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ON AN L^1-FORCED AUTONOMOUS DUFFING'S EQUATION WITH PERIODIC BOUNDARY CONDITIONS IN THE PRESENCE OF DAMPING

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ABSTRACT. Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function, $e : [0,1] \to \mathbf{R}$ be a function in $L^1[0,1]$ and $c \in \mathbf{R}$, $c \neq 0$, be given. Suppose that $\alpha \in \mathbf{R}$, $1 \leq \alpha < 2$ be such that $\lim_{|u|\to\infty} |\frac{g(u)}{u^\alpha}| < \infty$, and let $g_- = \limsup_{\alpha\to-\infty} g(u), g_+ = \liminf_{u\to\infty} g(u)$ so that $-\infty \leq g_- < g_+ \leq \infty$. Then if $g_- < \int_0^1 e(x)\,dx < g_+$, the boundary value problem $u'' + cu' + g(u) = e$, $u(0) = u(1)$, $u'(0) = u'(1)$ has at least one solution. It is also proved that if g is increasing in $\mathbf{R}$ (not necessarily strictly) and g is Lipschitz-continuous with Lipschitz constant k, such that $k < 4\pi^2 + c^2$ then the set of solutions of $u'' + cu' + g(u) = e, u(0) = u(1)$, $u'(0) = u'(1)$ is a non-empty, compact, connected and acyclic set.

1. Introduction. Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function, $e : [0,1] \to \mathbf{R}$ and $c \in \mathbf{R}$, $c \neq 0$, be given. This paper is devoted to the study of the forced autonomous Duffing's equation

$$\begin{gathered} u'' + cu' + g(u) = e(x), \qquad 0 < x < 1 \\ u(0) = u(1), \qquad u'(0) = u'(1). \end{gathered} \tag{1.1}$$

This equation was studied by the author in [**2**], when $e(x) \in L^2[0,1]$. It was proved in [**2**] that if $g_- = \limsup_{u\to-\infty} g(u)$, $g_+ = \limsup_{u\to\infty} g(u)$ and $-\infty \leq g_- < \int_0^1 e(x)\,dx < g_+ \leq \infty$, then the equation (1.1) has at least one solution. The motivation to study equation (1.1) came from the observation that, if $c \neq 0$, the linear boundary value problem

$$\begin{gathered} u'' + cu' = \lambda u, \qquad 0 < x < 1, \\ u(0) = u(1), \qquad u'(0) = u'(1), \end{gathered} \tag{1.2}$$

has $\lambda = 0$ as its only eigenvalue.

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The purpose of this paper is to study the equation (1.1) when $e \in L^1[0,1]$. It is shown that the equation (1.1) has at least one solution when $g(u)$ grows asymptotically as u^α with $\alpha < 2$ and $e \in L^1[0,1]$. This is rather different than the case $e \in L^2[0,1]$ when no asymptotic growth is required on g. Some results on the structure of the set of solutions of (1.1) are also presented.

2. Main results. Let X and Y denote the Banach spaces $X = C[0,1]$ and $Y = L^1[0,1]$ with their usual norms. Let Y_2 be the subspace of Y spanned by the constant function 1 on $[0,1]$, i.e.,

$$Y_2 = \{u \in Y \mid u(x) \equiv c, \text{ for a.e. } x \in [0,1], c \in \mathbf{R}\},$$

and let Y_1 be the subspace of Y such that $Y = Y_1 \oplus Y_2$. We note that for $u \in Y$ we can write

$$u(x) = \left(u(x) - \int_0^1 u(x)\,dx\right) + \left(\int_0^1 u(x)\,dx\right) \tag{2.1}$$

for $x \in [0,1]$. We define the canonical projection operators $P : Y \to Y_1$, $Q : Y \to Y_2$ by

$$\begin{aligned} P(u)(x) &= u(x) - \int_0^1 u(x)\,dx, \\ Q(u) &= \int_0^1 u(x)\,dx, \end{aligned} \tag{2.2}$$

for $u \in Y$. Clearly, $Q = I - P$, where I denotes the identity mapping on Y, and the projections P and Q are continuous. Now let $X_2 = X \cap Y_2$. Clearly, X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \oplus X_2$. We note that $P(X) \subset X_1$, $Q(X) \subset X_2$ and the projections $P|X : X \to X_1$, $Q|X : X \to X_2$ are continuous. In the following, X, Y, P and Q will refer to the Banach spaces and projections as defined, and we do not distinguish between $P, P|X$ (respectively $Q, Q|X$) and depend on the context for proper meaning.

Also, for $u \in X$, $v \in Y$, let $(u,v) = \int_0^1 u(x)v(x)\,dx$ denote the duality pairing between X and Y. We note that, for $u \in X$, $v \in Y$, so that $u = Pu + Qu$, $v = Pv + Qv$, we have

$$(u,v) = (Pu, Pv) + (Qu, Qv). \tag{2.3}$$

Let $c \in \mathbf{R}$, $c \neq 0$ be given. Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{u \in X \mid u'(x) \in AC[0,1], u(0) = u(1), u'(0) = u'(1)\}, \tag{2.4}$$

and for $u \in D(L)$,

$$Lu = u'' + cu'. \tag{2.5}$$

(Here $AC[0,1]$ denotes the space of real-valued absolutely continuous functions on $[0,1]$.) It is easy to see that L is a linear Fredholm mapping with $\ker L = X_2$, $\operatorname{Im} L = Y_1$. Further, the mapping $K : Y_1 \to X_1$, defined for $u \in Y_1$ by

$$(Ku)(x) = v(x) - \int_0^1 v(x)\, dx, \tag{2.6}$$

where

$$v(x) = \int_0^x \int_0^\xi e^{c(t-\xi)} u(t)\, dt\, d\xi - \frac{e^{-cx} - 1}{c(e^c - 1)} \int_0^1 e^{ct} u(t)\, dt, \tag{2.7}$$

(note that we have assumed $c \neq 0$), satisfies the following conditions:
(2.8)
(i) for $u \in Y$, $KP(u) \in D(L)$, $LKP(u) = P(u)$,
(ii) for $u \in L^2[0,1]$, $(KP(u), P(u)) \geq -\dfrac{1}{(4\pi^2 + c^2)} ||P(u)||^2_{L^2[0,1]}$.

Indeed, note for $v = KP(u) \in D(L)$,

$$(KP(u), P(u)) = (v, Lv) = -\int_0^1 v'^2 \geq -\frac{1}{4\pi^2 + c^2} ||Lv||^2_{L^2[0,1]}$$

and so $(KP(u), P(u)) \geq -(1/(4\pi^2 + c^2))||P(u)||^2_{L^2[0,1]}$; since

$$\begin{aligned} ||Lv||^2_{L^2[0,1]} &= \int_0^1 (v'' + cv')^2\, dx \\ &= \int_0^1 [(v'')^2 + 2cv'v'' + c^2(v')^2]\, dx \\ &= \int_0^1 [(v'')^2 + c^2(v')^2]\, dx \\ &\geq (4\pi^2 + c^2) \int_0^1 v'^2\, dx. \end{aligned}$$

Let, now, $g : \mathbf{R} \to \mathbf{R}$ be a given continuous function. Let $N : X \to X \subset Y$ be the nonlinear mapping defined by

$$(Nu)(x) = g(u(x)), \qquad x \in [0,1]$$

for $u \in X$. It is then easy to see, using the Arzela-Ascoli theorem, that $KPN : X \to X_1$ is continuous and compact.

Theorem 1. *Let $g : \mathbf{R} \to \mathbf{R}$ be a given continuous function. Let c, a, A, r and R with $a \le A$, $r < 0 < R$, $c \ne 0$ be such that*

$$g(u) \ge A, \quad \text{for } u \ge R, \tag{2.9}$$

and

$$g(u) \le a, \quad \text{for } u \le r.$$

Suppose further that $a \in \mathbf{R}$, $1 \le \alpha < 2$ and $\beta \in \mathbf{R}$, $\beta \ge 0$ are such that

$$\lim_{|u| \to \infty} \left| \frac{g(u)}{u^\alpha} \right| = \beta. \tag{2.10}$$

Then, for every given function $e(x) \in L^1[0,1]$ with $a \le \int_0^1 e(x)\,dx \le A$, the Duffing's equation

$$\begin{gathered} u'' + cu' + g(u) = e, \qquad 0 < x < 1, \\ u(0) = u(1), \qquad u'(0) = u'(1) \end{gathered} \tag{2.11}$$

has at least one solution.

Proof. Define functions $g_1 : \mathbf{R} \to \mathbf{R}$ and $e_1 : [0,1] \to \mathbf{R}$ by setting

$$g_1(u) = g(u) - \frac{A+a}{2},$$
$$e_1(x) = e(x) - \frac{A+a}{2}.$$

Then $g_1 : \mathbf{R} \to \mathbf{R}$ is a continuous function and $e_1 : [0,1] \to \mathbf{R}$ is such that $e_1(x) \in L^1[0,1]$. Furthermore,

$$g_1(u) \ge \frac{1}{2}(A-a) \ge 0, \quad \text{for } u \ge R,$$
$$g_1(u) \le \frac{1}{2}(a-A) \le 0, \quad \text{for } u \le r,$$

and

$$\frac{1}{2}(a-A) \leq \int_0^1 e_1(x)\,dx \leq \frac{1}{2}(A-a).$$

Now, the Duffing's equation (2.11) is equivalent to the equation

$$\begin{aligned} u'' + cu' + g_1(u) = e_1, \qquad 0 < x < 1, \\ u(0) = u(1), \qquad u'(0) = u'(1). \end{aligned} \tag{2.12}$$

Now, for $X = C[0,1]$ and $Y = L^1[0,1]$ we consider the Niemytski operator $N : X \to Y$ defined for $u \in X$ by

$$(Nu)(x) = g_1(u(x)), \qquad x \in [0,1],$$

and the linear operator $L : D(L) \subset X \to Y$ defined in (2.4) and (2.5).

Now the equation (2.12) is equivalent to the operator equation

$$Lu + Nu = e_1, \tag{2.13}$$

in X. Now to solve (2.13) it suffices to solve the system of equations

$$\begin{aligned} Pu + KPNu = KPe_1, \\ QNu = Qe_1, \end{aligned} \tag{2.14}$$

in X. Indeed, if $u \in X$ solves (2.14), then $u \in D(L)$ and

$$\begin{aligned} LPu + LKPNu = Lu + PNu = LKPe_1 = Pe_1, \\ QNu = Qe_1, \end{aligned}$$

which gives, on adding, that $Lu + Nu = e_1$.

Now (2.14) is clearly equivalent to the single equation

$$Pu + QNu + KPNu = KPe_1 + Qe_1, \tag{2.15}$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can, therefore, apply the version given in [**5**, Theorem 1, Corollary 1, **6**, Theorem IV.4, **4**] of the Leray-Schauder continuation theorem, which ensures the existence of a solution for (2.15) if the set of all possible solutions of the family of equations

$$Pu + (1-\lambda)Qu + \lambda QNu + \lambda KPNu = \lambda KPe_1 + \lambda Qe_1, \tag{2.16}$$

$\lambda \in]0,1[$, is a priori bounded, independently of λ. Now (2.16) is equivalent to the system of equations

$$\begin{aligned} Pu + \lambda KPNu &= \lambda KPe_1, \\ (1-\lambda)Qu + \lambda QNu &= \lambda Qe_1. \end{aligned} \tag{2.17}$$

Let, now, $u_\lambda \in X$ be a solution of (2.17) for some $\lambda \in]0,1[$, then $u_\lambda \in D(L)$ and

$$\begin{aligned} Pu_\lambda + \lambda KPNu_\lambda &= \lambda KPe_1, \\ (1-\lambda)Qu_\lambda + \lambda QNu_\lambda &= \lambda Qe_1. \end{aligned} \tag{2.18}$$

It follows that

$$Lu_\lambda + (1-\lambda)Qu_\lambda + Nu_\lambda = \lambda e_1,$$

i.e.,

$$\begin{gathered} u''_\lambda + cu'_\lambda + (1-\lambda)\int_0^1 u_\lambda(x)\,dx + g_1(u_\lambda) = \lambda e_1, \\ u_\lambda(0) = u_\lambda(1), \qquad u'_\lambda(0) = u'_\lambda(1). \end{gathered} \tag{2.19}$$

Now, multiplying the equation in (2.19) by u'_λ and integrating over $[0,1]$, we obtain that

$$c\int_0^1 u'^2_\lambda = \lambda \int_0^1 e_1(x) u'_\lambda(x)\,dx,$$

which implies, using Hölder's inequality, that

$$|c|\,||u'_\lambda||^2_{L^2[0,1]} \le ||e_1||_{L^1[0,1]} \cdot ||u'_\lambda||_{L^\infty[0,1]}. \tag{2.20}$$

Since now $u_\lambda(0) = u_\lambda(1)$, there exists a $\xi_\lambda \in (0,1)$ such that $u'_\lambda(\xi_\lambda) = 0$. It follows that, for $x \in [0,1]$,

$$|u'_\lambda(x)| = \left| \int_{\xi_\lambda}^x u''_\lambda(t)\,dt \right| \le ||u''_\lambda||_{L^1[0,1]}. \tag{2.21}$$

Now, we get from the equation in (2.19) that

$$\begin{aligned} ||u''_\lambda||_{L^1[0,1]} \le |c| \cdot ||u'_\lambda||_{L^1[0,1]} &+ ||u_\lambda||_{L^1[0,1]} \\ &+ ||g_1(u_\lambda)||_{L^1[0,1]} + ||e_1||_{L^1[0,1]}. \end{aligned} \tag{2.22}$$

Next, assumption (2.10) implies that there exists a constant $C \geq 0$ such that

$$|g_1(u)| \leq (\beta+1)|u|^\alpha + C,$$

for $u \in \mathbf{R}$; and, accordingly, we get

(2.23)
$$\begin{aligned}||g_1(u)||_{L^1[0,1]} &\leq (\beta+1)||u||^\alpha_{L^2[0,1]} + C \\ &\leq 2^{\alpha-1}(\beta+1)(||Pu||^\alpha_{L^2[0,1]} + ||Qu||^\alpha) + C \\ &\leq 2^{\alpha-1}(\beta+1)\left(\frac{1}{(2\pi)^\alpha}||u'||^\alpha_{L^2[0,1]} + ||Qu||^\alpha\right) + C.\end{aligned}$$

It now follows, from (2.20), (2.22) and (2.23), that

(2.24)
$$\begin{aligned}|c|\,||u'_\lambda||^2_{L^2[0,1]} \leq C_1||u'_\lambda||_{L^2[0,1]} + C_2||u'_\lambda||^\alpha_{L^2[0,1]} + C_3||Qu_\lambda|| \\ + C_4||Qu_\lambda||^\alpha + C_5,\end{aligned}$$

for some constants $C_1, C_2, C_3, C_4, C_5 \geq 0$, independent of $\lambda \in (0,1)$. Since, now, $1 \leq \alpha < 2$, it follows easily from (2.24) that there exist constants $C, D \geq 0$, independent of $\lambda \in (0,1)$ such that

(2.25)
$$||u'_\lambda||^2_{L^2[0,1]} \leq C + D||Qu_\lambda||^\alpha.$$

Now we claim that there exists a $\xi \in [0,1]$ such that $r \leq u_\lambda(\xi) \leq R$. Indeed, suppose that $u_\lambda(x) \geq R$ for every $x \in [0,1]$. Then we get from the second equation in (2.18) and our assumptions on g_1 and e_1 that

$$\begin{aligned}(1-\lambda)R + \lambda\cdot\frac{1}{2}(A-a) &\leq (1-\lambda)Qu_\lambda + \lambda QNu_\lambda \\ &= \lambda Qe_1 \leq \lambda\cdot\frac{1}{2}(A-a),\end{aligned}$$

so that $(1-\lambda)R \leq 0$, which is a contradiction since $\lambda \in\,]0,1[$ and $R > 0$. Similarly, $u_\lambda \leq r$ for $x \in [0,1]$ leads to a contradiction. This proves the claim.

It next follows that, for every $x \in [0,1]$,

(2.26)
$$\begin{aligned}|u_\lambda(x)| &\leq \max(-r, R) + \int_0^1 |u'_\lambda(x)|\,dx \\ &\leq \max(-r,R) + ||u'_\lambda||_{L^2[0,1]}.\end{aligned}$$

Noting that $||Qu|| \leq ||u||_{L^\infty[0,1]}$ for $u \in X$, we get from (2.25) and (2.26) that

$$||u_\lambda||_X^2 = ||u_\lambda||^2_{L^\infty[0,1]} \leq C + D||u_\lambda||_X^\alpha, \tag{2.27}$$

for some constants $C \geq 0$, $D \geq 0$, independent of $\lambda \in (0,1)$. (Note that the constants C and D in (2.27) are not the same as those in (2.25).) Since now $\alpha < 2$, it follows that there is a constant $C \geq 0$, independent of $\lambda \in (0,1)$, such that

$$||u_\lambda||_X \leq C.$$

This completes the proof of the Theorem. □

Corollary 1. *Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function and $c \in \mathbf{R}$, $c \neq 0$ be given. Let $g_- = \limsup_{u\to-\infty} g(u)$, $g_+ = \liminf_{u\to+\infty} g(u)$ be such that $-\infty \leq g_- < g_+ \leq \infty$. Suppose further that $\alpha \in \mathbf{R}$, $1 \leq \alpha < 2$ and $\beta \in \mathbf{R}$, $\beta \geq 0$ be such that*

$$\lim_{|u|\to\infty} \left| \frac{g(u)}{u^\alpha} \right| = \beta.$$

Then, for every $e(x) \in L^1[0,1]$ with $g_- < \int_0^1 e(x)\,dx < g_+$, the Duffing's equation

$$\begin{aligned} u'' + cu' + g(u) = e, &\qquad 0 < x < 1, \\ u(0) = u(1), &\qquad u'(0) = u'(1), \end{aligned} \tag{2.28}$$

has at least one solution.

Theorem 2. *Let $g : \mathbf{R} \to \mathbf{R}$ be a strictly increasing function and $c \in \mathbf{R}$, $c \neq 0$, given. Let $g_- = \lim_{u\to-\infty} g(u)$ and $g_+ = \lim_{u\to\infty} g(u)$. Suppose that g is a Lipschitz continuous function with constant k, i.e., for $u, v \in \mathbf{R}$,*

$$|g(u) - g(v)| \leq k|u - v|, \tag{2.29}$$

with

$$k < 4\pi^2 + c^2.$$

Then, for every $e \in L^1[0,1]$ *with* $-\infty \le g_- < \int_0^1 e(x)\,dx < g_+ \le \infty$, *the boundary value problem*

$$u'' + cu' + g(u) = e, \qquad 0 < x < 1, \qquad (2.30)$$
$$u(0) = u(1), \qquad u'(0) = u'(1),$$

has exactly one solution u *in* $X = C[0,1]$.

Proof. It suffices to show that (2.30) has exactly one solution u in X.

Let, now, u_1 and $u_2 \in X$ be two different solutions for (2.30). Then

$$u_1'' - u_2'' + c(u_1' - u_2') + g(u_1) - g(u_2) = 0, \qquad 0 < x < 1. \qquad (2.31)$$

Note that (2.23) implies $u_1'' - u_2'' \in L^2[0,1]$. It follows that

$$\begin{aligned} 0 &= -\int_0^1 (u_1' - u_2')^2\,dx + \int_0^1 (g(u_1) - g(u_2))(u_1 - u_2)\,dx \\ &= -\int_0^1 (u_1' - u_2')^2\,dx + \int_0^1 |g(u_1) - g(u_2)||u_1 - u_2|\,dx \\ &\ge -\frac{1}{4\pi^2 + c^2}||Lu_1 - Lu_2||^2_{L^2[0,1]} + \frac{1}{k}\int_0^1 |g(u_1) - g(u_2)|^2\,dx \\ &= \left(\frac{1}{k} - \frac{1}{4\pi^2 + c^2}\right)\int_0^1 |g(u_1) - g(u_2)|^2\,dx, \end{aligned}$$

in view of (2.31). Using, now, (2.29), we get that $g(u_1(x)) = g(u_2(x))$ for almost every $x \in [0,1]$, which implies $u_1(x) = u_2(x)$ for almost every $x \in [0,1]$, since g is strictly increasing on $\mathbf{R}$. Hence, $u_1(x) = u_2(x)$ for every $x \in [0,1]$ since u_1 and u_2 are continuous in $[0,1]$.

This completes the proof of the Theorem. □

We next present some results on the structure of the set of solutions for the Duffing's equation (1.1).

Theorem 3. *Let* $g : \mathbf{R} \to \mathbf{R}$ *be a continuous function and* $c \in \mathbf{R}$, $c \ne 0$, *given. Let* $g_- = \limsup_{u\to-\infty} g(u)$, $g_+ = \limsup_{u\to\infty} g(u)$ *be* *such that* $-\infty \le g_- < g_+ \le \infty$. *Suppose further that* $\alpha \in \mathbf{R}$, $1 \le \alpha < 2$ *and* $\beta \in \mathbf{R}$, $\beta \ge 0$, *such that*

$$\lim_{|u|\to\infty} \left|\frac{g(u)}{u^\alpha}\right| = \beta.$$

Then, for $e(x) \in L^1[0,1]$ *with* $g_- < \int_0^1 e(x)\,dx < g_+$, *the set of solutions of the Duffing's equation*

$$\begin{aligned} u'' + cu' + g(u) &= e, \qquad 0 < x < 1, \\ u(0) = u(1), &\qquad u'(0) = u'(1), \end{aligned} \tag{2.32}$$

is a nonempty, compact subset of $C^1[0,1]$.

Proof. The set of solutions of (2.32) is nonempty by Theorem 1. Further, following the lines of the proof of Theorem 1, it is easy to show that the set of solutions of (2.32) is bounded in $W^{2,1}(0,1)$. Accordingly, it follows by the compact embedding of $W^{2,1}(0,1) \subset C^1[0,1]$ it is a compact subset of $C^1[0,1]$.

Hence, the theorem. □

Theorem 4. *Let* $g : \mathbf{R} \to \mathbf{R}$ *be an increasing* (*not necessarily strictly*) *function and* $c \in \mathbf{R}$, $c \ne 0$, *be given. Suppose that* g *is Lipschitz-continuous with Lipschitz constant* k, *i.e., for* $u, v \in \mathbf{R}$,

$$|g(u) - g(v)| \le k|u - v|,$$

with

$$k < 4\pi^2 + c^2.$$

Let $g_\pm = \lim_{u\to\pm\infty} g(u)$ *so that* $-\infty \le g_- < g_+ \le \infty$. *Then, for every* $e(x) \in L^1[0,1]$ *with* $g_- < \int_0^1 e(x)\,dx < g_+$, *the set* S *of solutions of* (2.32) *is a nonempty, compact, connected and acyclic subset of* $D(L)$, *where* L *is defined in* (2.4) *and* (2.5).

Proof. The set S of solutions of (2.32) is a nonempty, compact subset of $C^1[0,1]$ by Theorem 3. Now to show that S is connected, for each $n = 1, 2, \ldots$, consider the functions $g_n : \mathbf{R} \to \mathbf{R}$ defined by

$g_n(u) = g(u) + (1/n)u$. Then, for each n, g_n is a strictly-increasing function with Lipschitz constant $k + 1/n$. Since now $k < 4\pi^2 + c^2$, there exists an n_0 such that $k + 1/n < 4\pi^2 + c^2$ for $n \geq n_0$. Now, for each $v \in S$, define

$$S_n(v) = \left\{ u \in D(L) \mid Lu + g(u) + \frac{1}{n}u = Lv + g(v) + \frac{1}{n}v \right\}.$$

For $n \geq n_0$, it follows by Theorem 3 that $S_n(v)$ is a set consisting of a singleton-point, since $(g_n)_- = -\infty$ and $(g_n)+ = \infty$; and so $S_n(v)$ is a connected set. Using, now, a result of Bebernes-Martelli [**1**], we conclude that S is connected. Finally, for $\rho > 0$, we see that

$$\begin{aligned} r_n &= \sup\{||g_n(u) - g(u)||_{L^2[0,1]} \mid u \in D(L), ||u||_{L^2[0,1]} = \rho\} \\ &= \frac{1}{n}\rho, \end{aligned}$$

so that $\lim_{n\to\infty} r_n = 0$. Also, for every $v \in D(L)$, the problem

$$Lu + N_n(u) = u'' + cu' + g(u) + \frac{1}{n}u = v,$$

has exactly one solution for every $v \in D(L)$. It follows, again using a result from [**1**], that S is acyclic.

This completes the proof of Theorem 4. □

Remark 1. Theorem 3 generalizes Corollary 10 of [**3**]. Also, Theorem 4 generalizes Theorem 9 of [**3**].

Remark 2. If g, in Theorem 4, satisfies the condition $g_- \leq g(u) \leq g_+$ for every $u \in \mathbf{R}$, instead of increasing, then it is easy to show, using the example of $g : \mathbf{R} \to \mathbf{R}$ used in Example 11 of [**3**], that the set of solutions of

$$\begin{gathered} u'' + u' + g(u) = 0, \\ u(0) = u(1), \qquad u'(0) = u'(1), \end{gathered} \tag{2.33}$$

is not a connected set. Indeed, g is defined by

$$g(u) = \begin{cases} -1 & \text{if } u \leq -2 \\ u + 1 & \text{if } -2 < u \leq 0 \\ -u + 1 & \text{if } 0 < u < 2 \\ u - 3 & \text{if } 2 < u \leq 4 \\ 1 & \text{if } u > 4. \end{cases}$$

We see that $g(-1) = g(1) = g(3) = 0$. Also, it is immediate by multiplying the equation in (2.33) by u' and integrating over $[0,1]$ that $\int_0^1 u'^2\,dx = 0$, so that $u(x) \equiv$ constant, for $x \in [0,1]$. Accordingly, the set S of solutions of (2.33) is given by

$$S = \{u_1, u_2, u_3 \mid u_1(x) \equiv -1, u_2(x) \equiv 1, u_3(x) \equiv 3, x \in [0,1]\}.$$

Clearly, S is not a connected set.

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