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NON SELF-ADJOINT QUASI-DIFFERENTIAL OPERATORS WITH DISCRETE SPECTRA

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1. Introduction. The minimal operators T_0 and T_0^+ generated by a general ordinary quasi-differential expressions M and its formal adjoint M^+, respectively, form an adjoint pair of closed, densely-defined operators in the underlying L_w^2-space, that is, $T_0 \subset (T_0^+)^*$. The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression M are those which are regularly solvable with respect to T_0 and T_0^+. Such an operator S satisfies $T_0 \subset S \subset (T_0^+)^*$ and for some $\lambda \in \mathbf{C}$, $(S - \lambda I)$ is a Fredholm operator of zero index; this means that S has the desirable Fredholm property that the equation $(S - \lambda I)u = f$ has a solution if and only if f is orthogonal to the solutions of $(S^* - \bar{\lambda} I)v = 0$, and, furthermore, the solution spaces of $(S - \lambda I)u = 0$ and $(S^* - \bar{\lambda} I)v = 0$ have the same finite dimension. This notion was originally due to Visik, the abstract theory concerning regularly solvable extensions of T_0, which replaces the Stone-von Neumann theory of symmetric extensions of a symmetric operator, was worked out by W.D. Evans in [**4**]; see also [**3**, Chapter 3].

The principal object of this paper is to investigate the spectral properties of those operators which are regularly solvable with respect to the minimal operators T_0 and T_0^+ generated by a general ordinary quasi-differential expression M and its formal adjoint M^+ in $L_w^2(a, b)$. Of special interest is the subclass of operators which are well-posed with respect to T_0 and T_0^+: see Definition 2.1 below. We are mainly concerned with the case where all solutions are in $L_w^2(a, b)$.

In the case where all the solutions of the equations $(M - \lambda w)u = 0$, $(M^+ - \bar{\lambda} w)v = 0$ are in $L_w^2(a, b)$ for some (and hence all $\lambda \in \mathbf{C}$) it is shown that the well-posed operators have resolvents which are Hilbert Schmidt integral operators and consequently have a wholly discrete spectrum. This implies that all the regularly solvable operators have

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all of the standard essential spectra to be empty. These results extend those for formally symmetric expressions M studied by Akhiezer and Glazman in [**1**] and Naimark in [**9**].

We deal throughout with a quasi-differential expression M of arbitrary order n defined by a general Shin-Zettl matrix, and the minimal operator T_0 is generated by $(1/w)M[\cdot]$ in $L^2_w(I)$, where w is a positive weight function on the underlying interval I. The lefthand endpoint of I is assumed to be regular, but the righthand endpoint may be either regular or singular.

2. Preliminaries. We begin with a brief survey of definitions of adjoint pairs of operators and their associated regularly solvable and well-posed operators which are stated in [**3**, Chapter 3] and [**4**].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$, respectively, and $N(T)$ will denote its null space. The *nullity* of T, written $\text{nul}\,(T)$, is the dimension of $N(T)$ and the *deficiency* of T, $\text{def}\,(T)$, is the codimension of $R(T)$ in H; thus, if T is densely defined and $R(T)$ is closed, then $\text{def}\,(T) = \text{nul}\,(T^*)$. The *Fredholm domain* of T is (in the notation of [**3**]) the open subset $\Delta_3(T)$ of $\mathbf{C}$ consisting of those values $\lambda \in \mathbf{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator on H. Thus, $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The *index* of $(T - \lambda I)$ is the number

$$\text{ind}\,(T - \lambda I) = \text{nul}\,(T - \lambda I) - \text{def}\,(T - \lambda I),$$

this being defined for $\lambda \in \Delta_3(T)$.

Two closely densely defined operators A and B acting in H are said to form an *adjoint pair* if $A \subset B^*$ and, consequently, $B \subset A^*$; equivalently,

$$(Ax, y) = (x, By), \quad \text{for all } x \in D(A) \text{ and } y \in D(B),$$

where $(\cdot, \cdot)$ denotes the inner product on H.

The field of regularity $\Pi(A)$ of A is the set of $\lambda \in \mathbf{C}$, for which there exists a positive constant $K(\lambda)$ such that

$$||(A - \lambda I)x|| \geq K(\lambda)||x| \quad \text{for all } x \in D(A),$$

or, equivalently, on using the closed-graph theorem,

$$\operatorname{nul}(A-\lambda I)=0 \quad \text{and} \quad R(A-\lambda I) \text{ is closed.}$$

The *joint field of regularity* $\Pi(A,B)$ of A and B is the set of $\lambda \in \mathbf{C}$ which are such that $\lambda \in \Pi(\Lambda)$, $\bar{\lambda} \in \Pi(B)$ and both $\operatorname{def}(A-\lambda I)$ and $\operatorname{def}(B-\bar{\lambda} I)$ are finite. An adjoint pair A and B is said to be *compatible* if $\Pi(A,B) \neq \varnothing$.

Definition 2.1. A closed operator S in H is said to be *regularly solvable* with respect to the compatible adjoint pair A and B if $A \subset S \subset B^*$ and $\Pi(A,B) \cap \Delta_4(S) \neq \varnothing$, where

$$\Delta_4(S)=\{\lambda : \lambda \in \Delta_3(S), \operatorname{ind}(S-\lambda I)=0\}.$$

If $A \subset S \subset B^*$ and the resolvent set $\rho(S)$ (see [**3, 4**] of S is nonempty, S is said to be *well-posed* with respect to A and B. Note that if $A \subset S \subset B^*$ and $\lambda \in \rho(S)$, then $\lambda \in \Pi(A)$ and $\bar{\lambda} \in \rho(S^*) \subset \Pi(B)$ so that if $\operatorname{def}(A-\lambda I)$ and $\operatorname{def}(B-\bar{\lambda} I)$ are finite, then A and B are compatible; thus in this case S is regularly solvable with respect to A and B. The terminology, "regularly solvable" comes from Vishik's paper [**10**], while the notion of "well-posed," was introduced by Zhikhar in his work on J-self-adjoint operators in [**17**].

An important subset of the spectrum of a closed densely defined operator T in H is the so-called *essential spectrum.* The various essential spectra of T are defined as follows: First, let

$$\begin{aligned}
&\Phi_+(T)=\{\lambda \in \mathbf{C} : R(T-\lambda I) \text{ closed and } \operatorname{nul}(T-\lambda I)<\infty\},\\
&\Phi_-(T)=\{\lambda \in \mathbf{C} : R(T-\lambda I) \text{ closed and } \operatorname{def}(T-\lambda I)<\infty\},\\
&\Delta_1(T)=\Phi_+(T)\cup\Phi_-(T), \qquad \Delta_2(T)=\Phi_+(T),
\end{aligned}$$

$\Delta_3(T)$ and $\Delta_4(T)$ have been defined earlier, and $\Delta_5(T)$ is the union of all the components of $\Delta_1(T)$ which intersect $\rho(T)$. Then the essential spectra of T are the sets

$$\sigma_{ek}(T)=\mathbf{C}\backslash\Delta_k(T), \qquad k=1,2,3,4,5. \tag{2.1}$$

The sets $\sigma_{ek}(T)$ are closed and $\sigma_{ek}(T) \subset \sigma_{ej}(T)$ if $k<j$. The inclusion is strict, in general. We refer the reader to [**3**, Chapter 9] for further information about the sets $\sigma_{ek}(T)$.

We now turn to the quasi-differential expressions defined in terms of a Shin-Zettl matrix A on an open interval I, where I denotes an open interval with left endpoint a and right endpoint b, $(-\infty \le a < b \le \infty)$. The set $Z_n(I)$ of Shin-Zettl matrices on I consists of $n \times n$-matrices $A = \{a_{rs}\}$ whose entries are complex-valued functions on I which satisfy the following conditions:

$$\begin{aligned} a_{rs} &\in L^1_{\text{loc}}(I) && 1 \le r,\ s \le n,\ n \ge 2 \\ a_{r,r+1} &\ne 0 \quad \text{a.e. on } I && 1 \le r \le n-1, \\ a_{rs} &= 0 \quad \text{a.e. on } I && 2 \le r+1 < s \le n. \end{aligned} \tag{2.2}$$

For $A \in Z_n(I)$ the *quasi-derivatives* associated with A are defined by

$$\begin{aligned} y^{[0]} &:= y, \\ y^{[r]} &:= a_{r,r+1}^{-1}\left\{(y^{[r-1]})' - \sum_{s=1}^{r} a_{rs} y^{[s-1]}\right\}, \qquad 1 \le r \le n-1, \\ y^{[n]} &:= (y^{[n-1]})' - \sum_{s=1}^{n} a_{ns} y^{[s-1]}, \end{aligned} \tag{2.3}$$

where the prime denotes differentiation.

The *quasi-differential expression* M associated with A is given by

$$M[y] := i^n y^{[n]}, \qquad n \ge 2 \tag{2.4}$$

this being defined on the set

$$V(M) := \{y : y^{[r-1]} \in AC_{\text{loc}}(I),\ r = 1, 2, \dots, n\}, \tag{2.5}$$

where $AC_{\text{loc}}(I)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I.

The *formal adjoint* M^+ of M is defined by the matrix $A^+ \in Z_n(I)$ given by

$$A^+ := -L^{-1} A^* L, \tag{2.6}$$

where A^* is the conjugate transpose of A and L is the nonsingular $n \times n$ matrix

$$L = \{(-1)^r \delta_{r,n+1-s}\}_{1 \le r \le n,\ 1 \le s \le n}, \tag{2.7}$$

δ being the Kronecker delta. If $A^+ = \{a^+_{rs}\}$, then it follows that

$$a^+_{rs} = (-1)^{r+s+1}\bar{a}_{n-s+1,n-r+1}, \quad \text{for each } r \text{ and } s. \tag{2.8}$$

The quasi-derivatives associated with A^+ are, therefore,

$$\begin{aligned} y_+^{[0]} &:= y, \\ y_+^{[r]} &:= \bar{a}^{-1}_{n-r,n-r+1}\{(y_+^{[r-1]})' \\ &\qquad - \sum_{s=1}^{r}(-1)^{r+s+1}\bar{a}_{n-s+1,n-r+1}y_+^{[s-1]}\}, \\ &\qquad\qquad 1 \le r \le n-1, \\ y_+^{[n]} &:= (y_+^{[n-1]})' - \sum_{s=1}^{n}(-1)^{n+s+1}\bar{a}_{n-s+1,1}y_+^{[s-1]}, \end{aligned} \tag{2.9}$$

and

$$M^+[y] := i^n y_+^{[n]}, \qquad n \ge 2 \tag{2.10}$$

for all y in

$$V(M^+) := \{y : y_+^{[r-1]} \in AC_{\mathrm{loc}}(I),\ r = 1, 2, \dots, n\}. \tag{2.11}$$

Note that $(A^+)^+ = A$ and so $(M^+)^+ = M$. We refer to [**6, 7** and **16**] for a full account of the above and subsequent results on quasi-differential expressions.

For $u \in V(M)$, $v \in V(M^+)$ and $[\alpha, \beta] \subset I$, we have Green's formula

$$\int_\alpha^\beta \{\bar{v}M[u] - u\overline{M^+[v]}\}\,dx = [u,v](\beta) - [u,v](\alpha), \tag{2.12}$$

where

$$\begin{aligned} [u,v](x) &= i^n\left\{\sum_{r=0}^{n-1}(-1)^{n+r+1}u^{[r]}(x)\bar{v}_+^{[n-r-1]}(x)\right\} \\ &= (-i^n)[u(x), \dots, u^{[n-1]}(x)]L\begin{bmatrix}\bar{v}(x) \\ \vdots \\ \bar{v}_+^{[n-1]}(x)\end{bmatrix}; \end{aligned} \tag{2.13}$$

see [**16**, Corollary 1].

Let the interval I have endpoints a, b, $-\infty \leq a < b \leq \infty$, and let w be a function which satisfies

$$w > 0 \quad \text{a.e. on } I, \qquad w \in L^1_{\text{loc}}(I). \tag{2.14}$$

The equation

$$M[u] = \lambda w u, \qquad \lambda \in \mathbf{C} \tag{2.15}$$

on I is said to be *regular* at the left endpoint $a \in \mathbf{R}$ if for all $X \in (a, b)$,

$$a \in \mathbf{R}; \quad w, a_{rs} \in L^1[a, X], \qquad r = 1, 2, \dots, n. \tag{2.16}$$

Otherwise (2.15) is said to be *singular* at a. Similarly, we define the terms regular and singular at b. If (2.15) is regular at both endpoints, then it is said to be regular; in this case we have

$$a, b \in \mathbf{R}, \quad w, a_{rs} \in L^1(a, b), \qquad r, s = 1, 2, \dots, n. \tag{2.17}$$

We shall be concerned with the case where a is a regular endpoint of (2.15), the endpoint b being allows to be either regular or singular. Note that, in view of (2.8), an endpoint of I is regular for (2.15) if and only if it is regular for the equation

$$M^+[v] = \bar{\lambda} w v, \qquad \lambda \in \mathbf{C}. \tag{2.18}$$

Let $L^2_w(a, b)$ denote the usual weighted L^2-space with inner-product,

$$(f, g) := \int_a^b f(x)\bar{g}(x)w(x)\, dx, \tag{2.19}$$

and norm $||f|| := (f, f)^{1/2}$; this is a Hilbert space on identifying functions which differ only on null sets. Set

$$\begin{aligned} D &:= \{u; u \in V(M), u \text{ and } (1/w)M[u] \in L^2_w(a, b)\}, \\ D^+ &:= \{v : v \in V(M^+), v \text{ and } (1/w)M^+[v] \in L^2_w(a, b)\}. \end{aligned} \tag{2.20}$$

The subspaces D and D^+ of $L^2_w(a, b)$ are domains of the so-called *maximal operators* T and T^+, respectively, defined by

$$Tu := (1/w)M[u], \quad u \in D \quad \text{and} \quad T^+v := (1/w)M^+[v], \quad v \in D^+.$$

For the regular problem the *minimal operators* T_0 and T_0^+ are the restrictions of $(1/w)M[\cdot]$ and $(1/w)M^+[\cdot]$ to the subspaces

(2.21)
$$D_0 := \{u : u \in D, u^{[r-1]}(a) = u^{[r-1]}(b) = 0,\ r = 1, 2, \dots, n\},$$
$$D_0^+ := \{v : v \in D^+, v_+^{[r-1]}(a) = v_+^{[r-1]}(b) = 0,\ r = 1, 2, \dots, n\},$$

respectively. The subspaces D_0 and D_0^+ are dense in $L_w^2(a,b)$ and T_0 and T_0^+ are closed operators (see [**16**, Section 3]). In the singular problem we first introduce operators T_0' and $(T_0^+)'$, T_0' being the restriction of $(1/w)M[\cdot]$ to

$$D_0' := \{u : u \in D, \operatorname{supp} u \subset (a,b)\},$$

and with $(T_0^+)'$ defined similarly. These operators are densely-defined and closable in $L_w^2(a,b)$, and we define the minimal operators T_0 and T_0^+ to be their respective closures (see [**16**, Section 5]). We denote the domains of T_0 and T_0^+ by D_0 and D_0^+, respectively. It can be shown that

$$\begin{aligned} &u \in D_0 \Longrightarrow u^{[r-1]}(a) = 0, \qquad r = 1, 2, \dots, n,\\ &v \in D^+ \Longrightarrow v_+^{[r-1]}(a) = 0, \qquad r = 1, 2, \dots, n, \end{aligned} \tag{2.22}$$

because we are assuming that a is a regular endpoint.

Moreover, in both the regular and singular problems, we have

$$T_0^* = T^+, \qquad T_0^+ = T^*; \tag{2.23}$$

see [**16**, Section 5] in the case where $M = M^+$, and compare with the treatment in [**3**, Section 3], [**8** and **12**] in the general case.

3. The regularity solvable operators. We see from (2.23) that $T_0 \subset T = (T_0^+)^*$, and hence, T_0 and T_0^+ form an adjoint pair of closed, densely-defined operators in $L_w^2(a,b)$. By [**3**, Corollary 3.3.2], $\operatorname{def}(T_0 - \lambda I) + \operatorname{def}(T_0^+ - \bar{\lambda} I)$ is constant on the joint field of regularity $\Pi(T_0, T_0^+)$, and we have shown in [**5**] that

$$n \le \operatorname{def}(T_0 - \lambda I) + \operatorname{def}(T_0^+ - \bar{\lambda} I) \le 2n \quad \text{for all } \lambda \in \Pi(T_0, T_0^+).$$

For $\Pi(T_0, T_0^+) \neq \varnothing$, the operators which are regularly solvable with respect to T_0 and T_0^+ are characterized by the following theorem which is proved for the general case in [**5**]; see also Theorem 10.5 of [**3**]. We shall use the notation

$$[u,v](b) = \lim_{x\to b^-}[u,v](x), \qquad u\in D \text{ and } v\in D^+,$$

if b is a singular endpoint of I, and similarly for $[u,v](a)$ if a is singular. Note that it follows from (2.12) that these limits exist for $u\in D$ and $v\in D^+$ since then $\bar{v}M[u]$ and $\overline{uM^+[v]}$ are both integrable by the Cauchy-Schwartz inequality.

Theorem 3.1. *Let T_0 and T_0^+ be compatible, and suppose that*

$$\text{def}\,(T_0-\lambda I) = \text{def}\,(T_0^+-\bar{\lambda} I) = n \quad \textit{for all } \lambda\in\Pi(T_0,T_0^+).$$

Then every closed operator S which is regularly solvable with respect to T_0 and T_0^+ is the restriction of T to the set of functions $u\in D$ which satisfy linearly independent boundary conditions

$$(3.1) \qquad [u,\phi_j](b)-[u,\phi_j](a)=0, \qquad j=1,2,\dots,n.$$

The set $\{\phi_1,\phi_2,\dots,\phi_n\}$ is a basis for $\{D(S^*)/D_0^+\}$ where $\dim\{D(S^*)/D_0^+\} = \text{def}\,(T_0^+-\bar{\lambda}I)$, *and S^* is the restriction of T^+ to the set of functions $v\in D^+$ which satisfy linearly independent boundary conditions*

$$(3.2) \qquad [\psi_j,v](b)-[\psi_j,v](a)=0, \qquad j=1,2,\dots,n.$$

The set $\{\psi_1,\psi_2,\dots,\psi_n\}$ is a basis for $\{D(S)/D_0\}$ where $\dim\{D(S)/D_0\} = \text{def}\,(T_0-\lambda I)$ *and*

$$(3.3) \qquad [\psi_j,\phi_k](b)-[\psi_j,\phi_k](a)=0, \qquad j,k=1,2,\dots,n.$$

Conversely, for arbitrary functions $\{\phi_1,\phi_2,\dots,\phi_n\}$ (respectively $\{\psi_1,\psi_2,\dots,\psi_n\}$) in $D^+(D)$ which are linearly independent satisfy (3.1) *(respectively* (3.2)*), and* (3.3) *is satisfied, then $S=T|D_1$ is regularly solvable with respect to T_0 and T_0^+, and $S^*=T^+|D_2$.*

S is self-adjoint (respectively J-self-adjoint) if, and only if, $M=M^+$ (respectively $M^+=\overline{M}$) and $\psi_j=\phi_j$, $\psi_j=\bar{\phi}_j$, for $j=1,2,\dots,n$.

We refer to [**12**] for further information on boundary conditions for nonselfadjoint problems.

4. Properties of quasi-differential equations. Given a function g, by a solution of

$$M[y] = wg, \quad \text{on } [a,b), \tag{4.1}$$

we mean a function y from $[a,b)$ to the complex numbers $\mathbf{C}$ such that $y^{[r]} \in AC_{\text{loc}}[a,b)$ for $r = 0,1,\dots,n-1$ (i.e., $y \in V(M)$) and (4.1) is satisfied almost everywhere on $[a,b)$.

Similarly, given a vector (matrix)-valued function G, we define a solution of

$$Y'(t) = A(t)Y(t) + (1/i^n)w(t)G(t) \quad \text{on } [a,b), \tag{4.2}$$

to be a vector (matrix)-valued function $Y \in AC_{\text{loc}}[a,b)$. It follows from the definition of $M[.]$ in terms of a Shin-Zettl matrix A that (4.1) is equivalent to (4.2) (see [**2**, Chapter 3] and [**16**]) where $Y = (y, y^{[1]}, \dots, y^{[n-1]})^\top$ ($\top$ indicates a transposed matrix) and G is the column vector $(0,\dots,0,g)^\top$, $g \in L^1_{\text{loc}}[a,b)$.

The evolution operator (or fundamental matrix) $M(t,t_0)$ is the unique matrix-valued solution of

$$Y'(t) = A(t)Y(t), \qquad Y(t_0) = I, \qquad t_0 \in [a,b). \tag{4.3}$$

Note that, if y is a vector-valued solution of (4.3) which satisfies $y(t_0) = \xi$, then

$$M(t,t_0)\xi = y(t), \qquad t, t_0 \in [a,b). \tag{4.4}$$

It also follows that

$$M(t,t_0)M(t_0,s) = M(t,s)$$

and, in particular,

$$M(t,t_0)^{-1} = M(t_0,t).$$

Let

$$M(t,t_0) = \Phi(t) = (M_{kj}(t,t_0)) \quad \text{for } t, t_0 \in [a,b), \tag{4.5}$$

$k,j = 1,2,\dots,n$. Denote by $N(t,t_0)$ the unique (matrix-valued) solution of

$$X'(t) = A^{+}(t)X(t), \qquad X(t_0) = I, \qquad t_0 \in [a,b), \tag{4.6}$$

where A^{+} is defined by (2.6). Let

$$N(t,t_0) = \psi(t) = (N_{kj}(t,t_0)) \quad \text{for } t, t_0 \in [a,b), \tag{4.7}$$

$k,j = 1,2,\dots,n$. From [**14**] and [**16**] we have that, for any $t, t_0 \in [a,b)$,

$$M(t,t_0) = L^{-1}N^{*}(t_0,t)L, \tag{4.8}$$

and

$$M_{kj}(t,t_0) = (-1)^{k+j}\overline{N}_{n+1-j,n+1-k^{(t_0,t)}}, \qquad k,j = 1,2,\dots,n. \tag{4.9}$$

Theorem 4.1. *Suppose that G is a locally integrable vector-valued function and $t_0 \in [a,b)$. If Y is a vector-valued solution of* (4.2), *then*

$$Y(t) = M(t,t_0)Y(t_0) + \frac{1}{i^n}\bigg\{\int_{t_0}^{t} M(t,s)w(s)G(s)\,ds\bigg\}. \tag{4.10}$$

Proof. The proof follows by a direct computation (see [**2**, Chapter 3]). Formula (4.10) is known as the variation of parameters formula. □

From [**11, 15**] and [**16**], we have the following immediate consequence of formula (4.10):

Corollary 4.2. *For g locally integrable, the solution ϕ of the quasi-differential equation*

$$M[\phi] - \lambda w\phi = wg \quad \textit{on } [a,b), \tag{4.11}$$

satisfying

$$\phi^{[r]}(t_0) = c_{r+1} \quad \textit{for } r = 0, 1, \dots, n-1, t_0 \in [a, b),$$

is given by

$$\phi(t) = \sum_{k=1}^{n} c_k \phi_k(t) + \frac{1}{i^n} \biggl\{ \sum_{k=1}^{n} \phi_k(t) \int_{t_0}^{t} \overline{\phi_k^+(s)} w(s) g(s)\, ds \biggr\}, \tag{4.12}$$

where $\phi_k(t)$, $k = 1, 2, \dots, n$, are the solutions of the homogeneous equation

$$M[\phi] - \lambda w \phi = 0 \quad \textit{on } [a, b) \tag{4.13}$$

determined by

$$\begin{gathered} \phi_k^{[r]}(t_0) = \delta_{k, r+1} \\ k = 1, 2, \dots, n;\ r = 0, 1, \dots, n-1 \end{gathered} \tag{4.14}$$

and $\phi_k^+(s)$, $k = 1, 2, \dots, n$, are the solutions of the adjoint equation

$$M^+[\psi] - \bar{\lambda} w \psi = 0 \quad \textit{on } [a, b) \tag{4.15}$$

determined by

$$\begin{gathered} \phi_k^{+^{[r]}}(t_0) = (-1)^{k+n} \delta_{k, n-r} \\ k = 1, 2, \dots, n;\ r = 0, 1, \dots, n-1. \end{gathered} \tag{4.16}$$

Proof. From the variation of parameters formula (4.10) with $G = (0, \dots, 0, g)^\top$, it follows that

$$\phi(t) = \sum_{k=1}^{n} M_{1k}(t, t_0) c_k + \frac{1}{i^n} \biggl\{ \int_{t_0}^{t} M_{1n}(t, s) w(s) g(s)\, ds \biggr\}, \tag{4.17}$$

where $M_{1k}(t, t_0)$, $k = 1, 2, \dots, n$ are bases for the solution space of (4.13) and $M_{1n}(t, s)$ is the element in the first row and nth column of the matrix $M(t, s)$.

As a consequence of (4.17) and the definitions of $M(.,.), N(.,.), \phi_k(.)$ and $\phi_k^+(.)$, we can obtain that

$$M_{1n}(t,s) = \sum_{k=1}^{n} M_{1k}(t,t_0)M_{kn}(t_0,s)$$
$$= \sum_{k=1}^{n} \phi_k(t)(-1)^{k+n}\overline{N}_{1,n+1-k^{(s,t_0)}},$$

by (4.5) and (4.9),

$$= \sum_{k=1}^{n} \phi_k(t)(-1)^{k+n}\bar{\psi}_{n+1-k^{(s)}}$$
$$= \sum_{k=1}^{n} \phi_k(t)\overline{\phi_k^+(s)};$$

where

$$\phi_k^+(s) = (-1)^{k+n}\psi_{n+1-k^{(s)}}, \qquad k = 1,2,\dots,n.$$

Hence (4.12) follows. □

Also, by [**9**] and [**16**], the solution ϕ of (4.11) is given by

$$\phi(t) = \sum_{k=1}^{n} c_k\phi_k(t) + \frac{1}{i^n}\bigg\{\sum_{k=1}^{n}\phi_k(t)\int_{t_0}^{t} V_k(s)w(s)g(s)\,ds\bigg\}, \tag{4.18}$$

for some constants $c_1, c_2, \dots, c_n \in \mathbf{C}$ and $t_0 \in [a,b)$, where

$$V_k(s) = W_k(\phi_1,\dots,\phi_n)(s)/W(\phi_1,\dots,\phi_n)(s), \tag{4.19}$$

$k = 1,2,\dots,n$; $W_k(\phi_1,\dots,\phi_n)(s)$ is the determinant obtained from the Wronskian, $W(\phi_1,\dots,\phi_n)(s)$ by replacing the kth column by $(0,\dots,0,1)$.

By comparing between the two formulas (4.12) and (4.18) we have that

$$V_k(s) = \overline{\phi_k^+(s)}, \qquad k = 1,2,\dots,n \quad \text{for all } s \in [a,b). \tag{4.20}$$

This result can also be proved directly (see [**2**, problem 19, p. 101]).

The following result was obtained by Zettl in [**15**, Section 3], see also [**12**]; for the case of a formally self-adjoint expression $M[\cdot]$ see Walker's paper [**11**, Section 3].

Proposition 4.3. *Suppose that, for some $\lambda_0 \in \mathbf{C}$ and $c \in [a,b)$, all solutions of*

$$M[\phi] = \lambda_0 w\phi \quad \textit{and} \quad M^+[\psi] = \bar{\lambda}_0 w\psi$$

are in $L^2_w(a,b)$. Then all solutions of

$$M[\phi] = \lambda w\phi \quad \textit{and} \quad M^+[\psi] = \bar{\lambda} w\psi$$

are in $L^2_w(a,b)$ for all $\lambda \in \mathbf{C}$.

Proof. Let $\{\phi_1(.,\lambda_0),\dots,\phi_n(.,\lambda_0)\}$ and $\{\phi_1^+(.,\lambda_0),\dots,\phi_n^+(.,\lambda_0)\}$ be two sets of linearly independent solutions of

$$M[\phi] = \lambda_0 w\phi \quad \text{and} \quad M^+[\psi] = \bar{\lambda}_0 w\psi$$

which are in $L^2_w(a,b)$ and satisfying (4.14) and (4.16), respectively. Let $\phi(t,\lambda)$ be any solution of $M[\phi] = \lambda w\phi$ which may be written as follows $M[\phi] = \lambda_0 w\phi + (\lambda - \lambda_0)w\phi$. Let c be in $[a,b)$. The variation of parameters formula (4.12) yields

(4.21)
$$\begin{aligned}\phi(t,\lambda) &= \sum_{j=1}^n c_j\phi_j(t,\lambda_0)\\ &\quad + \frac{1}{i^n}(\lambda-\lambda_0)\bigg\{\sum_{j=1}^n \phi_j(t,\lambda_0)\int_c^t \overline{\phi_j^+(s,\lambda_0)}\phi(s,\lambda)w(s)\,ds\bigg\}\\ &\quad \dots\end{aligned}$$

for some constants $c_1, c_2, \dots, c_n \in \mathbf{C}$. Let k_0 and k_1 be the smallest numbers such that,

$$||\phi_j(.,\lambda_0)||_{L^2 w(c,b)} \le k_0$$

and

$$||\phi_j^+(.,\lambda_0)||_{L^2_w(c,b)} \le k_1, \qquad j = 1,\dots,n.$$

Using the Cauchy-Schwartz inequality yields, for $a \le c \le t \le z < b$,

$$\begin{aligned}
|\phi(t,\lambda)| &\le \sum_{j=1}^{n} |c_j|\,|\phi_j(t,\lambda_0)| \\
&\quad + |\lambda-\lambda_0| \sum_{j=1}^{n} |\phi_j(t,\lambda_0)| \bigg\{ \bigg\{ \int_c^t \overline{\phi_j^+(s,\lambda_0)|}^2 w(s)\,ds \bigg\}^{1/2} \\
&\qquad \cdot \bigg\{ \int_c^z |\phi(s,\lambda)|^2 w(s)\,ds \bigg\}^{1/2} \bigg\} \\
&\le \sum_{j=1}^{n} |c_j|\,|\phi_j(t,\lambda_0) \\
&\quad + k_1|\lambda-\lambda_0| \sum_{j=1}^{n} |\phi_j(t,\lambda_0)| \bigg\{ \int_c^z |\phi(s,\lambda)|^2 w(s)\,ds \bigg\}^{1/2}
\end{aligned}$$

and, for $z \in [c,b)$,

$$\bigg\{ \int_c^z |\phi(.,\lambda)|^2 w \bigg\}^{1/2} \le k_0 \sum_{j=1}^{n} |c_j| + nk_1k_0|\lambda-\lambda_0| \bigg\{ \int_c^z |\phi(.,\lambda)|^2 w \bigg\}^{1/2}.$$

We now choose c to be sufficiently close to b in order that

$$k_0k_1|\lambda-\lambda_0| \le 1/(2n).$$

It follows that

$$\int_c^z |\phi(.,\lambda)|^2 w \le 4k_0^2 \bigg\{ \sum_{j=1}^{n} |c_j| \bigg\}^2. \tag{4.22}$$

Since the righthand side of (4.22) is independent of z, then we have that $\phi(\cdot,\lambda) \in L_w^2(a,b)$. Similarly, we also have that $\phi^+(\cdot,\lambda) \in L_w^2(a,b)$. Hence the proposition. □

5. The main results. We shall now begin the investigation of resolvents of well-posed extensions of the operator T_0; we shall see that in the maximal case, i.e., when

$$\text{def}\,(T_0 - \lambda I) = \text{def}\,(T_0^+ - \bar{\lambda} I) = n \quad \text{for all } \lambda \in \Pi(T_0, T_0^+),$$

these resolvents are integral operators. In fact, they are Hilbert-Schmidt integral operators. We refer to Walker's paper [**12**] for more details.

We denote by H' the set of all functions f which are quadratically integrable over a given fixed interval $\Delta = [\alpha, \beta] \subset (a, b)$ and which vanish outside Δ (the interval can be different for different functions f). The space H_Δ of all functions which are quadratically integrable over a given fixed interval Δ can be regarded as a part of H', $H_\Delta \subset H'$ on setting $f = 0$ outside Δ.

A special case of the following theorem was proved in Naimark [**9**, Volume 2] and in Akhiezer and Glazman [**1**, Volume 2], namely, the case of self-adjoint extensions of the minimal operator.

Theorem 5.1. *Let $\lambda \in \Pi(T_0, T_0^+)$ and suppose that*

$$\mathrm{def}\,(T_0 - \lambda I) = \mathrm{def}\,(T_0^+ - \bar{\lambda} I) = n.$$

Let S be an arbitrary closed operator which is well-posed with respect to T_0 and T_0^+ and $\lambda \in \rho(S)$. Then the resolvents of S and S^* are Hilbert-Schmidt integral operators, i.e., for $\lambda \in \rho(S)$,

$$(S - \lambda I)^{-1} f(x) = \int_a^b K(x, t, \lambda) w(t) f(t)\, dt, \tag{5.1}$$

$$(S^* - \bar{\lambda} I)^{-1} g(t) = \int_a^b K^+(t, x, \bar{\lambda}) w(x) g(x)\, dx, \tag{5.2}$$

$x, t \in [a, b)$ almost everywhere, where the kernels $K(x, t, \lambda)$ and $K^+(t, x, \bar{\lambda})$ are continuous functions on $[a, b) \times [a, b)$ and satisfy

$$\overline{K^+(t, x, \bar{\lambda})} = K(x, t, \lambda) \quad \textit{for all } x, t \in [a, b),$$

$$\int_a^b \int_a^b |K(x, t, \lambda)|^2 w(x) w(t)\, dx\, dt < \infty. \tag{5.3}$$

Remark. An example of a closed operator which is well posed with respect to a compatible adjoint pair is given by the Visik extension (see

[**3**, Theorem 3] and [**10**, Theorem 1]). Note that if S is well posed, then T_0 and T_0^+ are compatible and S is regularly solvable with respect to T_0 and T_0^+.

The proof follows closely that of Theorem 1 in Section 19.2 in Naimark [**9**], but, for completeness, we go through the main points to bring in the changes necessary for our problems.

Proof. Let $R_\lambda = (S - \lambda I)^{-1}$ be the resolvent of any well-posed extension S of the minimal operator T_0. Let $\Delta = [\alpha, \beta]$ be a fixed, finite interval in (a, b). For $f \in H_\Delta = L_w^2(\alpha, \beta)$ with $f \equiv 0$ outside Δ, we put $\phi = R_\lambda f$. Then

$$M[\phi] - \lambda w\phi = wf. \tag{5.4}$$

Consequently, by the method of variation of constants formula (4.12), ϕ has the form

$$\phi(x) = \sum_{k=1}^{n} c_k\phi_k(x) + \frac{1}{i^n}\left\{ \sum_{k=1}^{n} \phi_k(x) \int_a^x \overline{\phi_k^+(t)} w(t) f(t)\, dt \right\} \tag{5.5}$$

for some constants $c_1, c_2, \dots, c_n \in \mathbf{C}$, where $\phi_k^+(t)$, $k = 1, 2, \dots, n$ are the solutions of the equation $M^+[\psi] - \bar{\lambda} w\psi = 0$, satisfying (4.16). Let

$$\operatorname{def}(T_0 - \lambda I) = \operatorname{def}(T_0^+ - \bar{\lambda} I) = n \quad \text{for all } \lambda \in \Pi(T_0, T_0^+).$$

We choose a fundamental system of solutions $\phi_1, \phi_2, \dots, \phi_n$ of the equation

$$M[\phi] - \lambda w\phi = 0, \tag{5.6}$$

satisfying (4.14), so that the functions $\phi_1, \phi_2, \dots, \phi_n$ belong to $H = L_w^2(a, b)$. □

For $x > \beta$ the integrals on the righthand side of (5.5) will be constant and equal to

$$\int_a^\beta \overline{\phi_k^+(t)} w(t) f(t)\, dt;$$

hence, for $x > \beta$,

$$\phi(x) = \sum_{k=1}^{n} \left(c_k + \frac{1}{i^n} \left\{ \int_a^\beta \overline{\phi_k^+(t)} w(t) f(t)\, dt \right\} \right) \phi_k(x). \tag{5.7}$$

Next we determine the constants $c_1, c_2, \dots, c_n$. If $\{\psi_1^+, \psi_2^+, \dots, \psi_n^+\}$ is a basis for $\{D(S^*)/D_0^+\}$, then because $\phi \in D(S)$ and $\lambda \in \rho(S) \subset \Delta_4(S)$ we have from Theorem 3.1 that

$$\begin{gathered} [\phi, \psi_j^+](b) - [\phi, \psi_j^+](a) = 0, \\ j = 1, 2, \dots, n \quad \text{on } [a, b). \end{gathered} \tag{5.8}$$

We want to write these conditions out in detail. From (5.7), for $x > \beta$,

$$\phi^{[r]}(x) = \sum_{k=1}^{n} c_k \phi_k^{[r]}(x) + \frac{1}{i^n} \left\{ \sum_{k=1}^{n} \phi_k^{[r]}(x) \int_a^x \overline{\phi_k^+(t)} w(t) f(t)\, dt \right\}.$$

Hence

$$[\phi, \psi_j^+]_b = \sum_{k=1}^{n} \left(c_k + \frac{1}{i^n} \left\{ \int_a^b \overline{\phi_k^+(t)} w(t) f(t)\, dt \right\} \right) [\phi_k, \psi_j^+]_b.$$

Also

$$[\phi, \psi_j^+]_a = \sum_{k=1}^{n} c_k [\phi_k, \psi_j^+]_a, \qquad j = 1, 2, \dots, n,$$

since the integral on the righthand side of (5.5) vanishes in a neighborhood of a. By substituting these expressions into the conditions (5.8), we get

$$\sum_{k=1}^{n} \left(c_k + \frac{1}{i^n} \left\{ \int_a^b \overline{\phi_k^+(t)} w(t) f(t)\, dt \right\} \right) [\phi_k, \psi_j^+]_b = \sum_{k=1}^{n} c_k [\phi_k, \psi_j^+]_a,$$

$j = 1, 2, \dots, n$. This implies that

$$\begin{aligned} \sum_{k=1}^{n} c_k [\phi_k, \psi_j^+]_b &- \sum_{k=1}^{n} c_k [\phi_k, \psi_j^+]_a \\ &= -\frac{1}{i^n} \left\{ \sum_{k=1}^{n} [\phi_k, \psi_j^+]_b \int_a^b \overline{\phi_k^+(t)} w(t) f(t)\, dt \right\}. \end{aligned}$$

Hence we obtain the system of equations

$$\sum_{k=1}^{n} c_k([\phi_k, \psi_j^+])_a^b = -\frac{1}{i^n}\bigg\{\sum_{k=1}^{n}[\phi_k, \psi_j^+]_b \int_a^b \overline{\phi_k^+(t)}w(t)f(t)\,dt\bigg\}, \tag{5.9}$$

$j = 1, \dots, n$, in the variables $c_1, c_2, \dots, c_n$. The determinant of this system does not vanish, for otherwise the homogeneous system

$$\sum_{k=1}^{n} c_k([\phi_k, \psi_j^+])_a^b = 0, \qquad j = 1, 2, \dots, n,$$

would have a nontrivial solution $c_1, c_2, \dots, c_n$, and the function

$$\tilde{\phi} = \sum_{k=1}^{n} c_k \phi_k(x)$$

would satisfy the conditions

$$[\tilde{\phi}, \psi_j^+](b) - [\tilde{\phi}, \psi_j^+](a) = 0, \qquad j = 1, 2, \dots, n;$$

hence $\tilde{\phi}$ would belong to $D(S)$ and $(S - \lambda I)\tilde{\phi} = 0$. But this is impossible for $\tilde{\phi} \neq 0$ since it was assumed that $\lambda \in \rho(S)$.

If we solve the system (5.9), we obtain

$$c_k = \frac{1}{i^n}\bigg\{\int_a^b h_k(t)w(t)f(t)\,dt\bigg\}, \qquad k = 1, 2, \dots, n,$$

where $h_k(t)$ is a solution of the system

$$\sum_{k=1}^{n} h_k(t)([\phi_k, \psi_j^+])_a^b = -\sum_{k=1}^{n}[\phi_k \cdot \psi_j^+]_b \overline{\phi_k^+(t)}. \tag{5.10}$$

Since, as was proved above, the determinant of this last system does not vanish, and the functions $\phi_k^+(t)$ are continuous in the interval $[a, b)$, the functions $h_k(t)$ are also continuous in this interval. By substituting in formula (5.5) the expressions for c_k, $k = 1, 2, \dots, n$, we get

$$
\begin{aligned}
R_\lambda f(x) &= (S - \lambda I)^{-1} f(x) \\
&= \phi(x) \\
&= \frac{1}{i^n}\Bigg\{ \sum_{k=1}^{n} \phi_k(x) \int_a^x h_k(t) w(t) f(t)\, dt \\
&\qquad + \sum_{k=1}^{n} \phi_k(x) \int_x^b h_k(t) w(t) f(t)\, dt \\
&\qquad + \sum_{k=1}^{n} \phi_k(x) \int_a^x \overline{\phi_k^+(t)} w(t) f(t)\, dt \Bigg\} \\
&= \frac{1}{i^n}\Bigg\{ \sum_{k=1}^{n} \phi_k(x) \int_a^x [\overline{\phi_k^+(t)} + h_k(t)] w(t) f(t)\, dt \\
&\qquad + \sum_{k=1}^{n} \phi_k(x) \int_x^b h_k(t) w(t) f(t)\, dt \Bigg\}.
\end{aligned} \tag{5.11}
$$

Now we put

$$
K(x,t,\lambda) = \begin{cases} \frac{1}{i^n}\{\sum_{k=1}^n \phi_k(x) h_k(t)\}, & \text{for } x < t, \\ \frac{1}{i^n}\{\sum_{k=1}^n \phi_k(x)[h_k(t) + \overline{\phi_k^+(t)}]\} & \text{for } x > t. \end{cases} \tag{5.12}
$$

Formula (5.11) then takes the form

$$
R_\lambda f(x) = \int_a^b K(x,t,\lambda) w(t) f(t)\, dt, \tag{5.13}
$$

i.e., R_λ is an integral operator with the kernel $K(x,t,\lambda)$ operating on the functions $f \in H'$. Similarly, by the method of variation of constants formula, the solution $\phi^+(t)$ of

$$
M^+[\psi] - \bar{\lambda} w \psi = wg,
$$

has the form

$$
\phi^+(t) = \sum_{k=1}^{n} c_k \phi_k^+(t) + \frac{1}{i^n}\Bigg\{ \sum_{k=1}^{n} \phi_k^+(t) \int_a^t \overline{\phi_k(x)} w(x) g(x)\, dx \Bigg\},
$$

for some constants $c_1, c_2, \dots, c_n \in \mathbf{C}$. The argument as before leads to

$$\begin{aligned} R^*_\lambda g(t) &= (S^* - \bar{\lambda} I)^{-1} g(t) \\ &= \phi^+(t) \\ &= \frac{1}{i^n}\bigg\{ \sum_{k=1}^n \phi_k^+(t) \int_a^t [\overline{\phi_k(x)} + h_k^+(x)] w(x) g(x)\, dx \\ &\qquad + \sum_{k=1}^n \phi_k^+(t) \int_t^b h_k^+(x) w(x) g(x)\, dx \bigg\}, \end{aligned} \tag{5.14}$$

where h_k^+ is a solution of the system

$$\sum_{k=1}^n \overline{h_k^+(x)} ([\psi_j, \phi_k^+])_a^b = -\sum_{k=1}^n [\psi_j, \phi_k^+]_b \phi_k(x), \qquad j = 1, 2, \dots, n,$$

and $\{\psi_1, \psi_2, \dots, \psi_n\}$ is a basis of $\{D(S)/D_0\}$.

Now we put

$$K^+(t, x, \bar{\lambda}) = \begin{cases} \frac{1}{i^n}\{\sum_{k=1}^n \phi_k^+(t) h_k^+(x)\} & \text{for } t < x, \\ \frac{1}{i^n}\{\sum_{k=1}^n \phi_k^+(t)[h_k^+(x) + \overline{\phi_k(x)}]\} & \text{for } t > x. \end{cases} \tag{5.15}$$

Formula (5.14) then takes the form

$$R^*_\lambda g(t) = \int_a^b K^+(t, x, \bar{\lambda}) w(x) g(x)\, dx \tag{5.16}$$

$t \in [a, b)$ almost everywhere, i.e., R^*_λ is an integral operator with kernel $K^+(t, x, \bar{\lambda})$ operating on the functions $g \in H_\Delta \subset H'$.

Let us find the relation between the kernels $K(x, t, \lambda)$ and $K^+(t, x, \bar{\lambda})$ of $(S - \lambda I)^{-1}$ and $(S^* - \bar{\lambda} I)^{-1}$, respectively. Put

$$(S - \lambda I)^{-1} f = \phi \quad \text{and} \quad (S^* - \bar{\lambda} I)^{-1} g = \phi^+.$$

Then

$$f = (S - \lambda I)\phi, \qquad g = (S^* - \bar{\lambda} I)\phi^+,$$

and the equation

$$(\phi, (S^* - \bar{\lambda} I)\phi^+) = ((S - \lambda I)\phi, \phi^+),$$

can be written in the form,

$$((S-\lambda I)^{-1}f, g) = (f, (S^* - \bar{\lambda} I)^{-1} g),$$

i.e.,

$$\int_a^b \int_a^b K(x,t,\lambda) w(x) w(t) f(t) \overline{g(x)}\, dx\, dt = \int_a^b \int_a^b \overline{K^+(t,x,\bar{\lambda})} w(x) w(t) f(t) \overline{g(x)}\, dx\, dt,$$

for any continuous functions $f, g \in H_\Delta$, and by construction (see (5.12) and (5.15)) $K(x,t,\lambda)$ and $K^+(t,x,\bar{\lambda})$ are continuous functions on $[a,b) \times [a,b)$. This gives us

$$K(x,t,\lambda) = \overline{K^+(t,x,\bar{\lambda})} \quad \text{for all } x, t \in \Delta,$$

and hence, since Δ is arbitrary,

$$K(x,t,\lambda) = \overline{K^+(t,x,\bar{\lambda})} \quad \text{for all } x, t \in [a,b). \tag{5.17}$$

Also, $\phi_k(x), \phi_k^+(t) \in L_w^2(a,b)$ for $k = 1, 2, \dots, n$ and for fixed t, $K(x,t,\lambda)$ is a linear combination of $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$, while, for fixed x, $K^+(t,x,\bar{\lambda})$ is a linear combination of $\phi_1^+(t), \phi_2^+(t), \dots, \phi_n^+(t)$. Hence,

$$\int_a^b |K(x,t,\lambda)|^2 w(x)\, dx < \infty, \qquad a \le t < b,$$
$$\int_a^b |K^+(t,x,\bar{\lambda})|^2 w(t)\, dt < \infty, \qquad a \le x < b,$$

and (5.17) implies that

$$\int_a^b |K(x,t,\lambda)|^2 w(t)\, dt = \int_a^b |K^+(t,x,\bar{\lambda})|^2 w(t)\, dt < \infty,$$
$$\int_a^b |K^+(t,x,\bar{\lambda})|^2 w(x)\, dx = \int_a^b |K(x,t,\lambda)|^2 w(x)\, dx < \infty.$$

Now, we can construct R_λ and R_λ^* on $H = L_w^2(a,b)$ as follows: R_λ is a bounded operator, hence is continuous on H. Let $f \in H$. Then, if $\chi_{[a,\beta]}$ is the characteristic function of $[a,\beta]$, $f_\beta = \chi_{[a,\beta]} f \in H'$ and $f_\beta \to f$ pointwise and in $L^2_w(a,b)$ as $\beta \to b$, hence

$$\begin{aligned} R_\lambda f(x) &= \lim_{\beta\to b} R_\lambda f_\beta(x) \\ &= \lim_{\beta\to b}\left\{\int_b^\beta K(x,t,\lambda)w(t)f_\beta(t)\,dt\right\}, \end{aligned}$$

in the sense of $L^2_w(a,b)$, i.e.,

$$||R_\lambda f - R_\lambda f_\beta|| \to 0 \quad \text{as } \beta \to b.$$

But, for any $x \in [a,b)$, $K(x,.,\lambda)\ w(.)f(.) \in L^1(a,b)$, and by the "dominated convergence theorem,"

$$\begin{aligned} &\lim_{\beta\to b}\int_a^\beta K(x,t,\lambda)w(t)f_\beta(t)\,dt \\ &\qquad = \int_a^b K(x,t,\lambda)w(t)f(t)\,dt, \quad \text{a.e. } x\in[a,b). \end{aligned}$$

Hence, for any $f \in L^2_w(a,b)$, we can write

$$R_\lambda f(x) = \int_a^b K(x,t,\lambda)w(t)f(t)\,dt$$

for $x \in [a,b)$ almost everywhere. Similarly,

$$R^*_\lambda g(t) = \int_a^b K^+(t,x,\bar\lambda)w(x)g(x)\,dx$$

for $t \in [a,b)$ almost everywhere. We have thus proved that R_λ and R^*_λ are integral operators for any well-posed extension S.

Now, it is clear from (5.10) for all $\lambda \in \Pi(T_0, T_0^+)$ with

$$\text{def}\,(T_0 - \lambda I) = \text{def}\,(T_0^+ - \bar\lambda I) = n,$$

that the functions $h_k(t)$, $k = 1,2,\dots,n$, belong to $L^2_w(a,b)$ since $h_k(t)$ is a linear combination of the functions $\phi_1^+(t), \phi_2^+(t), \dots, \phi_n^+(t)$ which lie in $L^2_w(a,b)$. Similarly, $h^+_w(x)$, $k = 1, 2, \ldots, n$, belong to $L^2_w(a,b)$. By the upper half of the formula (5.12), we have

$$\int_a^b w(x)\,dx \int_w^b |K(x,t,\lambda)|^2 w(t)\,dt < +\infty.$$

For the inner integral exists and is a linear combination of products $\phi_k(x)\overline{\phi_j^+(t)}$, $j,k = 1, 2, \ldots, n$, and these products are in $L^1_w(a,b)$ because each of the factors belongs to $L^2_w(a,b)$. Also by (5.17) and by the upper half of (5.15),

$$\int_a^b w(x)\,dx \int_a^x |K(x,t,\lambda)|^2 w(t)\,dt \\ = \int_a^b w(x)\,dx \int_a^x |K^+(t,x,\bar{\lambda})|^2 w(t)\,dt < +\infty.$$

Hence, we also have

$$\int_a^b \int_a^b |K(x,t,\lambda)|^2 w(x) w(t)\,dx\,dt < +\infty,$$

and the theorem is completely proved for any well-posed extension. □

Remark 5.2. It follows immediately from Theorem 5.1 that, if

$$\operatorname{def}(T_0 - \lambda I) = \operatorname{def}(T_0^+ - \bar{\lambda} I) = n \quad \text{for all } \lambda \in \Pi(T_0, T_0^+),$$

and S is well posed with respect to T_0 and T_0^+ with $\lambda \in \rho(S)$, then $R_\lambda = (S - \lambda I)^{-1}$ is a Hilbert-Schmidt integral operator. Thus, it is a completely continuous operator, and consequently its spectrum is discrete and consists of isolated eigenvalues having finite algebraic (so geometric) multiplicity with zero as the only point of accumulation. Thus the spectra of all well-posed operators S are discrete, i.e.,

$$\sigma_{ek}(S) = \varnothing \quad \text{for } k = 1, 2, 3, 4, 5. \tag{5.18}$$

We refer to [**3**, Theorem 9.3.1] for more details.

Corollary 5.3. *Let* $\lambda \in \Pi(T_0, T_0^+)$ *with*

$$\operatorname{def}(T_0 - \lambda I) = \operatorname{def}(T_0^+ - \bar{\lambda} I) = n.$$

Then

$$\sigma_{ek}(S) = \varnothing, \quad \textit{for } k = 1, 2, 3, \tag{5.19}$$

for all operators S *which are regularly solvable with respect to the compatible adjoint pair* T_0 *and* T_0^+*.*

Proof. Since

$$\operatorname{def}(T_0 - \lambda I) = \operatorname{def}(T_0^+ - \bar{\lambda} I) = n, \quad \text{for all } \lambda \in \Pi(T_0, T_0^+),$$

then we have from [**3**, Theorem 3.3.5] that

$$\dim\{D(S)/D_0\} = \operatorname{def}(T_0 - \lambda I) = n,$$
$$\dim\{D(S^*)/D_0^+\} = \operatorname{def}(T_0^+ - \bar{\lambda} I) = n.$$

Thus, S is an n-dimensional extension of T_0 and so, by [**3**, Corollary 9.4.2] that

$$\sigma_{ek}(S) = \sigma_{ek}(T_0), \quad \text{for } k = 1, 2, 3. \tag{5.20}$$

In particular, if S is well posed (say the Visik extension) we get from (5.18) and (5.20) that $\sigma_{ek}(T_0) = \varnothing$ for $k = 1, 2, 3$. On applying (5.20) again to any of the regularly solvable operators S under consideration, we have that

$$\sigma_{ek}(S) = \varnothing \quad \text{for } k = 1, 2, 3. \qquad \square$$

Remark 5.4. From (2.1) and by (5.19), we conclude that $\Delta_3(S) = \mathbf{C}$. Hence $(S - \lambda I)$ is a Fredholm operator for all $\lambda \in \mathbf{C}$. Therefore λ belongs to the spectrum of S if and only if range $R(S - \lambda I)$ is closed, and the following conditions are satisfied:

(a) $\operatorname{nul}(S - \lambda I) = \operatorname{nul}(S^* - \bar{\lambda} I) \neq 0$,

(b) $\operatorname{nul}(S - \lambda I) \neq 0$, $\operatorname{nul}(S^* - \bar{\lambda} I) = 0$,

(c) $\operatorname{nul}(S - \lambda I) = 0$, $\operatorname{nul}(S^* - \bar{\lambda} I) \neq 0$, it is of interest to emphasize that if S is well-posed only (b) is true and $\operatorname{nul}(S-\lambda I) = \operatorname{nul}(S^* - \lambda I)$.

Remark 5.5. If $\operatorname{def}(T_0 - \lambda I) = \operatorname{def}(T_0^+ - \bar{\lambda} I) = n$ for some (and hence all) $\lambda \in \Pi(T_0, T_0^+)$, then $\Pi(T_0, T_0^+) = \mathbf{C}$.

Since T_0 and T_0^+ has no eigenvalues, then $(T_0 - \lambda I)^{-1}$ and $(T_0^+ - \bar{\lambda} I)^{-1}$ exist and their domains $R(T_0 - \lambda I)$ and $R(T_0^+ - \bar{\lambda} I)$ are closed subspaces of $L_w^2(a,b)$. Hence, since T_0 and T_0^+ are closed operators, then $(T_0 - \lambda I)^{-1}$ and $(T_0^+ - \bar{\lambda} I)^{-1}$ are also closed and so it follows from the closed graph theorem that $(T_0 - \lambda I)^{-1}$ and $(T_0^+ - \bar{\lambda} I)^{-1}$ are bounded, and hence $\Pi(T_0, T_0^+) = \Pi(T_0) = \Pi(T_0^+) = \mathbf{C}$.

Remark 5.6. If in Proposition 4.3 it is given that $\lambda_0 \in \Pi(T_0, T_0^+)$, then the proposition is a consequence of Remark 5.5.

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