for at least one t and, by Claims 2 and 4, $S < u(t) < R$ for every t. Hence, we conclude that $u \in \Omega$. Standard results in degree theory then yield the existence of at least a solution $u \in \overline{\Omega}$ to (2.1)–(2.2). $\square$

Remark 2.1. It is easy to give conditions on g so that it can be written as

$$g(s) = p(s)s^+ - q(s)s^- + r(s),$$

where p, q, r satisfy (i) and (iii). Precisely, we have the following result.

Proposition 2.1. *Let $g : \mathbf{R} \to \mathbf{R}$ be continuous. Then*

(j) $$\limsup_{s\to+\infty} \frac{g(s)}{s} \le a, \qquad \limsup_{s\to-\infty} \frac{g(s)}{s} \le b;$$

(jj) $$\liminf_{s\to\pm\infty} \frac{g(s)}{s} \ge 0;$$

hold if and only if there exist $p, q, r : \mathbf{R} \to \mathbf{R}$ continuous such that g can be written in the form

$$g(s) = p(s)s^+ - q(s)s^- + r(s),$$

with

$$\text{(k)} \qquad 0 \le p(s) \le a, \qquad 0 \le q(s) \le b;$$

$$\text{(kk)} \qquad \lim_{|s| \to +\infty} \frac{r(s)}{s} = 0.$$

Proof. We only need to prove that (j), (jj) imply that g can be written in the form

$$g(s) = p(s)s^+ - q(s)s^- + r(s),$$

where p, q, r satisfy (k), (kk). If $|s| \ge 1$, we take

$$p(s) = \min\left\{a, \max\left\{\frac{g(s)}{s}, 0\right\}\right\},$$
$$q(s) = \min\left\{b, \max\left\{\frac{g(s)}{s}, 0\right\}\right\}$$

and interpolate linearly p and q for $|s| \le 1$. To verify (kk), let us fix $\varepsilon > 0$ and choose $M > 0$ such that $s > M$ implies $-\varepsilon \le g(s)/s \le a + \varepsilon$ and $s < -M$ implies $-\varepsilon \le g(s)/s \le b + \varepsilon$. It follows that, for $s > M$,

$$-\varepsilon|s| \le r(s) = g(s) - p(s)s \le \varepsilon|s|$$

and, for $s < -M$,

$$-\varepsilon|s| \le r(s) = g(s) - q(s)s \le \varepsilon|s|.$$

Hence, (kk) follows. $\square$

Remark 2.2. Notice that if g is unbounded from below and from above, either the existence of a solution follows from the existence of (well-ordered) upper and lower solutions, or (jj) and (iv) hold. Indeed, if g is unbounded from above and from below, then either

$$\limsup_{s\to+\infty} g(s) = +\infty, \qquad \liminf_{s\to+\infty} g(s) = -\infty,$$

or

$$\limsup_{s\to-\infty} g(s) = +\infty, \qquad \liminf_{s\to-\infty} g(s) = -\infty,$$

or

$$\limsup_{s\to-\infty} g(s) = +\infty, \qquad \liminf_{s\to+\infty} g(s) = -\infty,$$

or else

$$\limsup_{s\to+\infty} g(s) = +\infty, \qquad \liminf_{s\to-\infty} g(s) = -\infty. \tag{2.14}$$

In the first three cases we can find $B < A$ such that (2.4) holds. The constants A and B are an upper and a lower solution, respectively; the existence then follows from known results (cf. [**2**]). If we assume that none of these three cases holds, we have

$$\liminf_{s\to+\infty} g(s) > -\infty, \qquad \limsup_{s\to-\infty} g(s) < +\infty$$

and (jj) is satisfied. Further, from (2.14), constants $A < 0 < B$ can be found satisfying (iv).

Remark 2.3. Whenever (i) is satisfied, condition (ii) is equivalent to the following one:

$$\liminf_{s\to+\infty} \frac{\int_{[0,s]} p(\xi)\phi(\xi)\,d\xi}{\int_{[0,s]} \phi(\xi)\,d\xi} < a \tag{l}$$

or

$$\liminf_{s\to-\infty} \frac{\int_{[0,s]} p(\xi)\phi(|\xi|)\,d\xi}{\int_{[0,s]} \phi(|\xi|)\,d\xi} < b,$$

where $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ is an (arbitrary) nondecreasing continuous function, with $\phi(s) > 0$, for $s > 0$, such that, for some $k > 0$ and $d > 0$,

$$s\phi(s) \leq k \int_{[0,s]} \phi(\xi)\,d\xi, \quad \text{for all } s \geq d.$$

The equivalence occurring between (ii) and (l), under (i), is proved in the Appendix. In particular, if we choose $\phi(s) \equiv s$ and Proposition 2.1 applies, condition (l) turns out to be equivalent to a condition on the potential $G(s) = \int_{[0,s]} g(\xi)\, d\xi$ of $g(s)$. Namely,

$$\liminf_{s\to+\infty} \frac{2G(s)}{s^2} < a \quad \text{or} \quad \liminf_{s\to-\infty} \frac{2G(s)}{s^2} < b. \tag{ll}$$

This follows from the fact that

$$\left| \frac{2G(s)}{s^2} - \left(\int_{[0,s]} p(\xi)\xi\, d\xi \right) \Big/ \left(\int_{[0,s]} \xi\, d\xi \right) \right| \leq \frac{2}{s^2} \left(\int_{[0,s]} r(\xi)\, d\xi \right) \to 0,$$

as $s \to +\infty$. A similar computation has to be performed for $s \to -\infty$.

From the above remarks we can write the following corollary.

Corollary 2.1. *Assume* $(a, b) \in \mathcal{C}_1$ *and*

$$\inf_{s\in\mathbf{R}} g(s) = -\infty \quad \textit{and} \quad \sup_{s\in\mathbf{R}} g(s) = +\infty; \tag{h}$$

$$\limsup_{s\to+\infty} \frac{g(s)}{s} \leq a, \qquad \limsup_{s\to-\infty} \frac{g(s)}{s} \leq b; \tag{j}$$

$$\liminf_{s\to+\infty} \frac{2G(s)}{s^2} < a \quad \textit{or} \quad \liminf_{s\to-\infty} \frac{2G(s)}{s^2} < b. \tag{ll}$$

Then problem (2.1)–(2.2) *has at least one solution for any given* $h \in L^\infty(0, 2\pi)$.

The final part of this section is devoted to the study of problem (2.1)–(2.2), in the case where $f \equiv 0$, i.e.,

$$u'' + g(u) = h(t), \tag{2.15}$$

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \tag{2.2}$$

Exploiting the variational structure of this problem, we are able to state a variant of Theorem 2.1, where the pointwise condition (iv) is replaced with the, in a sense more natural, Ahmad-Lazer-Paul condition at the eigenvalue 0 (cf. [**3**]):

$$\lim_{|s|\to+\infty}(G(s)-\bar{h}s)=+\infty, \tag{v}$$

where $\bar{h}=(1/2\pi)\int_{[0,2\pi]}h(t)\,dt$. Precisely, we have the following result.

Theorem 2.2. *Assume that* $(a,b)\in\mathcal{C}_1$ *and*

$$\lim_{|s|\to+\infty}(G(s)-\bar{h}s)=+\infty; \tag{v}$$

$$\limsup_{s\to+\infty}\frac{g(s)}{s}\le a,\qquad \limsup_{s\to-\infty}\frac{g(s)}{s}\le b; \tag{j}$$

$$\liminf_{s\to+\infty}\frac{2G(s)}{s^2}<a\quad \textit{or}\quad \liminf_{s\to-\infty}\frac{2G(s)}{s^2}<b. \tag{ll}$$

Then problem (2.15)–(2.2) *has at least one solution.*

Proof. This proof borrows some arguments from [**15**] and from [**10**]. As in [**15**], we distinguish between two cases:

$$\inf_{s\in\mathbf{R}}g(s)=-\infty\quad\text{and}\quad\sup_{s\in\mathbf{R}}g(s)=+\infty,$$

or

$$\inf_{s\in\mathbf{R}}g(s)>-\infty\quad\text{or}\quad\sup_{s\in\mathbf{R}}g(s)<+\infty.$$

In the former case, Corollary 2.1 applies and yields the existence of a solution. Therefore, let us consider the latter case and assume, for instance, that

$$\sup_{s\in\mathbf{R}}g(s)=:\gamma<+\infty. \tag{2.16}$$

We are going to prove the existence of a solution, under conditions (2.16) and (v), by a variational argument based on a variant of the mountain pass lemma, as stated in [**24**, Theorem 3.10]. Accordingly, we denote by H the Hilbert space of all functions $u \in H^1(0,2\pi)$, satisfying $u(0)=u(2\pi)$. H is endowed with the H^1-norm

$$||u|| = \left(\int_{[0,2\pi]} (|u(t)|^2 + |u'(t)|^2)\,dt \right)^{1/2}.$$

We consider the functional

$$\phi(u) = \int_{[0,2\pi]} [(1/2)|u'(t)|^2 - G(u(t)) + h(t)u(t)]\,dt.$$

Clearly, ϕ is well-defined on H, weakly lower semicontinuous and C^1. Moreover, its critical points are precisely the solutions of problem (2.15)–(2.2).

At first, following [**15**, Proposition 2], we prove that the functional ϕ satisfies the Palais-Smale condition. Let $\{u_n\}$ be a sequence in H such that, for some constant c_1 and all n,

$$\left| \int_{[0,2\pi]} [(1/2)|u_n'(t)|^2 - G(u_n(t)) + h(t)u_n(t)]\,dt \right| \le c_1, \tag{2.17}$$

and, for every $v \in H$,

$$\left| \int_{[0,2\pi]} [u_n'(t)v'(t) - g(u_n(t))v(t) + h(t)v(t)]\,dt \right| \le \varepsilon_n ||v||, \tag{2.18}$$

with $\varepsilon_n \to 0$. We will show that $\{u_n\}$ has a bounded subsequence, which suffices in the present situation to derive the Palais-Smale condition. Taking $v \equiv 1$ in (2.18), we have, for some constant c_2,

$$\left| \int_{[0,2\pi]} g(u_n(t))\,dt \right| \le c_2, \quad \text{for all } n. \tag{2.19}$$

Now, by (2.16), we get

$$\left| \int_{[g\le 0]} g(u_n(t))\,dt \right| \le c_2 + \int_{[g>0]} g(u_n(t))\,dt \le c_2 + 2\pi\gamma$$

and hence, for some constant c_3,

$$\int_{[0,2\pi]} |g(u_n(t))|\,dt \le c_3, \quad \text{for all } n. \tag{2.20}$$

Taking in (2.18), $v \equiv w_n := u_n - \bar{u}_n$, where $\bar{u}_n = (1/2\pi)\int_{[0,2\pi]} u_n(t)\,dt$, we have by (2.20) and standard inequalities for some constants c_4 and c_5,

$$\begin{aligned} c_4||w_n|| &\ge \left| \int_{[0,2\pi]} [|w_n'(t)|^2 - g(u_n(t))w_n(t) + h(t)w_n(t)]\,dt \right| \\ &\ge |w_n'|_2^2 - (c_3 + |h|_1)|w_n|_\infty \\ &\ge (1/2)||w_n||^2 - c_5||w_n||. \end{aligned}$$

Accordingly, we conclude that there exists a constant c_6 such that

$$|u_n'|_2 \le ||w_n|| \le c_6. \tag{2.21}$$

Assume now by contradiction that

$$||u_n|| \to +\infty, \quad \text{as } n \to +\infty.$$

Because of (2.21) we have, possibly passing to a subsequence, that either

$$m_n = \min u_n \to +\infty, \quad \text{as } n \to +\infty,$$

or

$$M_n = \max u_n \to -\infty, \quad \text{as } n \to +\infty.$$

In either case we get, by (v),

$$G(M_n) - \bar{h}M_n \to +\infty, \quad \text{as } n \to +\infty. \tag{2.22}$$

On the other hand, from (2.17), (2.21) and (2.16), it follows that

$$\begin{aligned} c_1 &\geq -\int_{[0,2\pi]} (1/2)|u_n'(t)|^2 \\ &\quad + \int_{[0,2\pi]} (G(u_n(t)) - h(t)u_n(t))\, dt \\ &= -(1/2)|u_n'|_2^2 + 2\pi(G(M_n) - \bar{h}M_n) \\ &\quad - \int_{[0,2\pi]} \int_{[u_n(t),M_n]} [g(\xi) - h(t)]\, d\xi\, dt \\ &\geq -\frac{c_6^2}{2} + 2\pi(G(M_n) - \bar{h}M_n) \\ &\quad - \int_{[0,2\pi]} \int_{[u_n(t),M_n]} |\gamma - h(t)|\, d\xi\, dt \\ &\geq -\frac{c_6^2}{2} + 2\pi(G(M_n) - \bar{h}M_n) - |\gamma - h|_1 |M_n - u_n|_\infty \\ &\geq -\frac{c_6^2}{2} + 2\pi(G(M_n) - \bar{h}M_n) - \sqrt{2\pi} c_6 |\gamma - h|_1. \end{aligned}$$

Hence, we reach a contradiction with (2.22).

Now we look at the shape of the functional ϕ; we will see that it has a mountain pass geometry. Precisely, following [**10**, Theorem 2], we define in H the open set

$$B = \{u \in H : \min u > 0\}$$

and we show that

$$\inf_{u \in \partial B} \phi(u) > -\infty. \tag{2.23}$$

Indeed, if $u \in \partial B$, i.e., $\min u = u(t_u) = 0$, extending the involved functions by 2π-periodicity on $\mathbf{R}$, we can write

$$\phi(u) = \int_{[tu,tu+2\pi]} [(1/2)|u'(t)|^2 - G(u(t)) + h(t)u(t)]\, dt.$$

Hence, using condition (2.16), we get

$$\phi(u) \geq \int_{[tu,tu+2\pi]} [(1/2)|u'(t)|^2 - |\gamma - h(t)|u(t)]\, dt$$

and, by Poincaré inequality,

$$\phi(u) \geq (1/2)[|u'|_2^2 - |\gamma - h|_2 |u'|_2].$$

This implies that

$$\phi(u) \to +\infty, \quad \text{as } ||u|| \to +\infty, \quad u \in \partial B,$$

because, when $\min u = 0$, $||u|| \to +\infty$ if and only if $|u'|_2 \to +\infty$. Then the weakly lower semicontinuity of ϕ yields (2.23). Moreover, by condition (v) we know that

$$\phi(r) \to -\infty, \quad \text{as } ||r|| \to \infty, \quad r \in \mathbf{R},$$

and then there exists a number $R > 0$ such that

$$\max\{\phi(-R), \phi(R)\} < \inf_{u \in \partial B} \phi(u),$$

where obviously the constant functions R and $-R$ are such that $R \in B$ and $-R \in H \setminus \overline{B}$. Hence, Theorem 3.10 in [**24**] applies and yields the existence of a critical point of ϕ, which corresponds to a solution of (2.15)–(2.2). Similarly, one should work in the case where, instead of (2.16), the condition $\inf_{s \in \mathbf{R}} g(s) > -\infty$ holds. □

Remark 2.4. Theorem 2.2 extends and improves a similar result obtained in [**15**], which was also motivated by a question raised in [**21**, Remark 10]. We notice that in [**15**] only the symmetric case $a = b = 1$ could be dealt with, and the more restrictive condition

$$\limsup_{s \to +\infty} \frac{2G(s)}{s^2} < 1 \quad \text{or} \quad \limsup_{s \to -\infty} \frac{2G(s)}{s^2} < 1$$

was assumed in place of (ll).

3. Nonsymmetric nonlinearities with coefficients between $\mathcal{C}_m$ and $\mathcal{C}_{m+1}$. In this section we study the solvability of the following periodic problem

$$u'' + g(u) = h(t), \tag{3.1}$$

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \tag{3.2}$$

with $g : \mathbf{R} \to \mathbf{R}$ continuous and $h \in L^1(0, 2\pi)$. As in the preceding section, for any real number s, the notation $s^+ = \max\{s, 0\}$ and $s^- = \max\{-s, 0\}$ is used.

Theorem 3.1. *Assume that* $(a, b) \in \mathcal{C}_m = \{(\mu, \nu) : 1/\sqrt{\mu} + 1/\sqrt{\nu} = 2/m\}$ *and* $(c, d) \in \mathcal{C}_{m+1} = \{(\mu, \nu) : 1/\sqrt{\mu} + 1/\sqrt{\nu} = 2/(m+1)\}$. *Suppose that* g *can be written in the form*

$$g(s) = p(s)s^+ - q(s)s^- + r(s),$$

where $p, q, r : \mathbf{R} \to \mathbf{R}$ *are continuous functions such that*

(i) $a \leq p(s) \leq c$, $b \leq q(s) \leq d$, *for all* s,

and

(ii) $\lim_{|s| \to +\infty} r(s)/s = 0$.

Suppose also that at least one of the following conditions holds:

$$a < \limsup_{s \to +\infty} \frac{1}{s} \int_{[0,s]} p(\xi)\, d\xi \quad \text{and} \quad \liminf_{s \to +\infty} \frac{1}{s} \int_{[0,s]} p(\xi)\, d\xi < c, \tag{3.3}$$

$$b < \limsup_{s \to -\infty} \frac{1}{s} \int_{[0,s]} q(\xi)\, d\xi \quad \text{and} \quad \liminf_{s \to -\infty} \frac{1}{s} \int_{[0,s]} q(\xi)\, d\xi < d, \tag{3.4}$$

$$a < \limsup_{s \to +\infty} \frac{1}{s} \int_{[0,s]} p(\xi)\, d\xi \quad \text{and} \quad \liminf_{s \to -\infty} \frac{1}{s} \int_{[0,s]} q(\xi)\, d\xi < d, \tag{3.5}$$

$$b < \limsup_{s \to -\infty} \frac{1}{s} \int_{[0,s]} q(\xi)\, d\xi \quad \text{and} \quad \liminf_{s \to +\infty} \frac{1}{s} \int_{[0,s]} p(\xi)\, d\xi < c. \tag{3.6}$$

Then problem (3.1)–(3.2) *has at least one solution, for any given* $h \in L^1(0, 2\pi)$.

To prove Theorem 3.1, we shall need the following lemma.

Lemma 3.1. *Assume that* $(a, b) \in \mathcal{C}_m$ *and* $(c, d) \in \mathcal{C}_{m+1}$. *Let* $p_0, q_0 \in L^\infty(0, 2\pi)$ *be such that, for almost every* $t \in [0, 2\pi]$,

$$a \leq p_0(t) \leq c, \qquad b \leq q_0(t) \leq d.$$

Suppose that $v \in W^{2,\infty}(0, 2\pi)$ *is a nonzero solution of*

$$v'' + p_0 v^+ - q_0 v^- = 0, \qquad v(0) = v(2\pi), \qquad v'(0) = v'(2\pi). \tag{3.7}$$

Then, extending p_0, q_0, v *by* 2π*-periodicity on* $\mathbf{R}$, *there exists an interval* $[t_0, t_0 + 2\pi]$ *and a constant* $\sigma > 0$ *such that either* v *has* m *positive humps* $[t_{2i-2}, t_{2i-1}]$ *and* m *negative humps* $[t_{2i-1}, t_{2i}]$ *on* $[t_0, t_0 + 2\pi]$, $i = 1, \dots, m$, *with*

(j) $t_{2i-1} - t_{2i-2} = \pi/\sqrt{a}$ *and, for almost every* $t \in [t_{2i-2}, t_{2i-1}]$,

$$p_0(t) = a, \qquad v(t) = \frac{\sigma}{\sqrt{a}} \sin \sqrt{a}(t - t_{2i-2}), \tag{3.8}$$

(jj) $t_{2i} - t_{2i-1} = \pi/\sqrt{b}$ *and, for almost every* $t \in [t_{2i-1}, t_{2i}]$,

$$q_0(t) = b, \qquad v(t) = \frac{-\sigma}{\sqrt{b}} \sin \sqrt{b}(t - t_{2i-1}); \tag{3.9}$$

or v *has* $m+1$ *positive humps* $[t_{2i-2}, t_{2i-1}]$ *and* $m+1$ *negative humps* $[t_{2i-1}, t_{2i}]$ *on* $[t_0, t_0 + 2\pi]$, $i = 1, \dots, m+1$, *with*

(jjj) $t_{2i-1} - t_{2i-2} = \pi/\sqrt{c}$ *and, for almost every* $t \in [t_{2i-2}, t_{2i-1}]$,

$$p_0(t) = c, \qquad v(t) = \frac{\sigma}{\sqrt{c}} \sin \sqrt{c}(t - t_{2i-2}), \tag{3.10}$$

(jv) $t_{2i} - t_{2i-1} = \pi/\sqrt{d}$ *and, for almost every* $t \in [t_{2i-1}, t_{2i}]$,

$$q_0(t) = d, \qquad v(t) = \frac{-\sigma}{\sqrt{d}} \sin \sqrt{d}(t - t_{2i-1}). \tag{3.11}$$

Proof of Lemma 3.1. Notice first that the function v cannot be one sign. Indeed, if we assume, for example, that $v(t) = v^+(t)$, integration of (3.7) on $[0, 2\pi]$ leads to the contradiction

$$0 < a|v|_1 \leq \int_{[0,2\pi]} p_0(t) v(t)\, dt = 0.$$

The same argument holds if $v(t) = v^-(t)$.

Let $[\alpha, \beta]$ be a positive hump of v, i.e., $v(\alpha) = v(\beta) = 0$ and $v(t) > 0$ for all $t \in]\alpha, \beta[$. We must have $\beta \le \alpha + \pi/\sqrt{a}$, since otherwise from (3.7), we easily get

$$\begin{aligned}
0 &\ge \int_{[\alpha, \alpha+\pi/\sqrt{a}]} (a - p_0(t))v(t) \sin \sqrt{a}(t-\alpha)\, dt \\
&= \int_{[\alpha, \alpha+\pi/\sqrt{a}]} (v'' + av) \sin \sqrt{a}(t-\alpha)\, dt \\
&= [v'(t) \sin \sqrt{a}(t-\alpha) - \sqrt{a} v(t) \cos \sqrt{a}(t-\alpha)]_\alpha^{\alpha+\pi/\sqrt{a}} \\
&= \sqrt{a} v(\alpha + \pi/\sqrt{a}) > 0.
\end{aligned}$$

This formula also shows that $\beta = \alpha + \pi/\sqrt{a}$ if and only if, for almost every $t \in [\alpha, \beta]$,

$$p_0(t) = a \quad \text{and} \quad v(t) = \frac{\sigma}{\sqrt{a}} \sin \sqrt{a}(t-\alpha),$$

for some constant $\sigma > 0$.

On the other hand, one sees that $\beta \ge \alpha + \pi/\sqrt{c}$. Indeed, otherwise from (3.7), we easily get

$$\begin{aligned}
0 &\le \int_{[\alpha, \beta]} (c - p_0(t))v(t) \sin \sqrt{c}(t-\alpha)\, dt \\
&= \int_{[\alpha, \beta]} (v'' + cv) \sin \sqrt{c}(t-\alpha)\, dt \\
&= [v'(t) \sin \sqrt{c}(t-\alpha) - \sqrt{c} v(t) \cos \sqrt{c}(t-\alpha)]_\alpha^{\beta} \\
&= v'(\beta) \sin \sqrt{c}(\beta - \alpha) < 0.
\end{aligned}$$

(Note that $v'(\beta) = 0$, together with $v(\beta) = 0$, would imply $v \equiv 0$.) Moreover, using this formula, we also have that $\beta = \alpha + \pi/\sqrt{c}$ if and only if, for almost every $t \in [\alpha, \beta]$,

$$p_0(t) = c \quad \text{and} \quad v(t) = \frac{\sigma}{\sqrt{c}} \sin \sqrt{c}(t-\alpha),$$

for some constant $\sigma > 0$.

Similarly, one proves that, for a negative hump $[\alpha, \beta]$,

$$\frac{\pi}{\sqrt{d}} \leq \beta - \alpha \leq \frac{\pi}{\sqrt{b}}.$$

Further, we have that $\beta - \alpha = \pi/\sqrt{b}$ if and only if, for almost every $t \in [\alpha, \beta]$,

$$q_0(t) = b \quad \text{and} \quad v(t) = \frac{-\sigma}{\sqrt{b}} \sin \sqrt{b}(t - \alpha),$$

for some $\sigma > 0$ and $\beta - \alpha = \pi/\sqrt{d}$ if and only if, for almost every $t \in [\alpha, \beta]$,

$$q_0(t) = d \quad \text{and} \quad v(t) = \frac{-\sigma}{\sqrt{d}} \sin \sqrt{d}(t - \alpha),$$

for some $\sigma > 0$.

Since v is a nonzero solution of (3.7), it has a finite number of zeros, which are simple. Hence, the number of positive humps is equal to the number of negative humps and they alternate. Assume that v has at most m positive humps $[t_{2i-2}, t_{2i-1}]$ and m negative humps $[t_{2i-1}, t_{2i}]$. Then

$$2\pi = \sum_i [(t_{2i-1} - t_{2i-2}) + (t_{2i} - t_{2i-1})] \leq m\left(\frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}}\right) = 2\pi$$

and v is a solution if and only if the equality holds, i.e., v has exactly m positive and m negative humps such that (3.8) and (3.9) hold, for the same constant σ. The same argument applies with

$$2\pi = \sum_i [(t_{2i-1} - t_{2i-2}) + (t_{2i} - t_{2i-1})] \geq (m+1)\left(\frac{\pi}{\sqrt{c}} + \frac{\pi}{\sqrt{d}}\right) = 2\pi$$

and v is a solution if and only if it has exactly $m+1$ positive and $m+1$ negative humps such that (3.10) and (3.11) hold for the same constant σ. □

Proof of Theorem 3.1. Assume that (3.3) holds. Let $\mu = (a+c)/2$ and $v = (b+d)/2$ and consider the homotopy

$$u'' + p_\lambda(u)u^+ - q_\lambda(u)u^- + r_\lambda(t, u) = 0, \qquad (3.12_\lambda)$$

with $\lambda \in [0,1]$ and

$$\begin{aligned} p_\lambda(s) &:= \lambda p(s) + (1-\lambda)\mu, \\ q_\lambda(s) &:= \lambda q(s) + (1-\lambda)\nu, \\ r_\lambda(t,s) &:= \lambda r(s) - \lambda h(t). \end{aligned}$$

From (3.3) we can find a constant $\eta > 0$ and a sequence $(R_n)_n$ such that $R_n \to +\infty$ and

$$(a+\eta)R_n \le \int_{[0,R_n]} p(\xi)\, d\xi \le (c-\eta)R_n.$$

Claim 1. *For any $n_0 \ge 0$, there exists $n \ge n_0$ such that, for all $\lambda \in [0,1]$, (3.12_λ)–(3.2) has no solution u such that*

(3.13) $$\max u = R_n.$$

Proof of Claim 1. Suppose on the contrary that there exists n_0 such that, for all $n \ge n_0$, we can find a solution u_n of (3.12_λ)–(3.2) for some $\lambda_n \in [0,1]$, that satisfies

$$\max u_n = R_n.$$

The function

$$v_n = \frac{u_n}{|u_n|_\infty}$$

solves the equation

$$v_n'' + p_{\lambda n}(u_n)v_n^+ - q_{\lambda n}(u_n)v_n^- + r_{\lambda_n}\frac{tu_n}{|u_n|_\infty} = 0$$

and satisfies the boundary conditions (3.2). We can assume, going to subsequences, that

(3.14) $$v_n \to v \not\equiv 0, \quad \text{in } C^1([0,2\pi]),$$

(3.15)

$$p_{\lambda n}(u_n) \to p_0, \qquad q_{\lambda n}(u_n) \to q_0, \quad \text{in the } L^\infty\text{-weak}^* \text{ topology},$$

and

$$\frac{r_{\lambda_n}(\cdot, u_n)}{||u_n||} \to 0, \quad \text{in } L^1(0, 2\pi).$$

Further, $v \in W^{2,1}(0, 2\pi)$ satisfies the equation

$$v'' + p_0 v^+ - q_0 v^- = 0 \tag{3.16}$$

and the boundary conditions (3.2), where by (3.15)

$$a \leq p_0(t) \leq c, \qquad b \leq q_0(t) \leq d,$$

for almost every $t \in [0, 2\pi]$. By Lemma 3.1 either v has exactly m positive and m negative humps, such that (3.8) and (3.9) are satisfied or v has exactly $m+1$ positive and $m+1$ negative humps, such that (3.10) and (3.11) are satisfied.

In any case (3.14), (3.8) and (3.9) imply that u_n takes positive and negative values. In the first case we consider, for each n, a positive hump $[\alpha_n, \beta_n]$ such that $u_n(\gamma_n) = \max_{[0,2\pi]} u_n = \max_{[\alpha_n,\beta_n]} u_n = R_n$. Going to subsequences, we can assume $\alpha_n \to \alpha$, $\beta_n \to \beta$ and $[\alpha, \beta]$ is a positive hump of v. One has

$$(0 \leq) \int_{[\alpha,\beta]} (p_{\lambda n}(u_n) - p_0) = \int_{[\alpha,\beta]} \left[\left(\lambda_n p(u_n) + (1-\lambda_n) \frac{a+c}{2} \right) - a \right] \to 0.$$

Hence, $\lambda_n \to 1$ and $p(u_n) \to a$ in $L^1(\alpha, \beta)$. It follows that, for any $\varepsilon > 0$ and for n large enough, we can write

$$\begin{aligned} \eta R_n &\leq \int_{[0,R_n]} (p(\xi) - a)\, d\xi \\ &= \int_{[\alpha_n,\gamma_n]} (p(u_n(t)) - a) u_n'(t)\, dt \\ &\leq |u_n'|_\infty \left[\int_{[\alpha_n,\alpha]} (p(u_n) - a) + \int_{[\alpha,\gamma_n]} (p(u_n) - a) \right] \\ &\leq \varepsilon |u_n'|_\infty, \end{aligned}$$

i.e.,

$$R_n \leq \frac{\varepsilon}{\eta} |u_n'|_\infty. \tag{3.17}$$

On the other hand, one has

$$|u_n''(t)| \le p_{\lambda n}(u_n(t))u_n^+(t) + q_{\lambda n}(u_n(t))u_n^-(t) + |r_{\lambda n}(t, u_n(t))|,$$

for almost every t and, therefore, for some constant $c_1 > 0$,

$$|u_n''|_1 \le c_1|(|u_n|_\infty + 1).$$

Hence, it follows

$$|u_n'|_\infty \le c_2(|u_n|_\infty + 1)$$

for some constant $c_2 > 0$. Using (3.14) and Lemma 3.1, we deduce that, for $K > 0$ small enough and n large,

$$\begin{aligned} \frac{1}{2}\sqrt{\frac{b}{a}} < \frac{1/\sqrt{a} - K}{1/\sqrt{b} + K} &\le -\frac{\max v_n}{\min v_n} \\ &= -\frac{\max u_n}{\min u_n} \le \frac{1/\sqrt{a} + K}{1/\sqrt{b} - K} < 2\sqrt{\frac{b}{a}}. \end{aligned}$$

Hence, for some $c_3 > 0$, $|u_n|_\infty \le c_3 R_n$. Accordingly, we obtain a contradiction from (3.17), choosing $\varepsilon > 0$ small enough. The proof is similar in the second case. □

Claim 2. *For n large enough, every solution u of* (3.12_λ)–(3.2), *for any* $\lambda \in [0, 1]$, *with* $\max u \le R_n$, *is such that*

$$\min u > -2 \max\left\{\sqrt{\frac{a}{b}}, \sqrt{\frac{c}{d}}\right\} R_n =: S_n.$$

The proof repeats the argument of Claim 4 in the proof of Theorem 2.1 (see also the last part of the proof of Claim 1 above).

Consider now the set

$$\Omega = \{u \in C^0([0, 2\pi]) : S < \min u \le \max u < R\},$$

where we choose $R = R_n$ and $S = S_n$ with n so large that Claims 1 and 2 hold. Hence there is no solution of (3.12_λ)–(3.2) on the boundary of

Ω and the proof of the theorem follows from a classical continuation argument.

Assume now that (3.5) holds. Consider the homotopy (3.12_λ)–(3.2). First we notice that if

$$\liminf_{s\to+\infty} \frac{1}{s}\int_{[0,s]} p(\xi)\,d\xi < c,$$

the proof follows from (3.3). Hence, we can assume that

$$\lim_{s\to+\infty} \frac{1}{s}\int_{[0,s]} p(\xi)\,d\xi = c > a.$$

This implies that there exist constants $\eta > 0$ and $R^* > 0$ such that, for every $R > R^*$,

$$(1/R)\int_{[0,R]} p(\xi)\,d\xi \geq a + \eta.$$

Following the proof of Claim 1, we obtain:

Claim 3. *There exists $R_0 > 0$ such that, for any $R \geq R_0$ and $\lambda \in [0,1]$, problem (3.12_λ)–(3.2) has no solution u such that $\max u = R$.*

From the second part of (3.5) we can find $\eta > 0$ and a sequence $(S_n)_n$ such that $S_n \to -\infty$ and

$$(1/S_n)\int_{[0,S_n]} p(\xi)\,d\xi \geq d - \eta.$$

The argument of Claim 1 proves the following

Claim 4. *For any $n_0 \geq 0$, there exists $n \geq n_0$ such that, for all $\lambda \in [0,1]$, (3.12_λ)–(3.2) has no solution u such that*

$$\min u = S_n < 0.$$

We are now in a position to apply a continuation argument with the set

$$\Omega = \{u \in C^0([0,2\pi]) : S < \min u \leq \max u < R\},$$

where $S = S_n < 0$, for $n \geq n_0$, is given by Claim 4 and R is chosen from Claim 3 such that $R > \max\{-S, R_0\}$.

Since cases (3.4) and (3.6) are symmetric to (3.3) and (3.5), respectively, the proof is complete. □

Remark 3.1. As in the preceding section, we can easily give conditions on g so that (i) and (ii) hold. Moreover, using the results stated in the Appendix we can rewrite conditions (3.3)–(3.6) in terms of the potential $G(s) = \int_{[0,s]} g(\xi)\, d\xi$. More precisely, one has the following corollary.

Corollary 3.1. *Assume that* $(a, b) \in \mathcal{C}_m = \{(\mu, \nu) : 1/\sqrt{\mu} + 1/\sqrt{\nu} = 2/m\}$, $(c, d) \in \mathcal{C}_{m+1} = \{(\mu, \nu) : 1/\sqrt{\mu} + 1/\sqrt{\nu} = 2/(m+1)\}$ *and*

$$a \leq \liminf_{s\to+\infty} \frac{g(s)}{s} \leq \limsup_{s\to+\infty} \frac{g(s)}{s} \leq c,$$

$$b \leq \liminf_{s\to-\infty} \frac{g(s)}{s} \leq \limsup_{s\to-\infty} \frac{g(s)}{s} \leq d.$$

Suppose that at least one of the following conditions holds

$$b < \limsup_{s\to-\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \liminf_{s\to-\infty} \frac{2G(s)}{s^2} < d,$$

$$b < \limsup_{s\to-\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \liminf_{s\to+\infty} \frac{2G(s)}{s^2} < c,$$

$$a < \limsup_{s\to+\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \liminf_{s\to-\infty} \frac{2G(s)}{s^2} < d,$$

$$a < \limsup_{s\to+\infty} \frac{2G(s)}{s^2} \quad \text{and} \quad \liminf_{s\to+\infty} \frac{2G(s)}{s^2} < c.$$

Then problem (3.1)–(3.2) *has at least one solution, for any given* $h \in L^1(0, 2\pi)$.

Appendix

Let $\phi : \mathbf{R}^+ = [0, +\infty[\to \mathbf{R}^+$ be a nondecreasing continuous function with $\phi(s) > 0$ for $s > 0$, and define

$$\Phi(s) = \int_{[0,s]} \phi(\xi)\, d\xi.$$

According to [**1**, page 232], we say that Φ satisfies the Δ_2-condition near infinity if there exist $c > 1$ and $d \geq 0$ such that

$$\Phi(2s) \leq c\Phi(s), \quad \text{for all } s \geq d.$$

It can be easily seen that for a function Φ as above this happens if and only if there is a constant $k > 1$ such that

$$\Phi(s) \leq s\phi(s) \leq k\Phi(s), \quad \text{for all } s \geq d.$$

Then we can prove:

Lemma 4.1. *Let Φ satisfy the Δ_2-condition near infinity and suppose that $f : \mathbf{R}^+ \to [0, L]$ is a continuous function. Then*

$$\lim_{s \to +\infty} \frac{1}{s} \int_{[0,s]} f(\xi)\, d\xi = 0$$

if and only if

$$\lim_{s \to +\infty} \frac{1}{\Phi(s)} \int_{[0,s]} f(\xi)\phi(\xi)\, d\xi = 0.$$

Proof. Assume that

$$\lim_{s \to +\infty} \frac{1}{s} \int_{[0,s]} f(\xi)\, d\xi = 0$$

and fix $s > d$. Then we have

$$\begin{aligned} 0 &\leq \frac{1}{\Phi(s)} \int_{[0,s]} f(\xi)\phi(\xi)\, d\xi \\ &\leq \frac{s\phi(s)}{s\Phi(s)} \int_{[0,s]} f(\xi)\, d\xi \\ &\leq \frac{k}{s} \int_{[0,s]} f(\xi)\, d\xi \end{aligned}$$

and the first part of the conclusion follows letting $s \to +\infty$.

Conversely, assume

$$\lim_{s\to+\infty} \frac{1}{\Phi(s)} \int_{[0,s]} f(\xi)\phi(\xi)\,d\xi = 0. \tag{4.1}$$

Fix $\varepsilon \in]0, 2L[$, and let $\alpha = \varepsilon/(2L) < 1$. Taking $s > d/\alpha$, we have

$$0 \leq \frac{1}{s}\int_{[0,s]} f(\xi)\,d\xi = \frac{1}{s}\int_{[0,\alpha s]} f(\xi)\,d\xi + \frac{1}{s}\int_{[\alpha s,s]} f(\xi)\,d\xi.$$

Since f is upper bounded by L, we immediately obtain

$$\frac{1}{s}\int_{[0,\alpha s]} f(\xi)\,d\xi \leq \alpha L = \frac{\varepsilon}{2}. \tag{4.2}$$

On the other hand, for $\alpha s \leq \xi \leq s$, we have that $\phi(\xi)/\Phi(\xi) \leq k/\xi$, so that integration on $[\alpha s, s]$ yields

$$\frac{\Phi(s)}{\Phi(\alpha s)} \leq \alpha^{-k}.$$

Hence, we have

$$\Phi(s) \leq \alpha^{-k}\Phi(\alpha s) \leq s\phi(\alpha s)\alpha^{1-k}$$

and then, as ϕ is nondecreasing,

$$\begin{aligned}\frac{1}{s}\int_{[\alpha s,s]} f(\xi)\,d\xi &\leq \frac{1}{s\phi(\alpha s)}\int_{[\alpha s,s]} f(\xi)\phi(\xi)\,d\xi\\ &\leq \frac{\alpha^{1-k}}{\Phi(s)}\int_{[\alpha s,s]} f(\xi)\phi(\xi)\,d\xi\\ &\leq \frac{\alpha^{1-k}}{\Phi(s)}\int_{[0,s]} f(\xi)\phi(\xi)\,d\xi.\end{aligned}$$

Using (4.1), we get, for s large enough,

$$\frac{1}{s}\int_{[\alpha s,s]} f(\xi)\,d\xi \leq \frac{\varepsilon}{2}. \tag{4.3}$$

In conclusion, from (4.2) and (4.3) we have, for s large enough,

$$\frac{1}{s}\int_{[0,s]} f(\xi)\,d\xi \le \varepsilon,$$

and therefore,

$$\lim_{s\to+\infty}\frac{1}{s}\int_{[0,s]} f(\xi)\,d\xi = 0.$$

The equivalence of the two conditions is thus proved. □

Lemma 4.2. *Under the same assumptions of Lemma* 4.1, *we have*

$$\lim_{s\to+\infty}\frac{1}{s}\int_{[0,s]} f(\xi)\,d\xi = L$$

if and only if

$$\lim_{s\to+\infty}\frac{1}{\Phi(s)}\int_{[0,s]} f(\xi)\phi(\xi)\,d\xi = L.$$

Proof. Apply Lemma 4.1 to the function $L - f(s)$ and the result immediately follows. □

Now we can conclude with

Lemma 4.3. *Under the same assumptions of Lemma* 4.1, *we have*

$$\liminf_{s\to+\infty}\frac{1}{s}\int_{[0,s]} f(\xi)\,d\xi < L$$

if and only if

$$\liminf_{s\to+\infty}\frac{1}{\Phi(s)}\int_{[0,s]} f(\xi)\phi(\xi)\,d\xi < L$$

and

$$\limsup_{s\to+\infty}\frac{1}{s}\int_{[0,s]} f(\xi)\,d\xi > 0$$

if and only if

$$\limsup_{s\to+\infty}\frac{1}{\Phi(s)}\int_{[0,s]} f(\xi)\phi(\xi)\,d\xi > 0.$$

Proof. It is sufficient to observe that $0 \leq \liminf \leq \limsup \leq L$, for all the ratios considered above and then to apply Lemma 4.2 and Lemma 4.1, respectively. $\square$

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Université Catholique de Louvain, Institut de Mathématique Pure et Appliquée, Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgique

Università di Trieste, Dipartimento di Scienze Matematiche, Piazzale Europa 1, I-34127 Trieste, Italia

Università di Udine, Dipartimento di Matematica e Informatica, Via Zanon 6, I-33100 Udine, Italia